An Introduction to Physical Layer Network Coding: Lattice Codes as Groups

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Cooperative Wireless Networks

Wireless networks must deal with interference & noise

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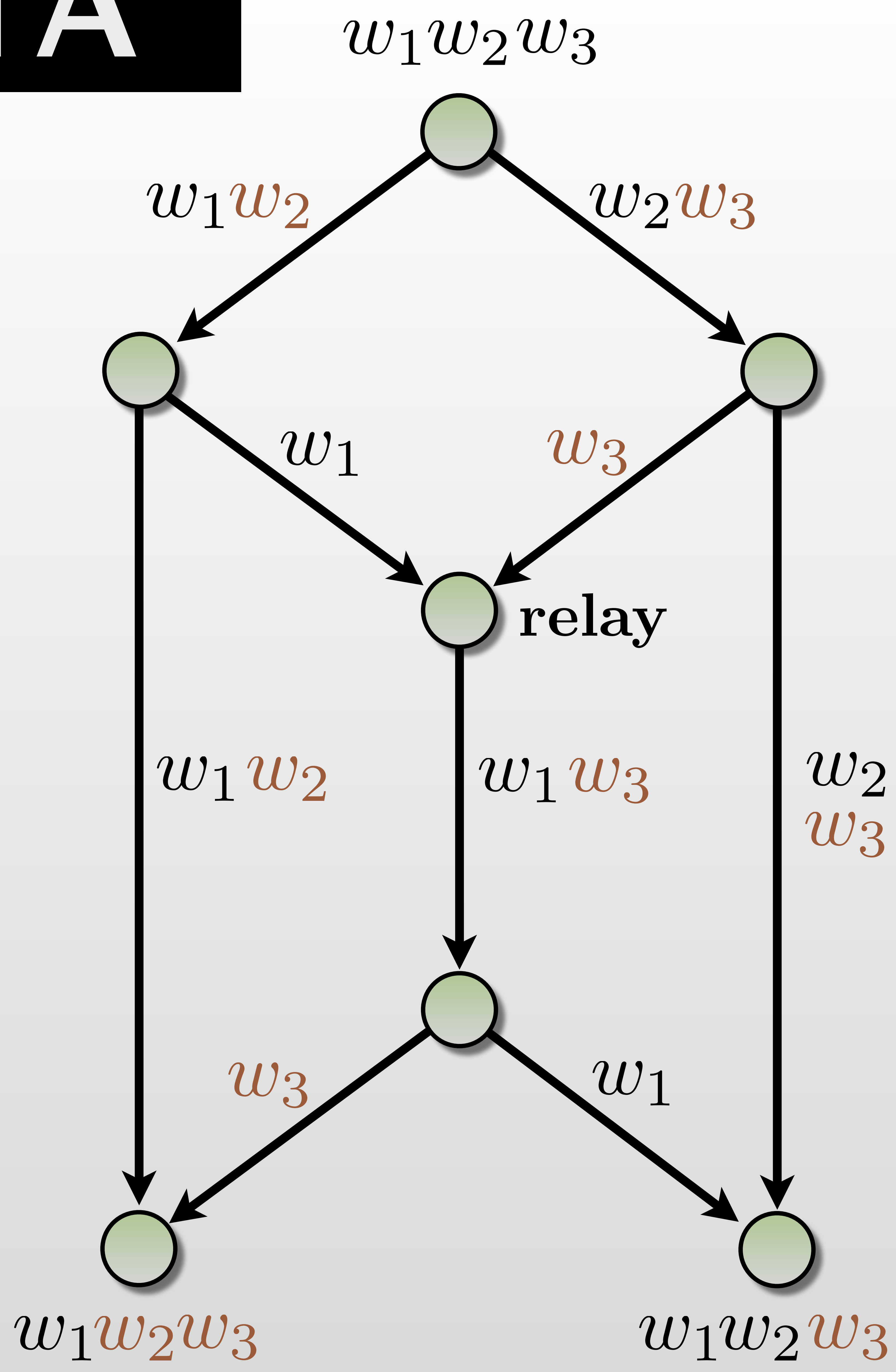
Wireless networks must deal with interference & noise

Overview

1A

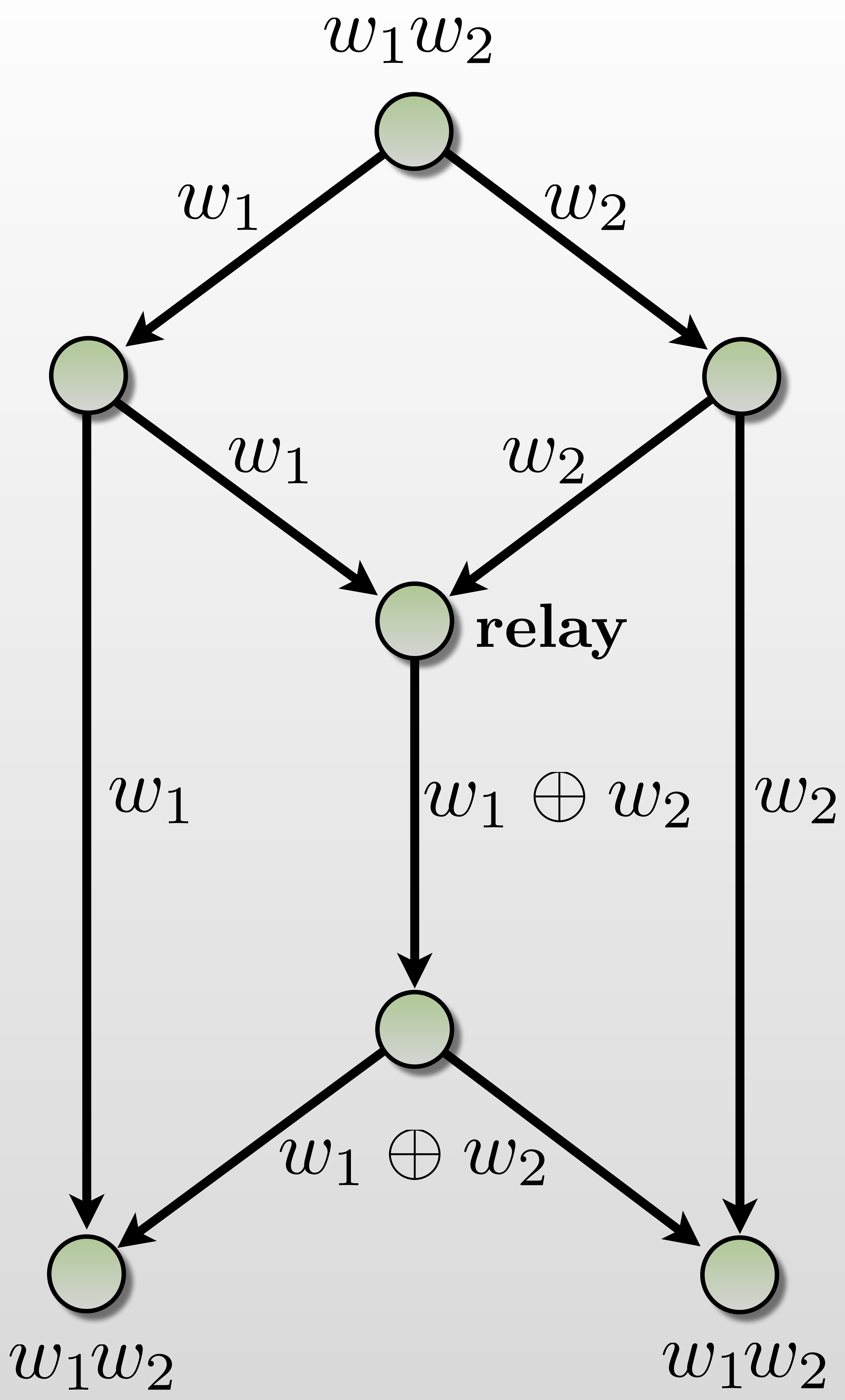


Routing vs. Network Coding

Capacity: max rate from source to destination

Routing

- Capacity $= 3/2$



Routing vs. Network Coding

Capacity: max rate from source to destination

Routing

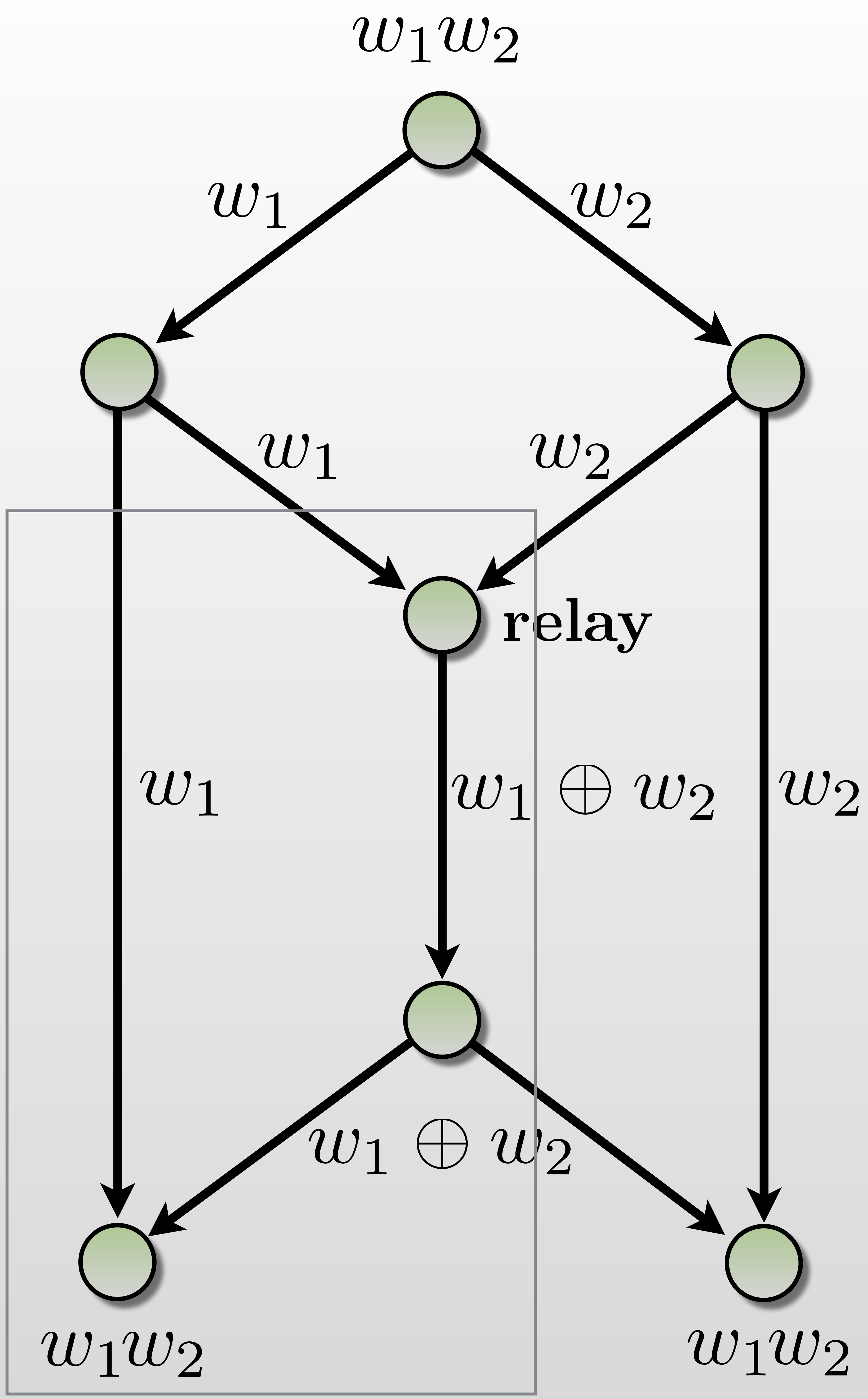
- Capacity $= 3/2$

Network Coding

- Internal nodes perform linear operations
- Capacity $= 2$

Forwarding combinations of messages can increase capacity

Routing vs. Network Coding



matrix form...

Capacity: max rate from source to destination

Routing

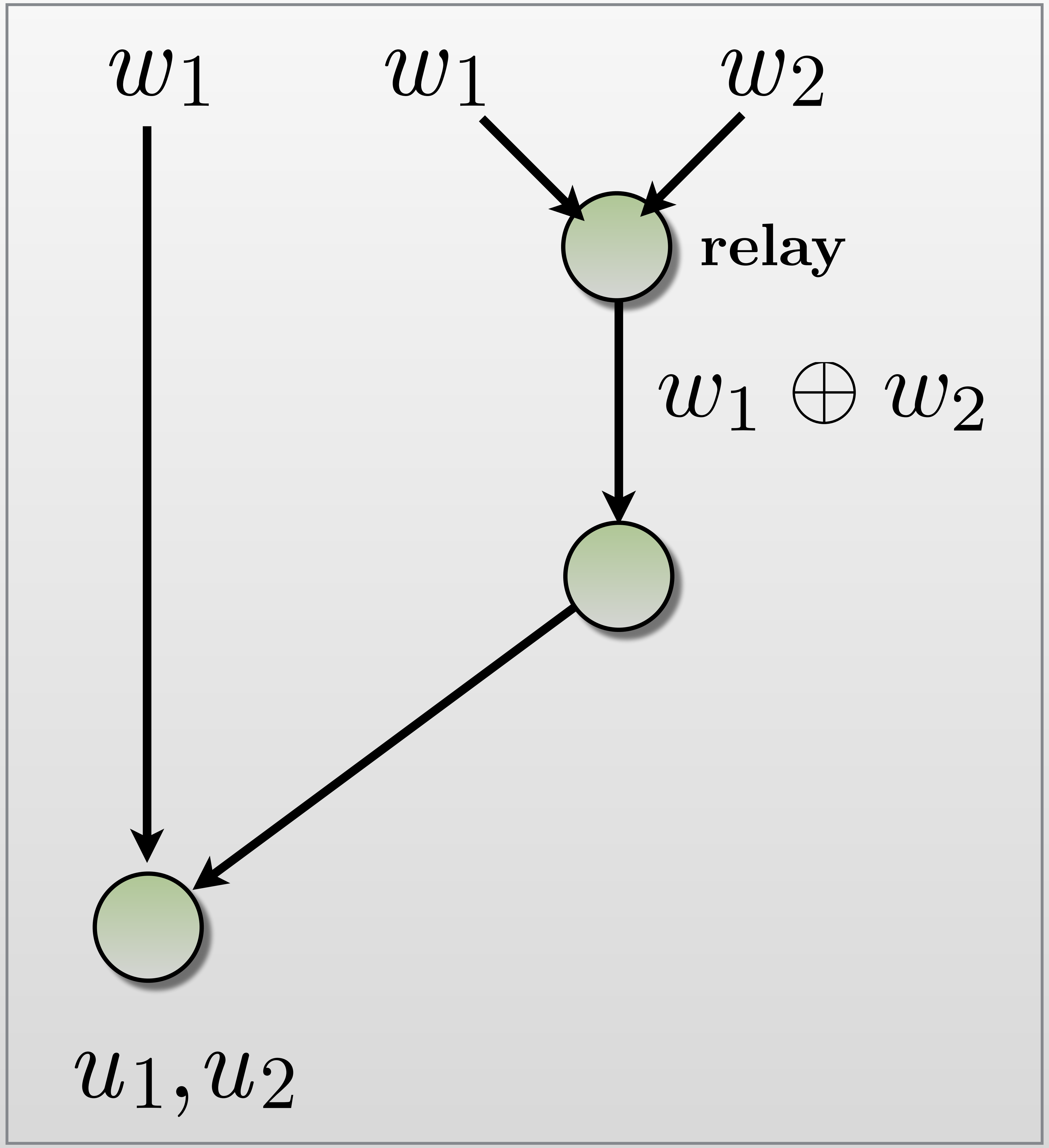
- Capacity $= 3/2$

Network Coding

- Internal nodes perform linear operations
- Capacity $= 2$

Forwarding combinations of messages can increase capacity

Matrix Form Recovery of Messages



2 received messages and

2 desired messages:

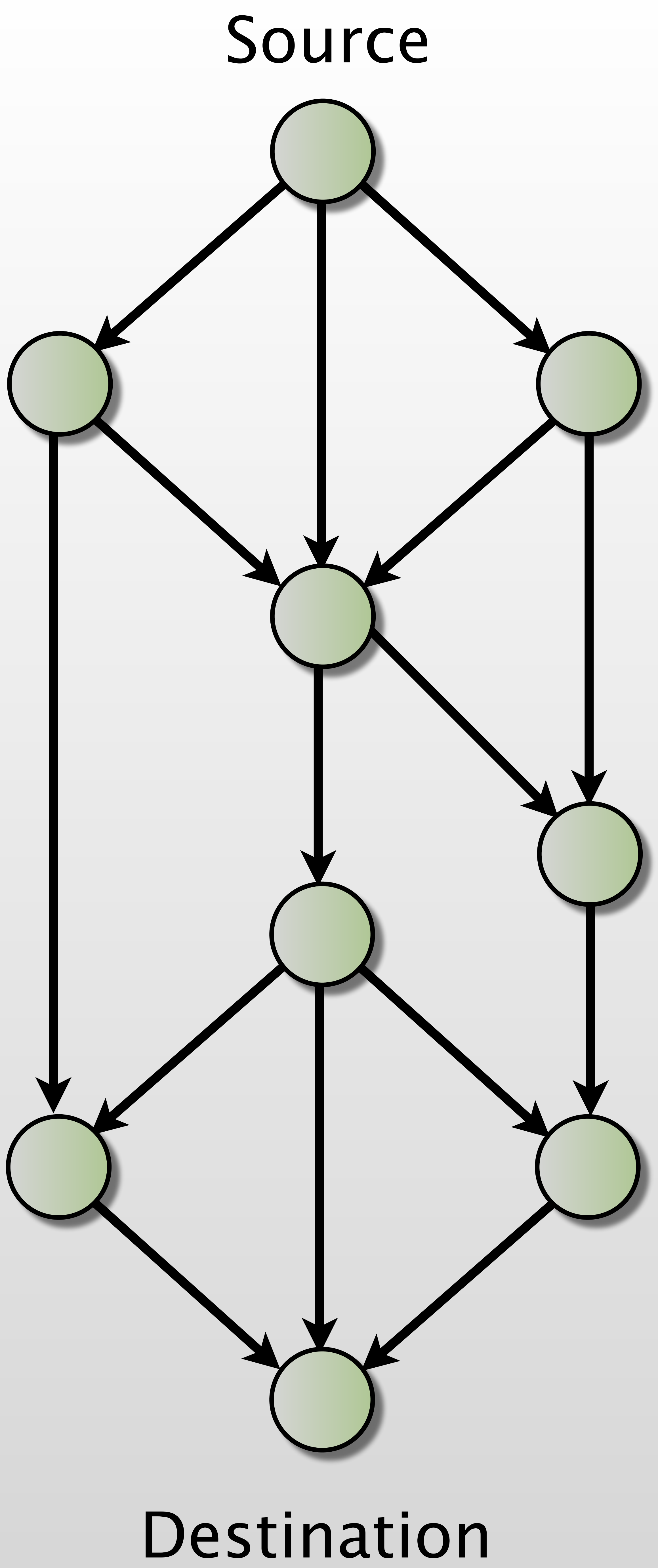
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{Q} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

received messages

desired messages

Generalized Network Coding



w, u, q in a field. Allow relay to multiply by q

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1L} \\ q_{21} & q_{22} & \cdots & q_{2L} \\ & \vdots & & \\ q_{M1} & q_{M2} & \cdots & q_{ML} \end{bmatrix}$$

If Q has rank L, then all messages $\mathbf{w}$ recoverable

How to design Q ?

- Algorithmic approach (Jaggi et al.)

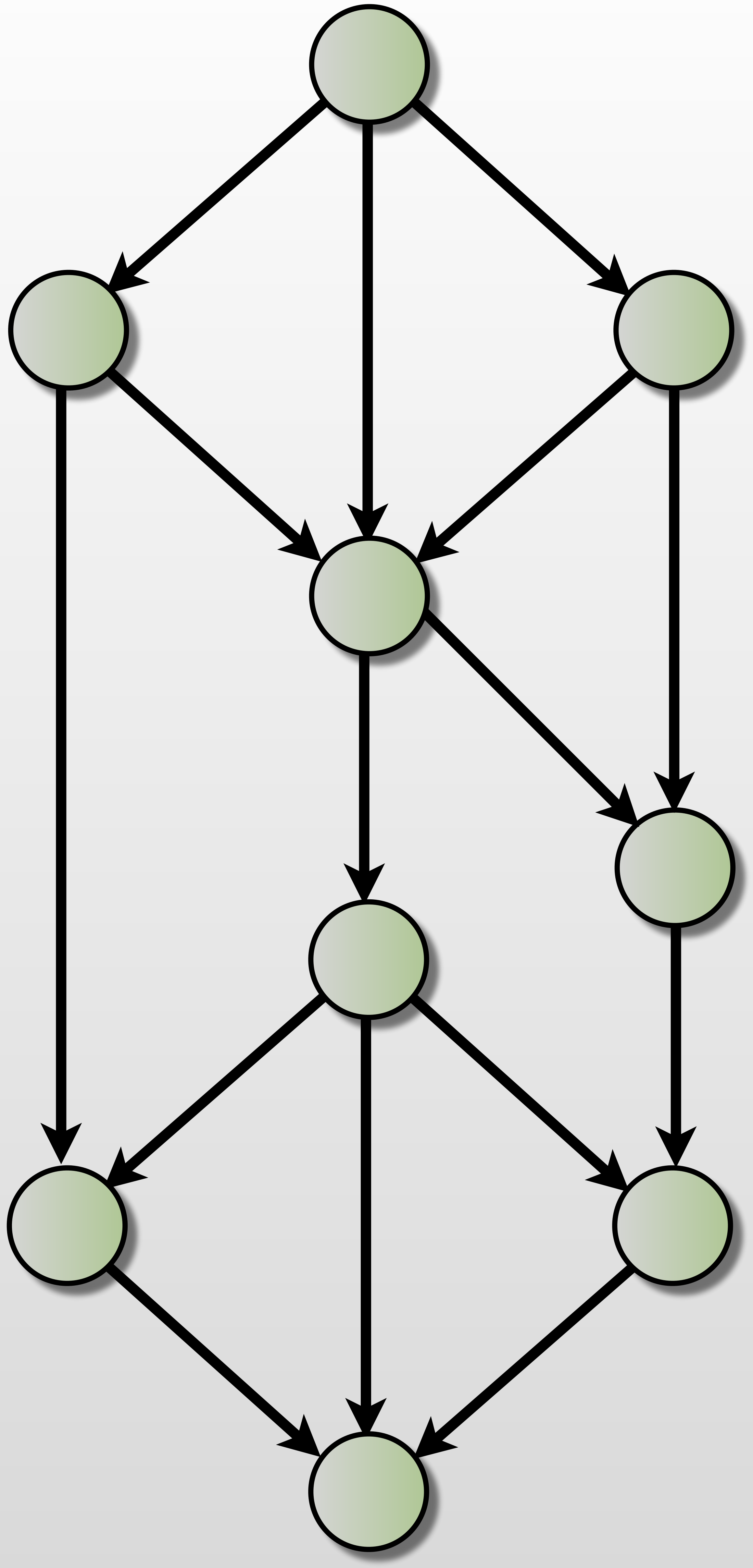
  Success if field size $p >$ number of destinations

- Random approach (Kotter and Medard. Ho et al.)

  Probability of valid solutions increases with p

Action of One Row
A "Relay"

Source has w_1, w_2, ...



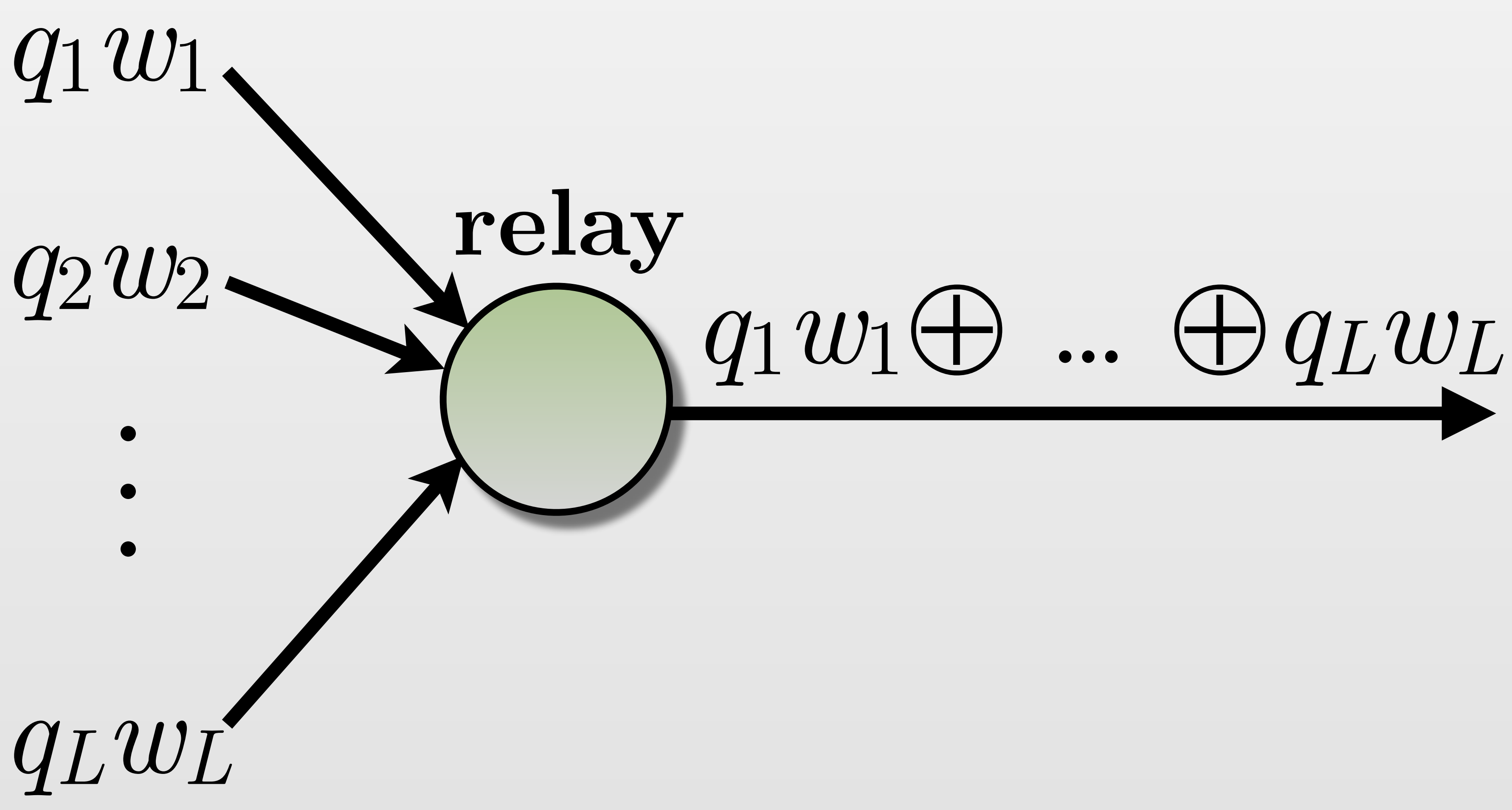
Destination has u_1, u_2, ...

$$\begin{bmatrix} u_1 \\ \boxed{u_2} \\ \vdots \\ u_M \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1L} \\ q_{21} & q_{22} & \cdots & q_{2L} \\ & \vdots & & \\ q_{M1} & q_{M2} & \cdots & q_{ML} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix}$$

received messages

desired messages

Action of One Row
A "Relay"



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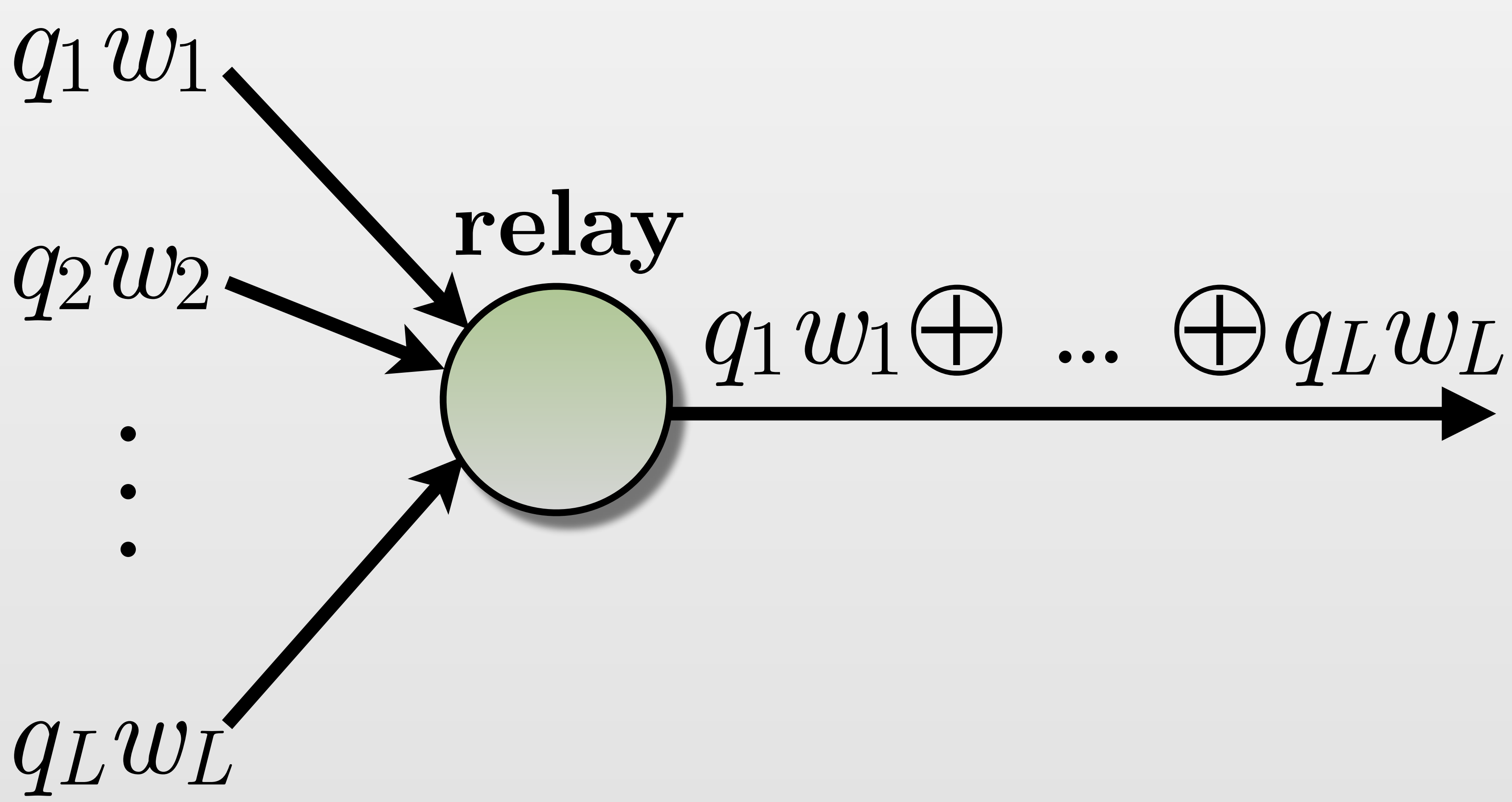
received messages

desired messages

Action of One Row
A "Relay"



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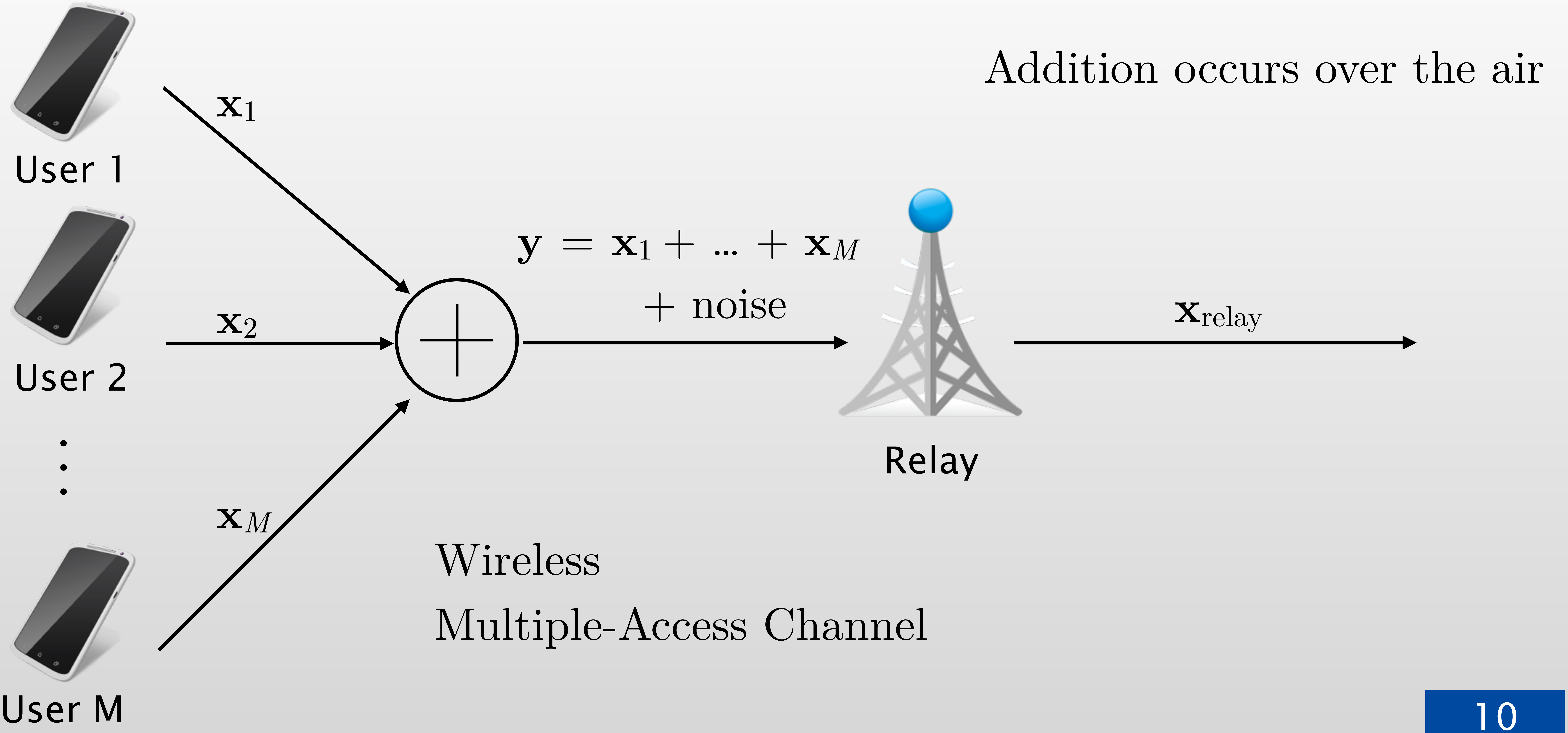
received messages

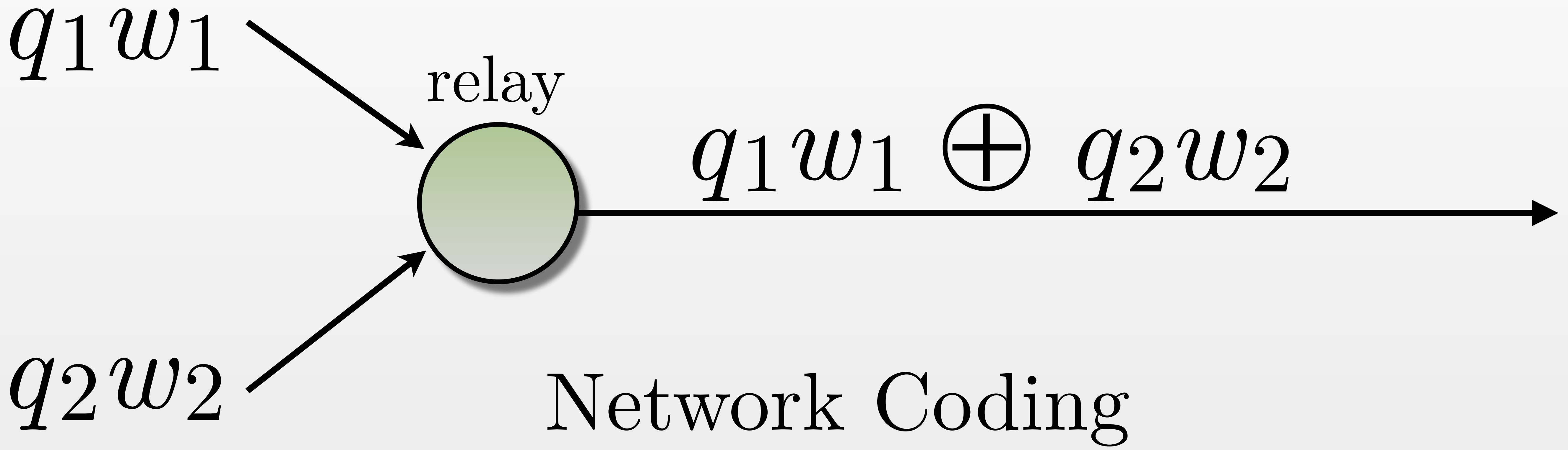
desired messages

What if the relay is wireless...?

1B PLNC = Physical Layer Network Coding



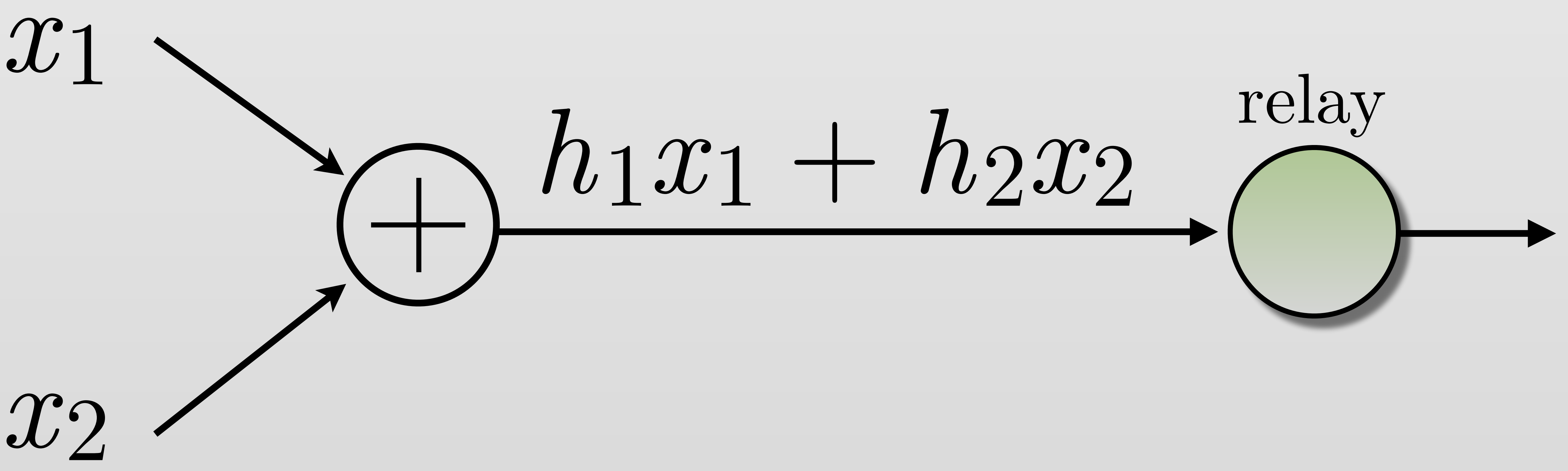
Network Coding vs. PLNC



Network Coding

<u>Network Coding</u>:

relay adds incoming messages



Physical Layer Network Coding

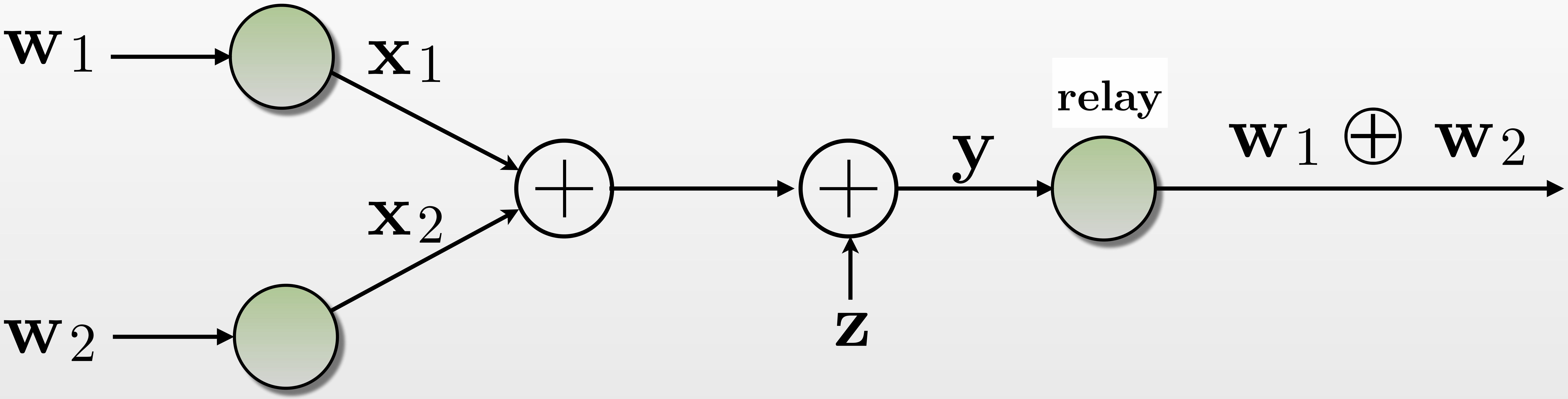
<u>PLNC</u>:

addition over the air

fading plays a role

combat noise

PLNC with Error-Correction



Perform error-correction coding on vectors:

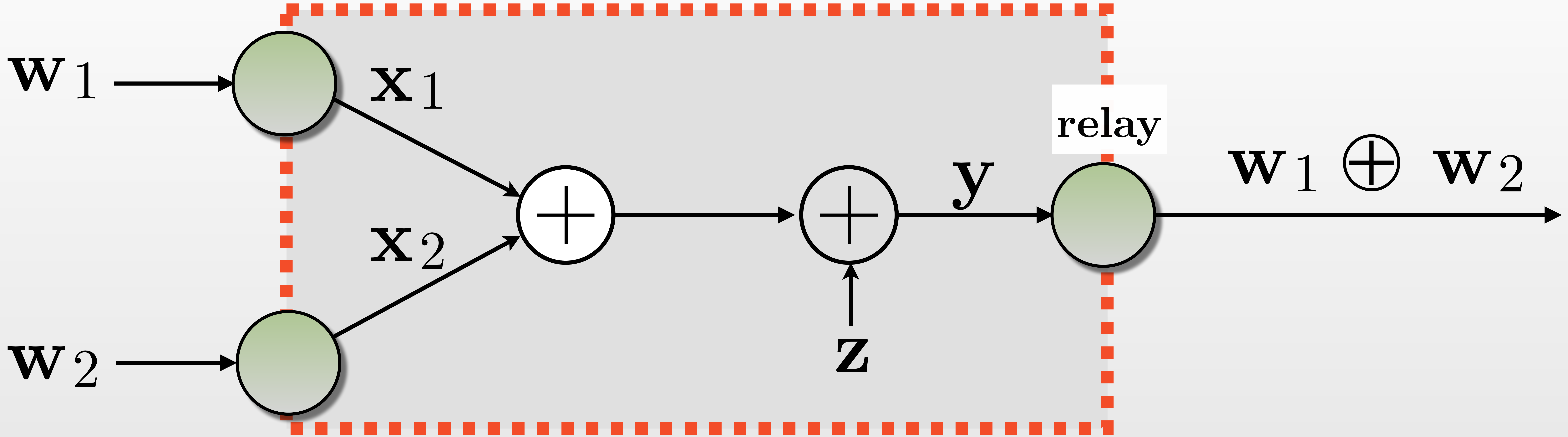
$$\mathbf{x}_i = \mathrm{Enc}(\mathbf{w}_i)$$

Relay performs two functions:

$$\mathbf{x}_1 + \mathbf{x}_2 = \mathrm{Decoder}(\mathbf{y})$$

$$\mathbf{w}_1 \oplus \mathbf{w}_2 = \mathrm{Enc}^{-1}(\mathbf{x}_1 + \mathbf{x}_2)$$

PLNC with Error-Correction



Powerful idea:

- Relay only eliminates noise
- Relay does not need to separate inference
- Converted a noisy network into a <u>noiseless</u> network

We Need A Code to Perform PLNC

Code must correct errors, for noisy wireless channels

- Code must satisfy a power constraint.

Code must form a group over addition

- so addition over the channel makes sense.

Code must have a group isomorphism: $\mathrm{Enc}(\mathbf{w}_1 \oplus \mathbf{w}_2) = \mathbf{x}_1 + \mathbf{x}_2$,

- so network coding can be performed

These properties are satisfied by nested lattice codes.

Quotient Groups

$$G/H$$

group

subgroup

$$\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$$

Definition of a Coset

Definition Let G be a group and let H be a subgroup of G. For any $a \in G$, the set $a + H = \{a + H \mid h \in H\}$ is called the *coset of H in G containing a*.

Quotient Groups

Let G/H be the set of all cosets of H in G, that is:

$$G/H = \{a + H | a \in G\}$$

Note that G/H is a set of sets. The set G/H is called a *quotient group*.

Example

- Integers $\mathbb{Z}$ are a group under addition
- $4\mathbb{Z}$ is a subgroup: $4\mathbb{Z} \subset \mathbb{Z}$.
- The quotient group $\mathbb{Z}/4\mathbb{Z}$, has four sets:

$$
\begin{aligned}
0 + 4\mathbb{Z} &= \{\ldots, -8, -4, 0, 4, 8, \cdots\} \\
1 + 4\mathbb{Z} &= \{\ldots, -7, -3, 1, 5, 9, \cdots\} \\
2 + 4\mathbb{Z} &= \{\ldots, -6, -2, 2, 6, 10, \cdots\} \\
3 + 4\mathbb{Z} &= \{\ldots, -5, -1, 3, 7, 11, \cdots\}
\end{aligned}
$$

The quotient group is closed under addition:

	$0 + 4\mathbb{Z}$	$1 + 4\mathbb{Z}$	$2 + 4\mathbb{Z}$	$3 + 4\mathbb{Z}$
$0 + 4\mathbb{Z}$	$0 + 4\mathbb{Z}$	$1 + 4\mathbb{Z}$	$2 + 4\mathbb{Z}$	$3 + 4\mathbb{Z}$
$1 + 4\mathbb{Z}$	$1 + 4\mathbb{Z}$	$2 + 4\mathbb{Z}$	$3 + 4\mathbb{Z}$	$0 + 4\mathbb{Z}$
$2 + 4\mathbb{Z}$	$2 + 4\mathbb{Z}$	$3 + 4\mathbb{Z}$	$0 + 4\mathbb{Z}$	$1 + 4\mathbb{Z}$
$3 + 4\mathbb{Z}$	$3 + 4\mathbb{Z}$	$0 + 4\mathbb{Z}$	$1 + 4\mathbb{Z}$	$2 + 4\mathbb{Z}$

Coset Leader (Coset Representative)

A coset leader is a single representative element from each coset.

Continue $\mathbb{Z}/4\mathbb{Z}$ example:

Coset leaders: $\{0, 1, 2, 3\}$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Coset leaders: $\{-2, -1, 0, 1\}$

$+$	0	1	-2	-1
0	0	1	-2	-1
1	1	-2	-1	0
-2	-2	-1	0	1
-1	-1	0	1	-2

2B Lattice: Linear code over real numbers

A lattice Λ is a linear additive subgroup of $\mathbb{R}^n$.

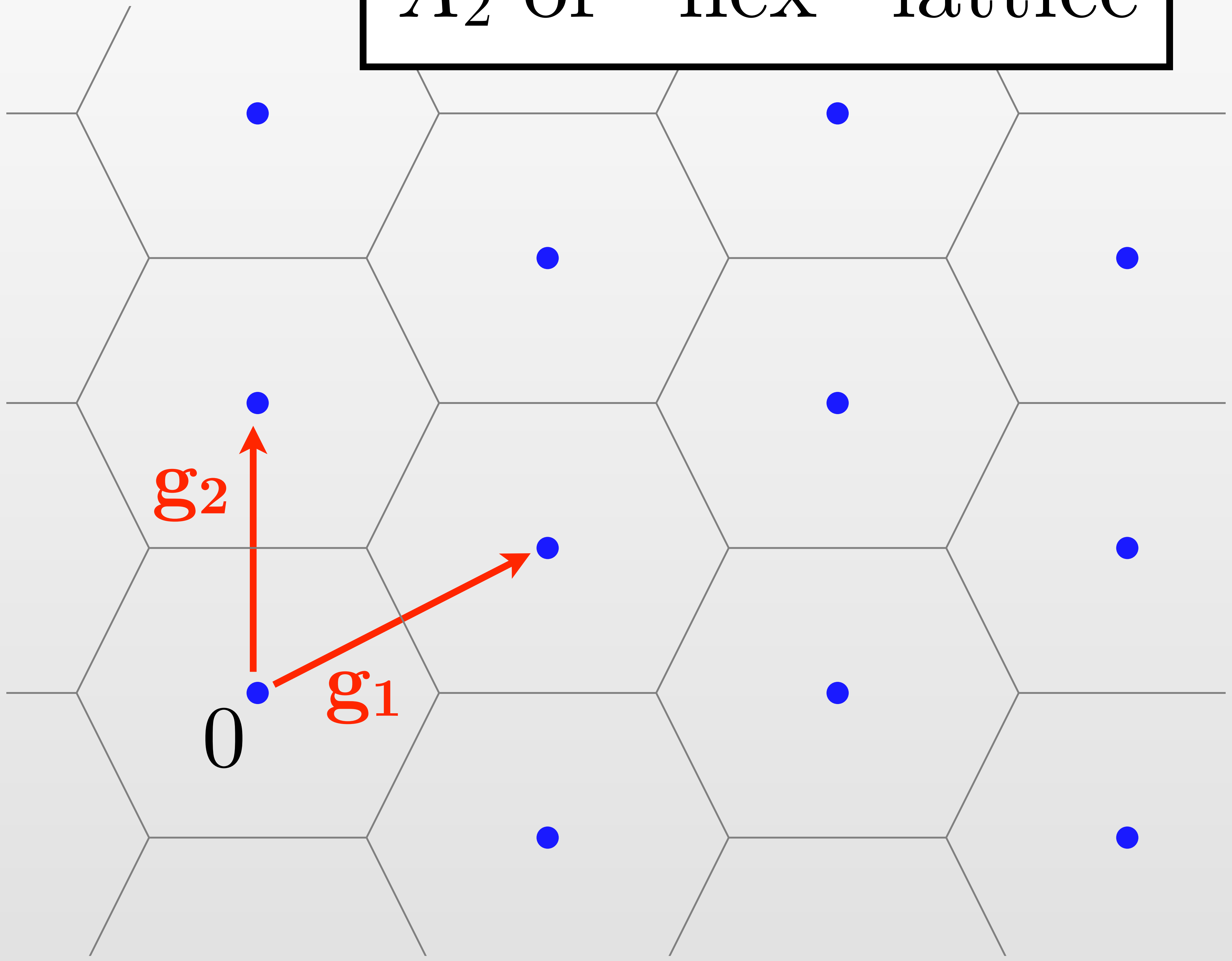
Λ may be represented by a basis of $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$.

A lattice point $\mathbf{x} \in \Lambda$ is an integral, linear combination of the basis vectors:

$$\mathbf{x} = \mathbf{g}_1 b_1 + \mathbf{g}_2 b_2 + \cdots + \mathbf{g}_n b_n,$$

where the b_i are integers.

A_2 or "hex" lattice



$$G = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}$$

$\mathbf{g}_1$ $\mathbf{g}_2$

Quotient Groups Based on Lattices

Let Λ_c be a lattice

- "coding lattice" corrects errors. Also called fine lattice.

Let Λ_s be a sublattice: $\Lambda_s \subset \Lambda_c$.

- "shaping lattice" enforces power constraint.
  Also called coarse lattice.

$K\Lambda_c$ is a lattice expanded by K.

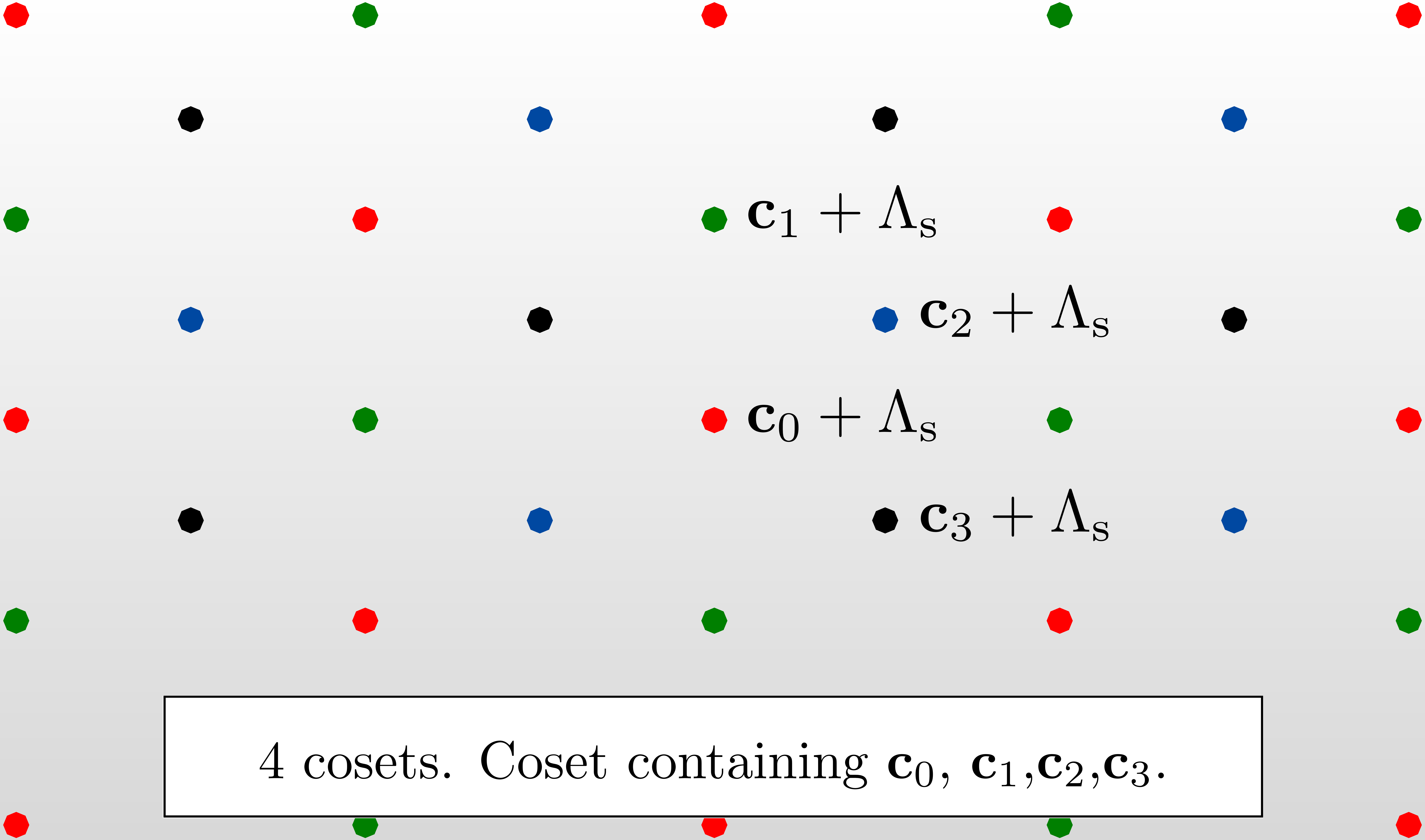
- Choosing $\Lambda_s = K\Lambda_c$ results in $\Lambda_s \subseteq \Lambda_c$ for $K = 1, 2, 3, \cdots$


Lattice (group) $\Lambda_c = A_2$


Sublattice (subgroup)
$\Lambda_{\mathrm{s}} = 2A_2$

Sublattice condition:

$$\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$$



4 cosets. Coset containing $\mathbf{c}_0$, $\mathbf{c}_1$,$\mathbf{c}_2$,$\mathbf{c}_3$.

Cosets form a group under addition

The set $\Lambda_\mathrm{c}/\Lambda_\mathrm{s}$ is a quotient group.

This table expresses group addition:

	$\mathbf{c}_0 + \Lambda_\mathrm{s}$	$\mathbf{c}_1 + \Lambda_\mathrm{s}$	$\mathbf{c}_2 + \Lambda_\mathrm{s}$	$\mathbf{c}_3 + \Lambda_\mathrm{s}$
$\mathbf{c}_0 + \Lambda_\mathrm{s}$	$\mathbf{c}_0 + \Lambda_\mathrm{s}$	$\mathbf{c}_1 + \Lambda_\mathrm{s}$	$\mathbf{c}_2 + \Lambda_\mathrm{s}$	$\mathbf{c}_3 + \Lambda_\mathrm{s}$
$\mathbf{c}_1 + \Lambda_\mathrm{s}$	$\mathbf{c}_1 + \Lambda_\mathrm{s}$	$\mathbf{c}_0 + \Lambda_\mathrm{s}$	$\mathbf{c}_3 + \Lambda_\mathrm{s}$	$\mathbf{c}_2 + \Lambda_\mathrm{s}$
$\mathbf{c}_2 + \Lambda_\mathrm{s}$	$\mathbf{c}_2 + \Lambda_\mathrm{s}$	$\mathbf{c}_3 + \Lambda_\mathrm{s}$	$\mathbf{c}_0 + \Lambda_\mathrm{s}$	$\mathbf{c}_1 + \Lambda_\mathrm{s}$
$\mathbf{c}_3 + \Lambda_\mathrm{s}$	$\mathbf{c}_3 + \Lambda_\mathrm{s}$	$\mathbf{c}_2 + \Lambda_\mathrm{s}$	$\mathbf{c}_1 + \Lambda_\mathrm{s}$	$\mathbf{c}_0 + \Lambda_\mathrm{s}$

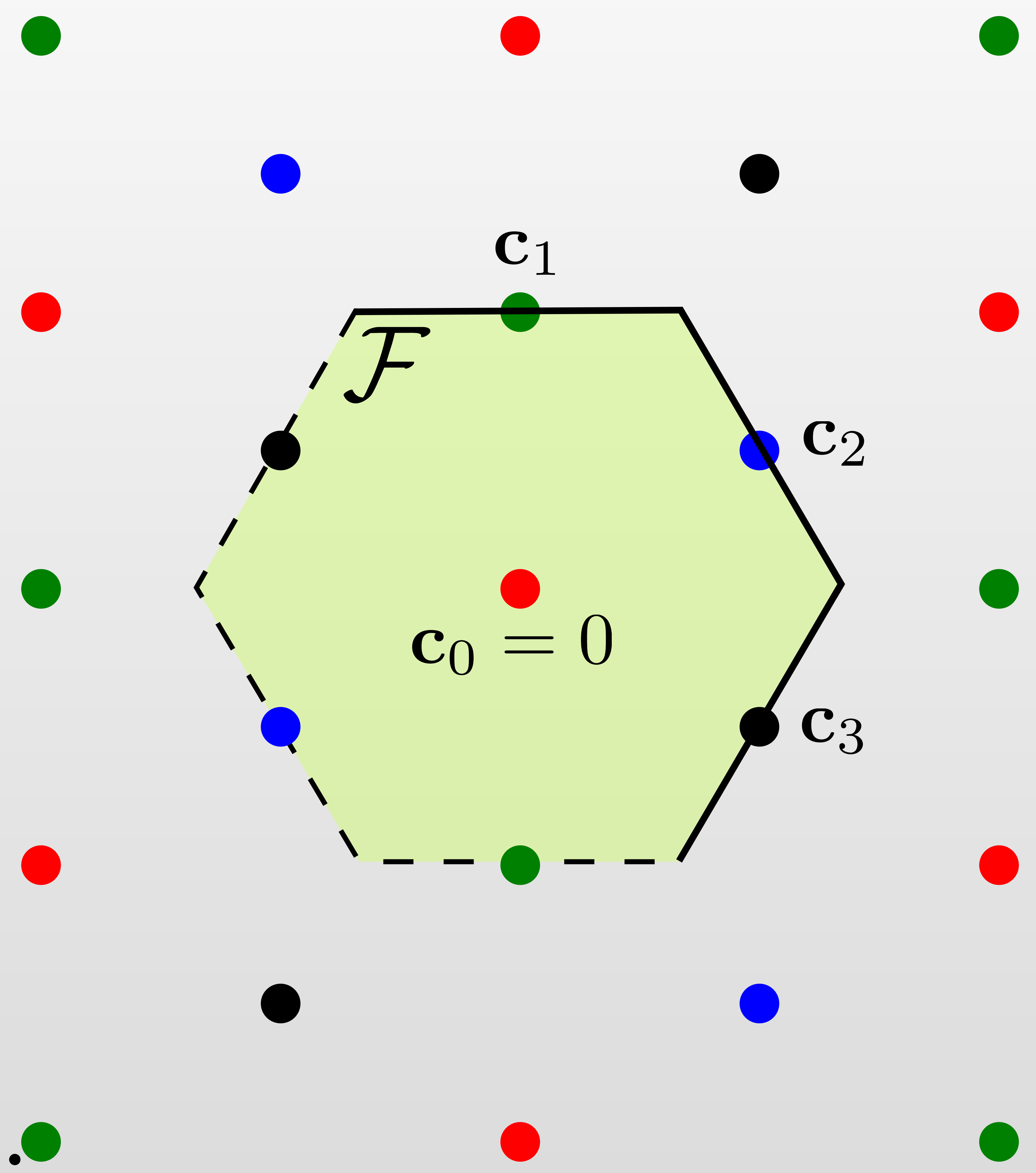
Nested Lattice Codes

Construct a lattice code $\mathcal{C}$:

$$\mathcal{C} = \Lambda_c \cap \mathcal{F}$$

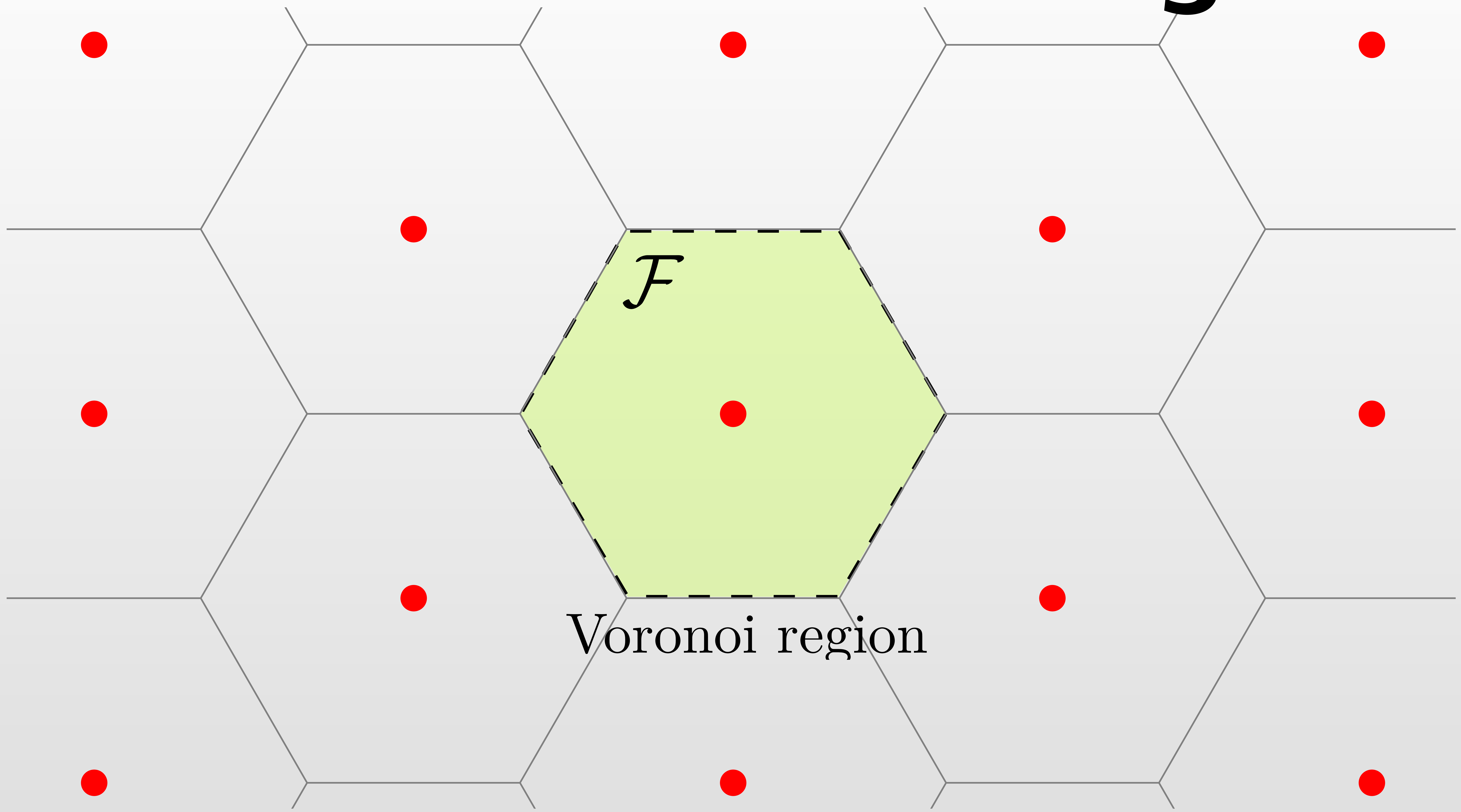
We need:

- Quotient Group Λ_c / Λ_s
- $\mathcal{F}$ is a fundamental region for Λ_s



The code $\mathcal{C}$ are coset leaders Λ_c / Λ_s, and is a group.

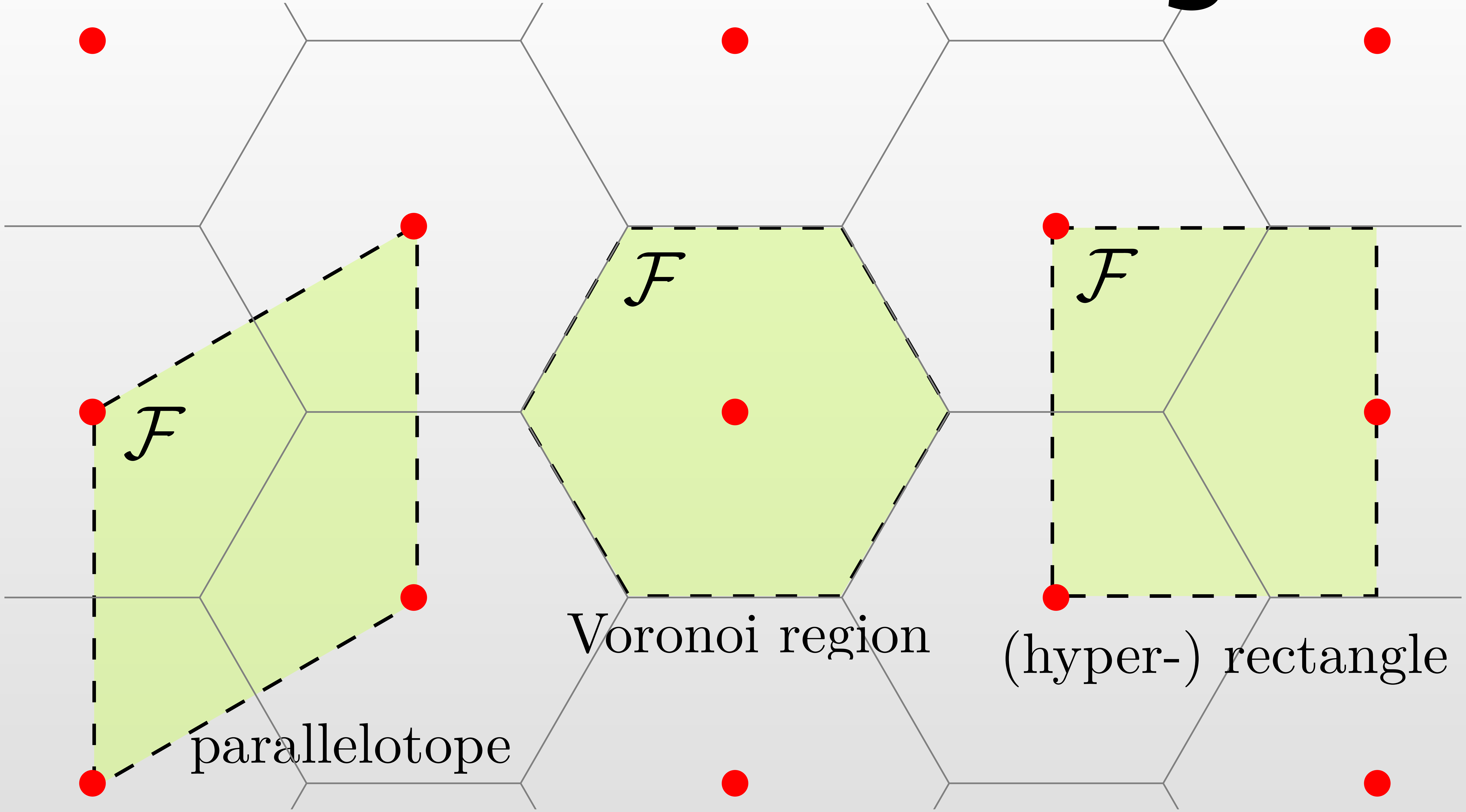
Fundamental Regions



A fundamental region $\mathcal{F} \subset \mathbb{R}^n$ is a shape that, if shifted by each lattice point, will exactly cover the whole real space.

Volume $V(\mathcal{F})$ of $\mathcal{F}$ is constant and $V(\mathcal{F}) = |\det \Lambda|$

Fundamental Regions



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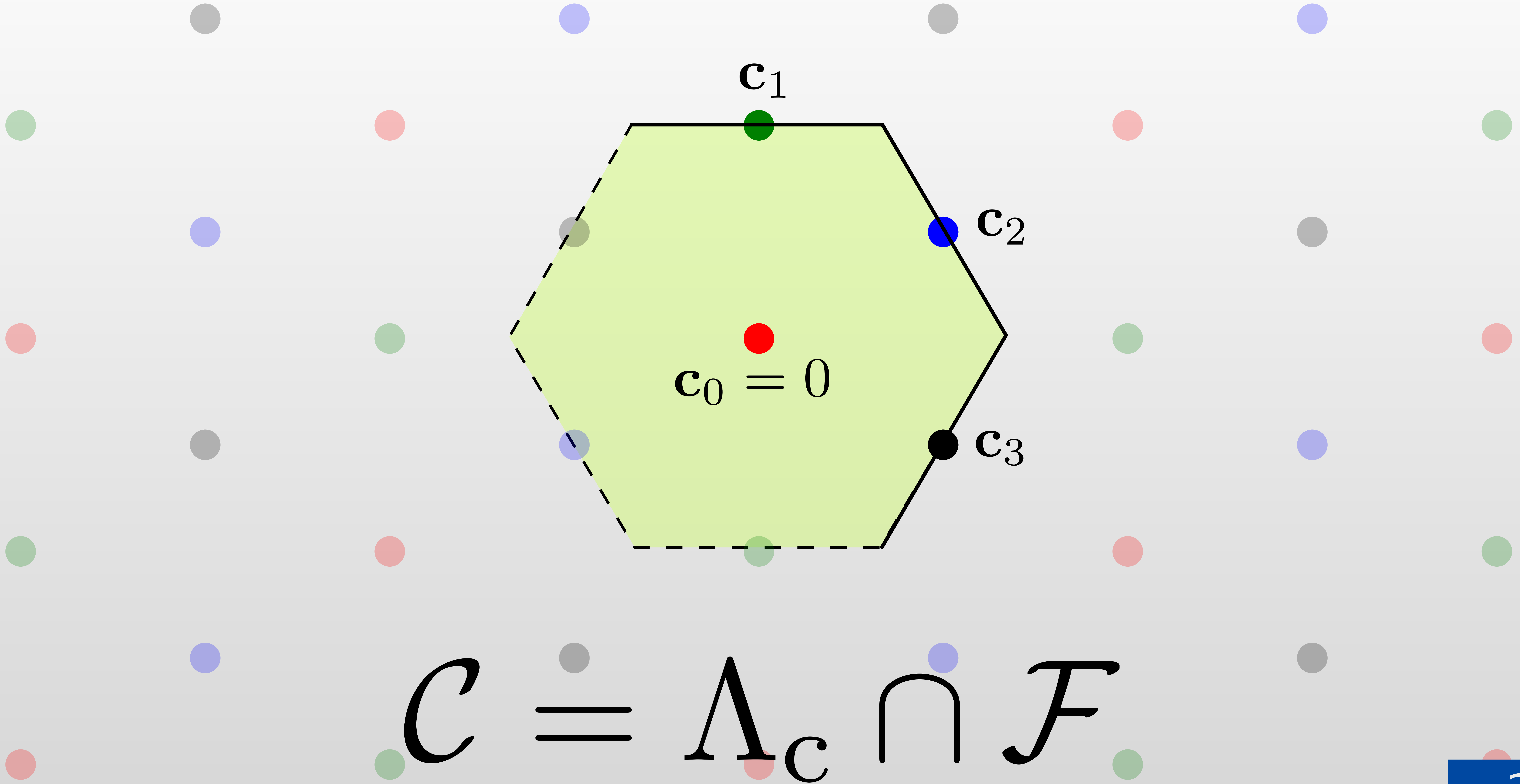
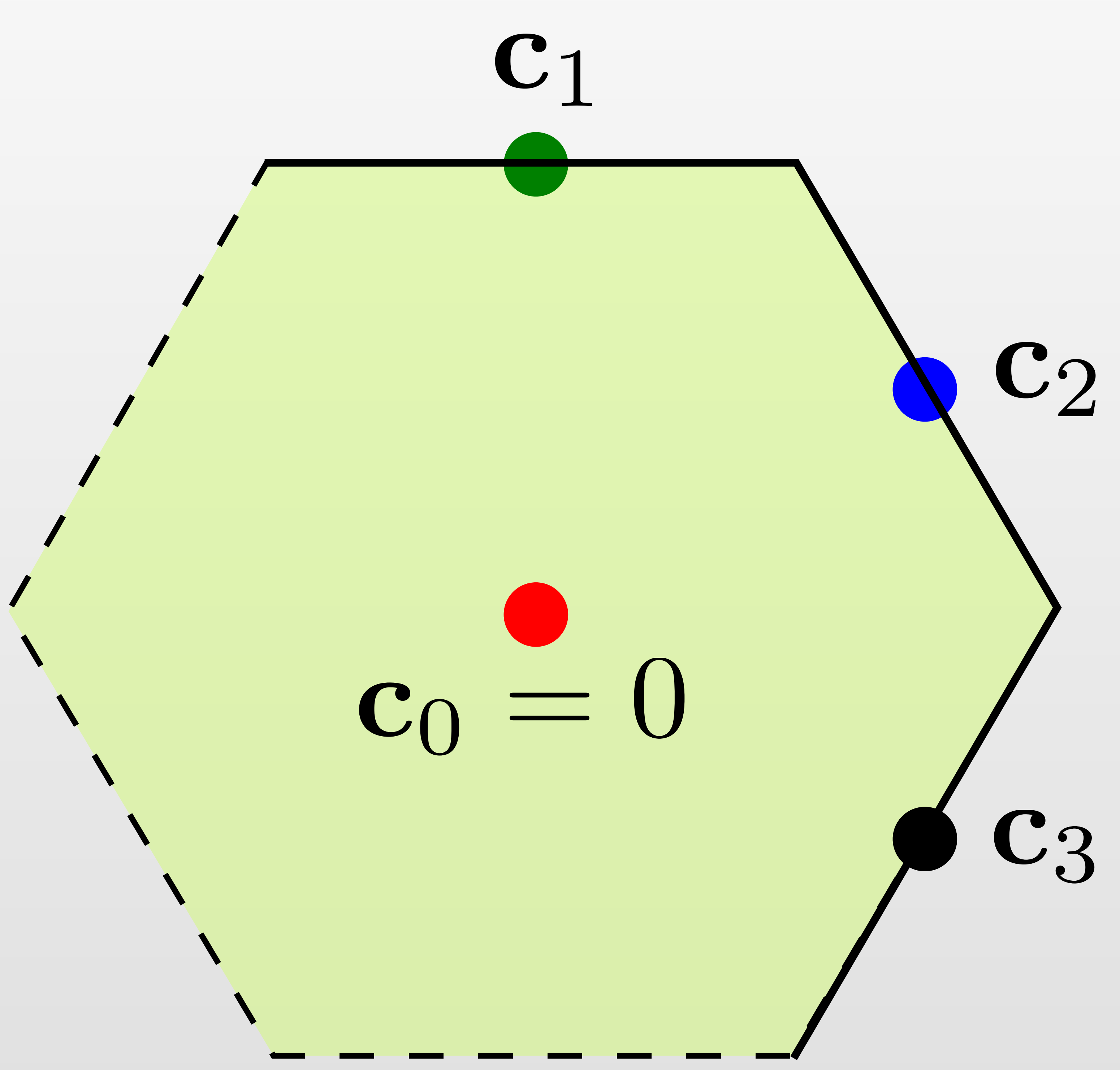
Volume $V(\mathcal{F})$ of $\mathcal{F}$ is constant and $V(\mathcal{F}) = |\det \Lambda|$

Selecting the Coset Leaders



$$\mathcal{C} = \Lambda_{\mathrm{C}} \cap \mathcal{F}$$

Nested Lattice Codes Form a Group



$$\mathcal{C} = \Lambda_{\mathrm{c}} \cap \mathcal{F}$$

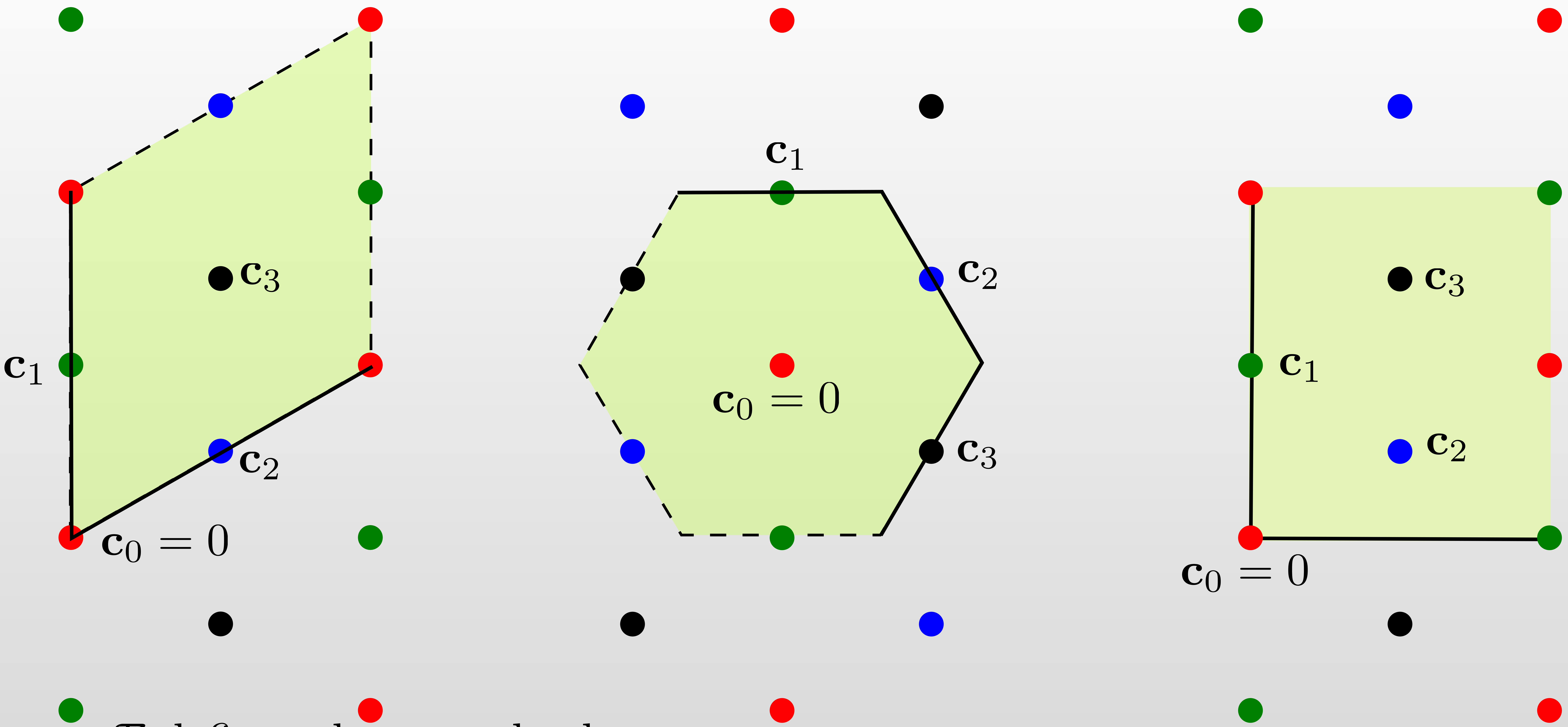
$+$	c_0	c_1	c_2	c_3
c_0	c_0	c_1	c_2	c_3
c_1	c_1	c_0	c_3	c_2
c_2	c_2	c_3	c_0	c_1
c_3	c_3	c_2	c_1	c_0

codebook $\mathcal{C} = \{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$

$\mathbf{c}_i$ are coset leaders of $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$.

$\mathcal{C}$ is closed under addition

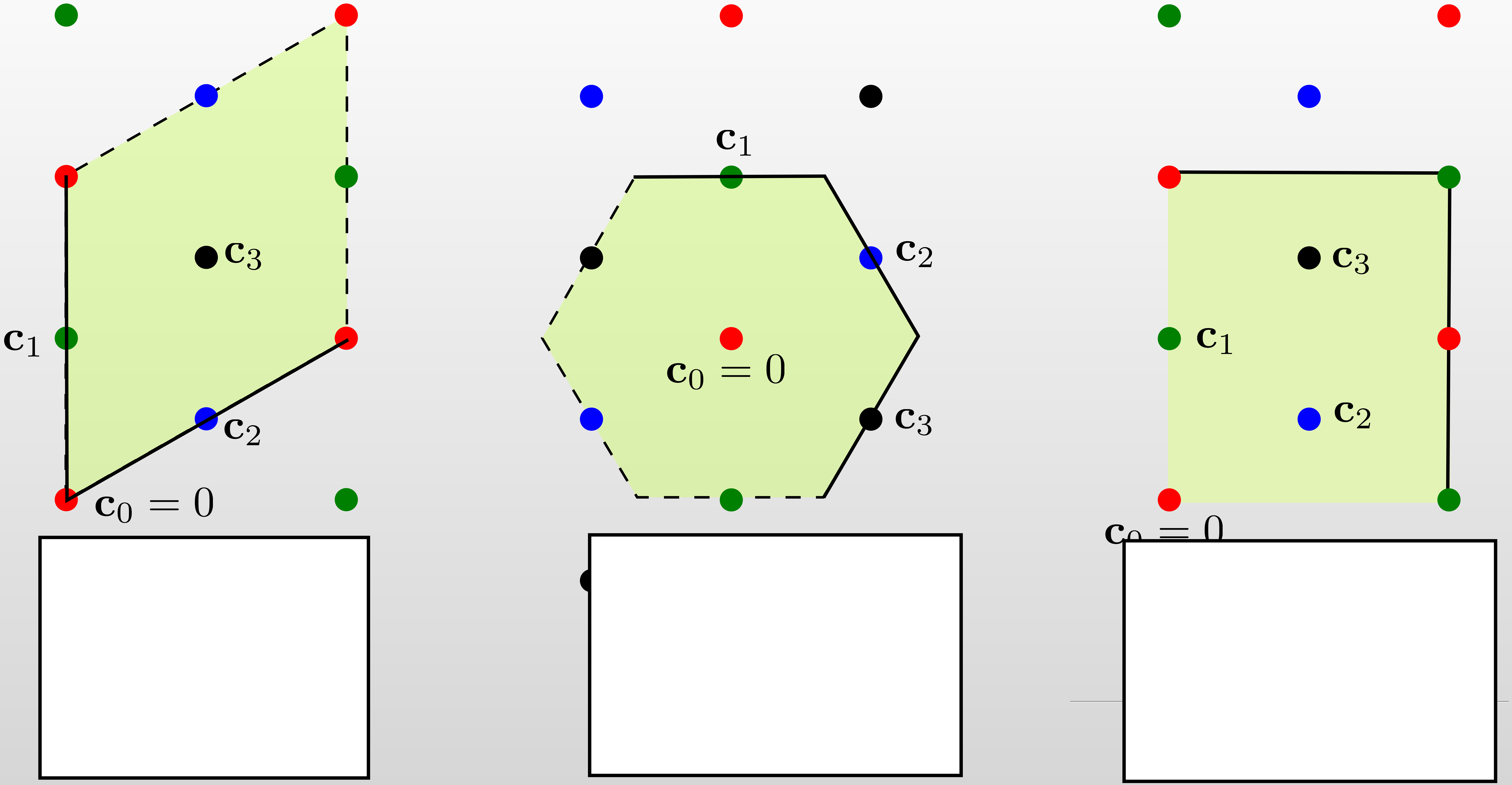
Various Nested Lattice Codes



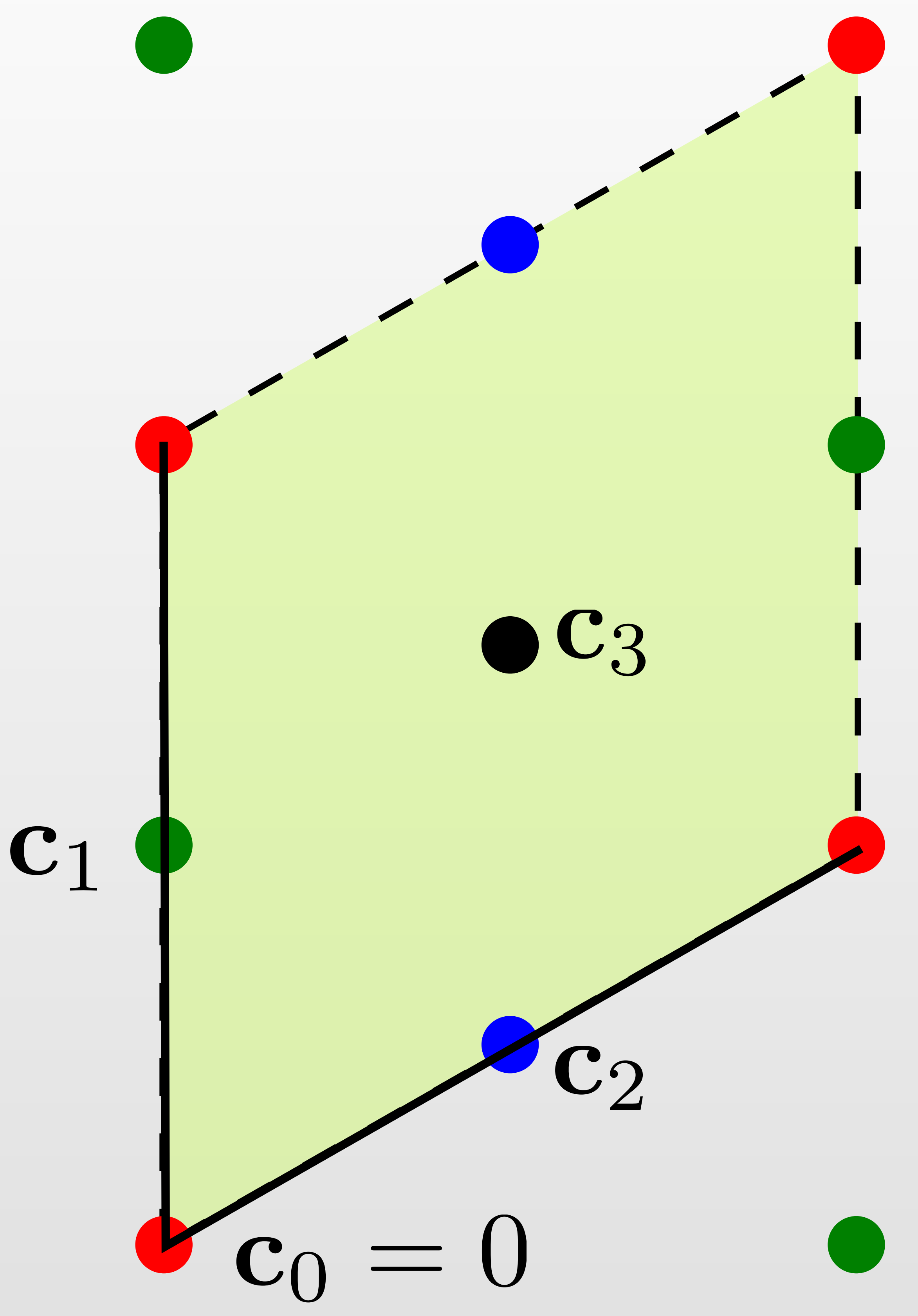
- $\mathcal{F}$ defines the coset leaders:

3 Different Voronoi regions — All codes form Groups

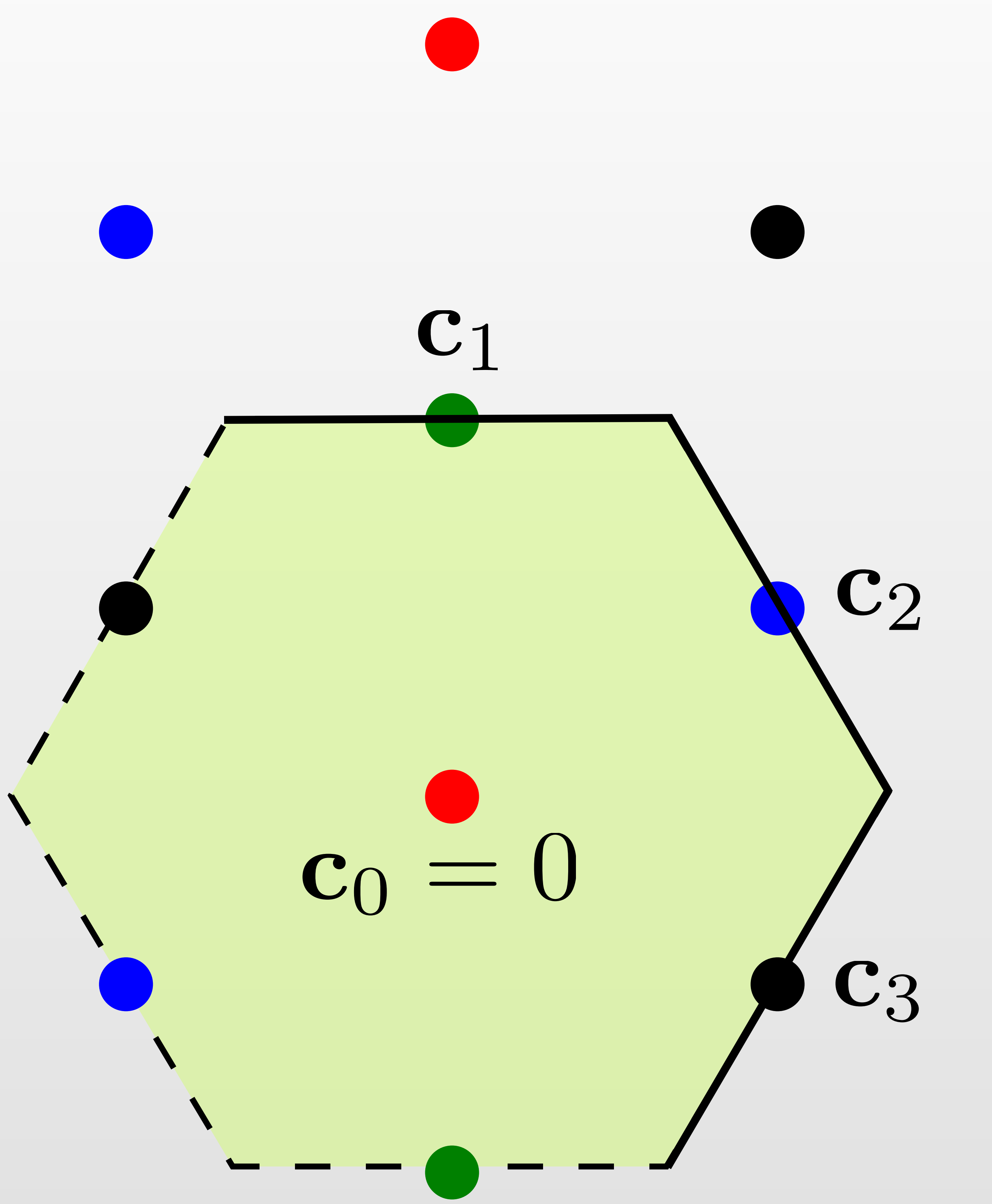
Various Nested Lattice Codes



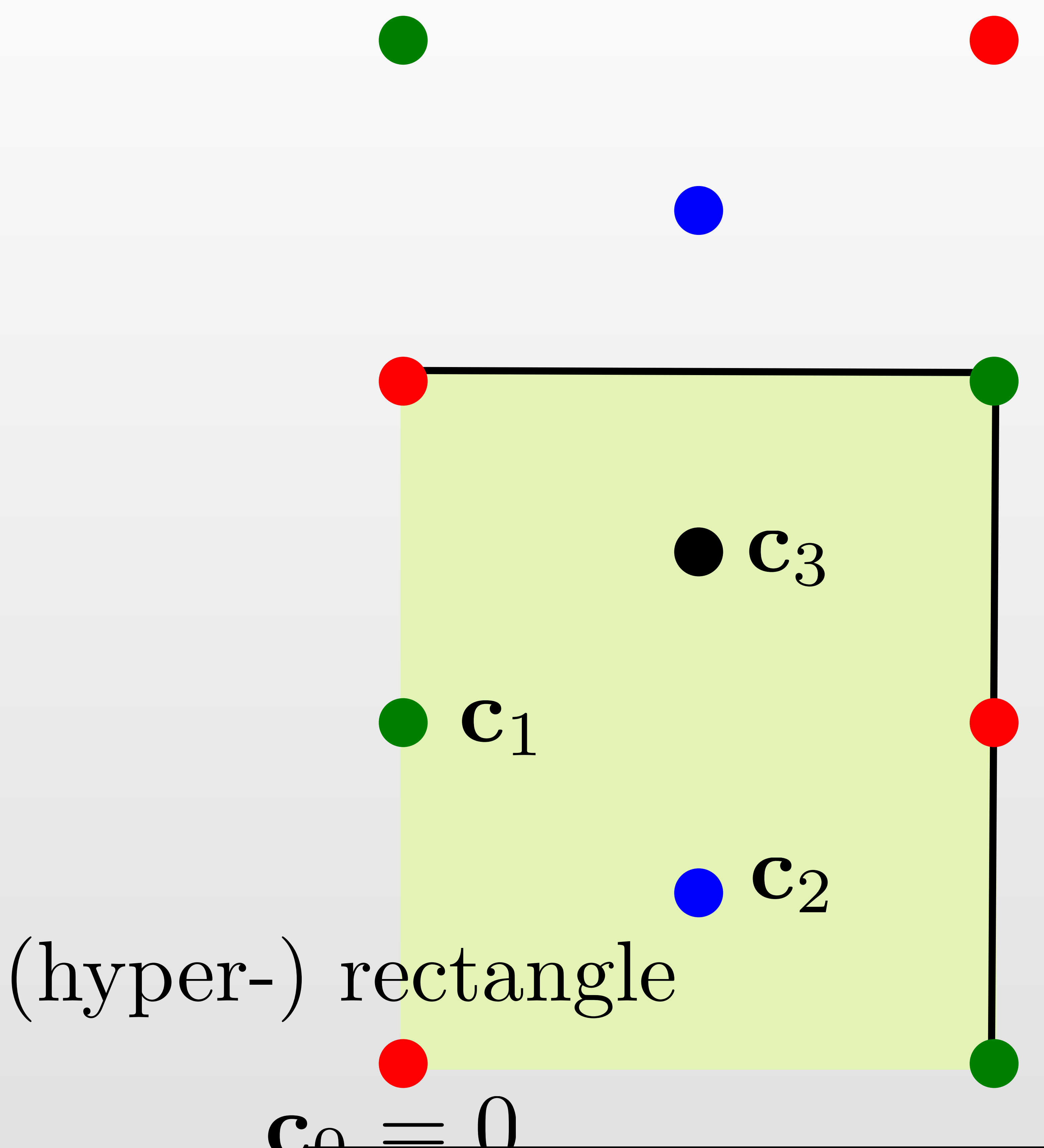
Various Nested Lattice Codes



parallelotope
Important for theory, not very practical

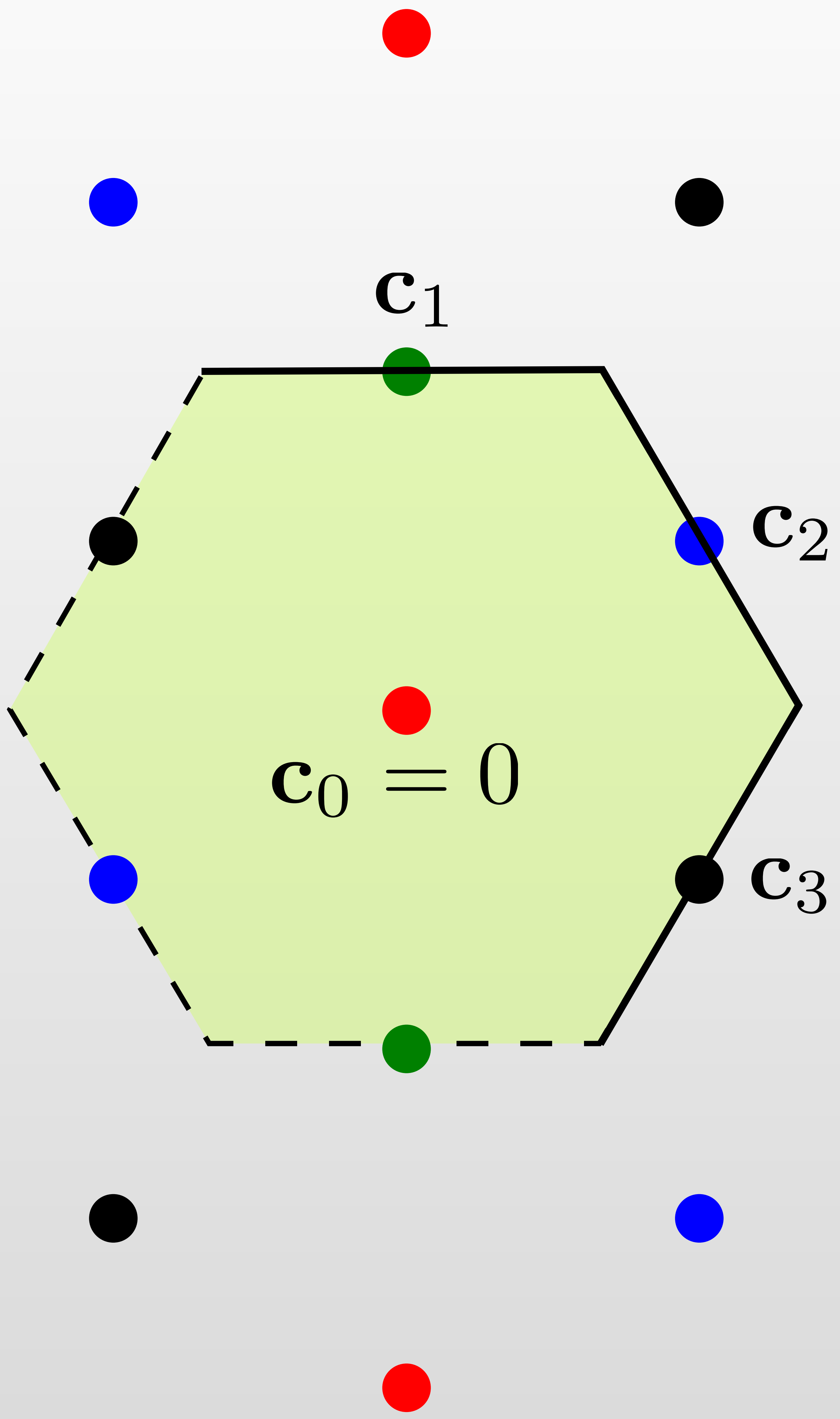


Voronoi region
Best transmit power, not always easy to implement



Rectangle
Easier to implement, no shaping gain

Voronoi is Best for AWGN Channel



Voronoi regions are sphere-like in high dimension.

A sphere satisfies the AWGN power constraint

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P$$

Encoding and Isomorphism

Encoding: mapping information to codewords

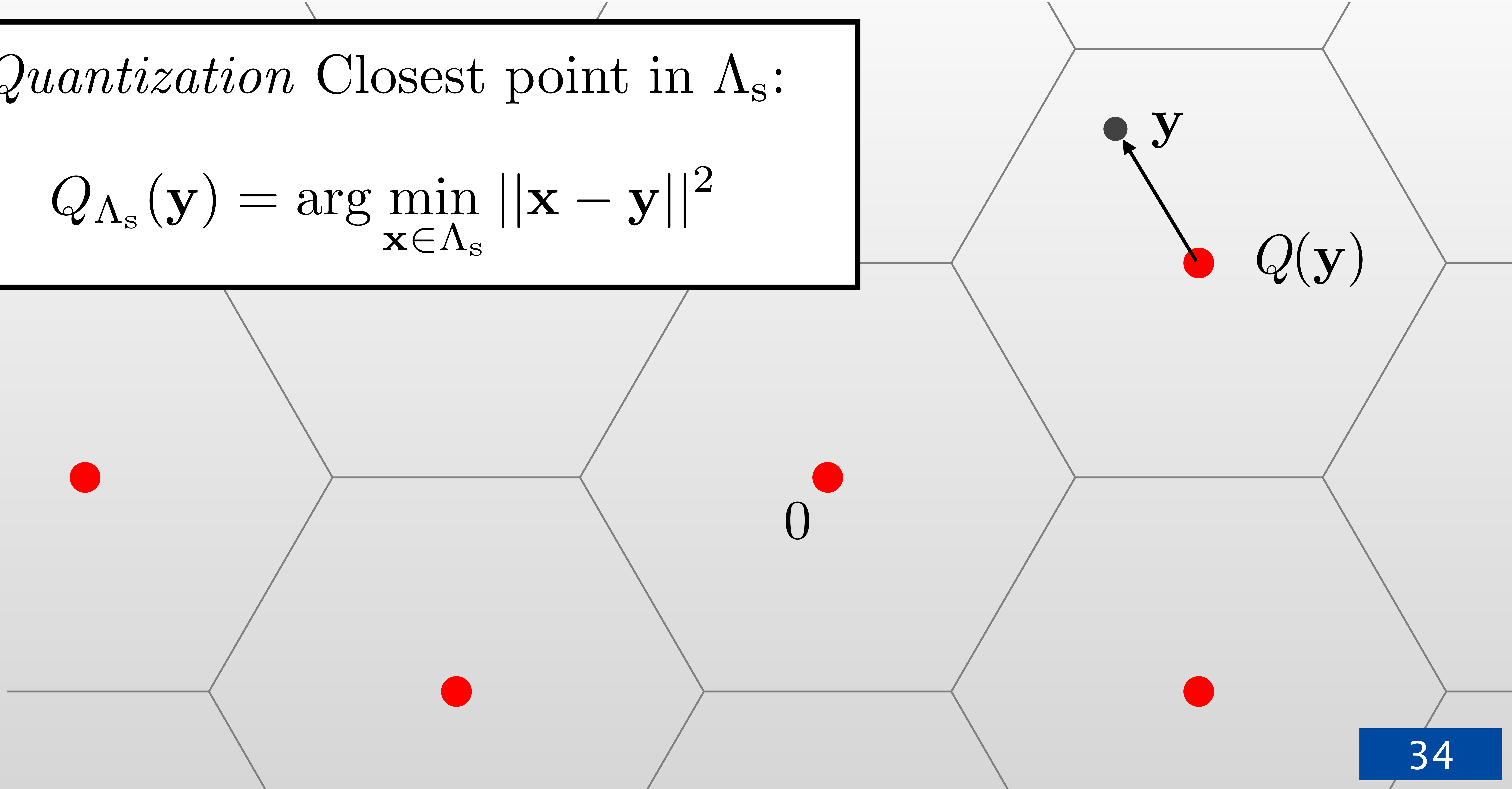
Indexing: mapping codewords to information

Isomorphism between information (ring) and codewords (group)

Quantization and Modulo

Quantization Closest point in Λ_s:

$$Q_{\Lambda_s}(\mathbf{y}) = \arg\min_{\mathbf{x}\in\Lambda_s} ||\mathbf{x}-\mathbf{y}||^2$$



Quantization and Modulo

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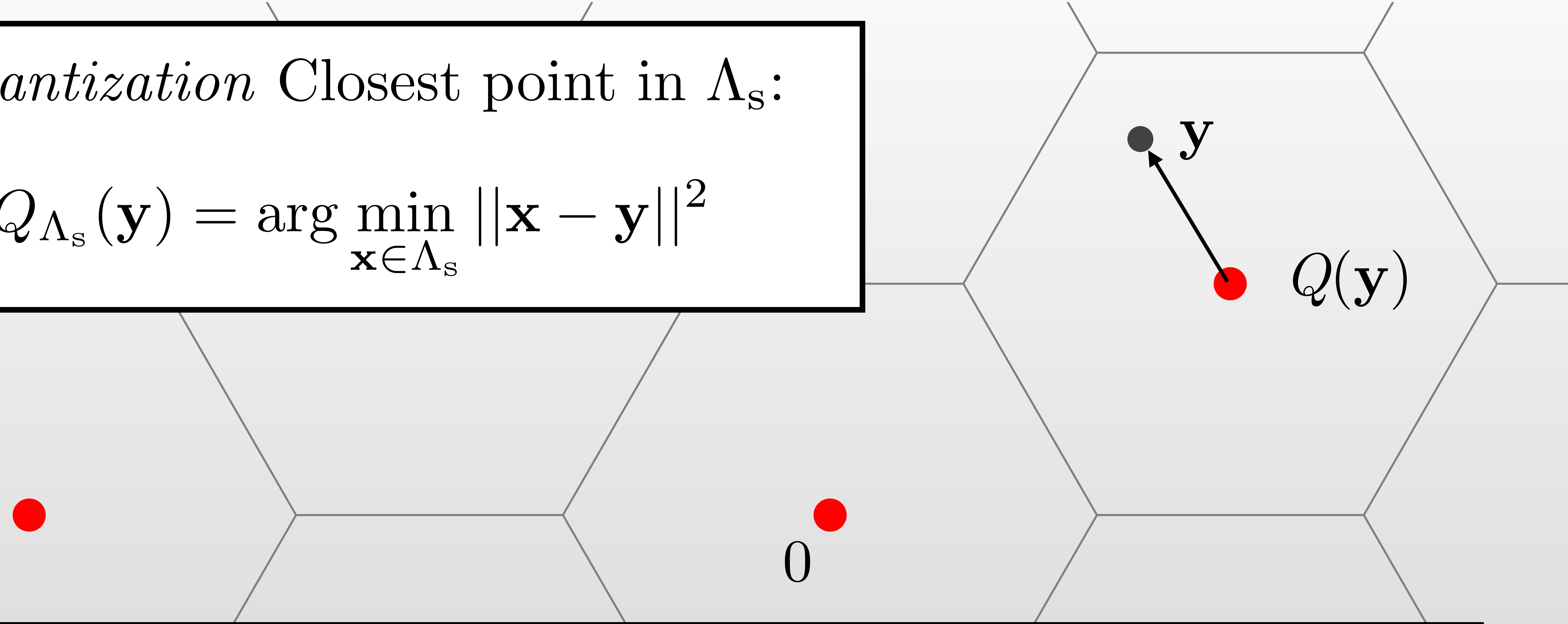
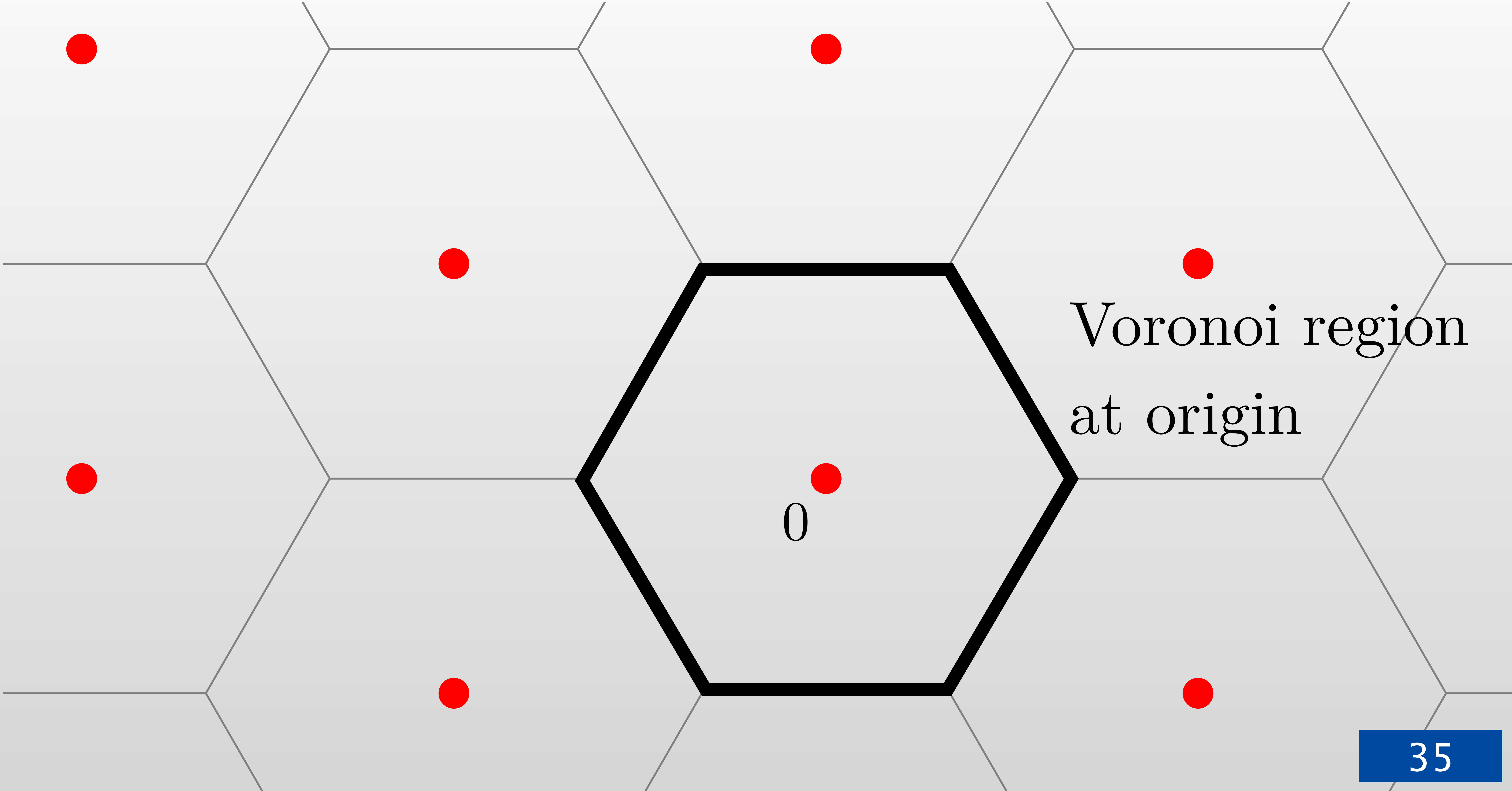


Quantization has exponential complexity in general

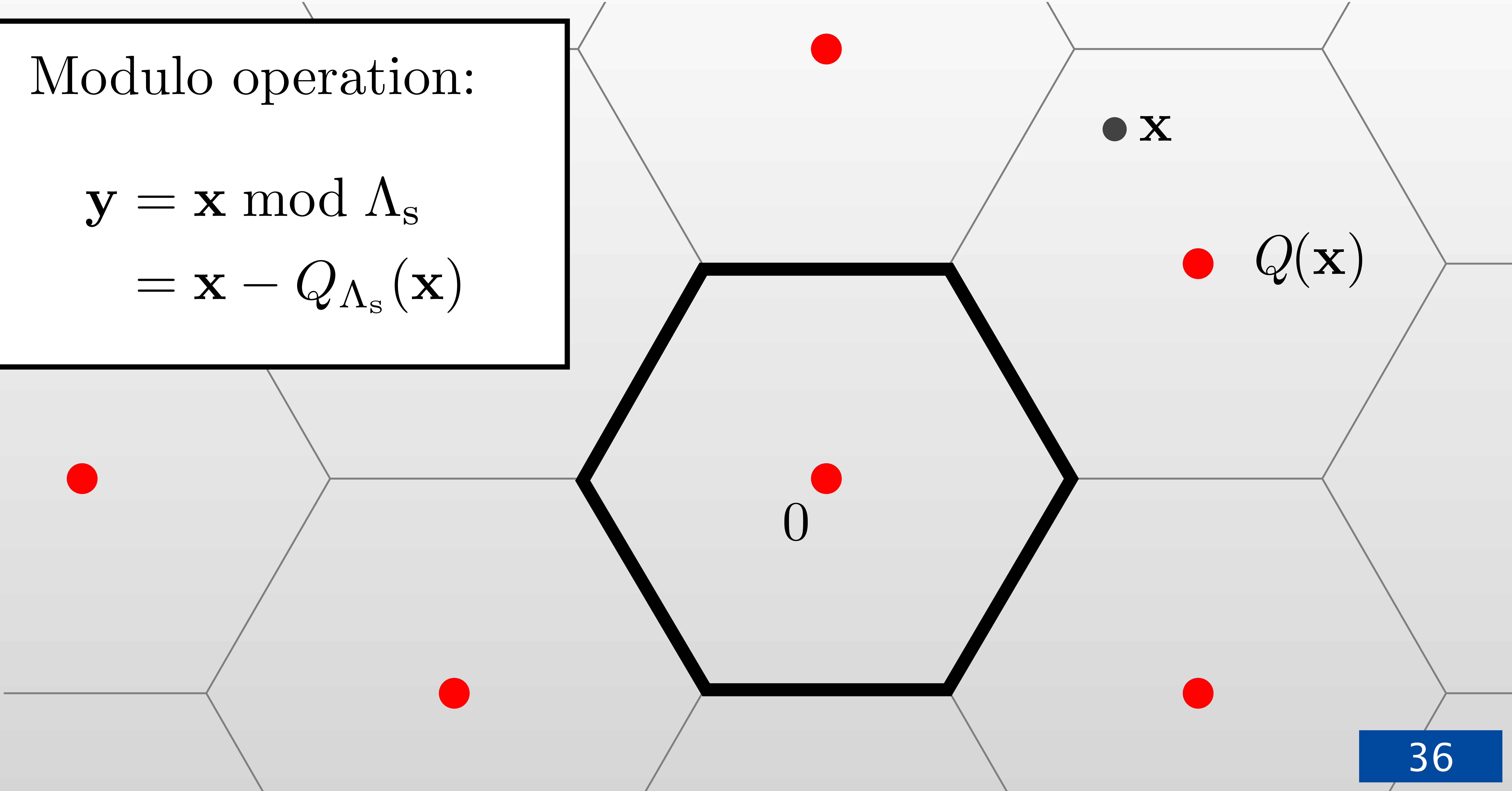
Quantization and Modulo



Quantization and Modulo

Modulo operation:

$$\begin{aligned} \mathbf{y} &= \mathbf{x} \bmod \Lambda_{\mathrm{s}} \\ &= \mathbf{x} - Q_{\Lambda_{\mathrm{s}}}(\mathbf{x}) \end{aligned}$$



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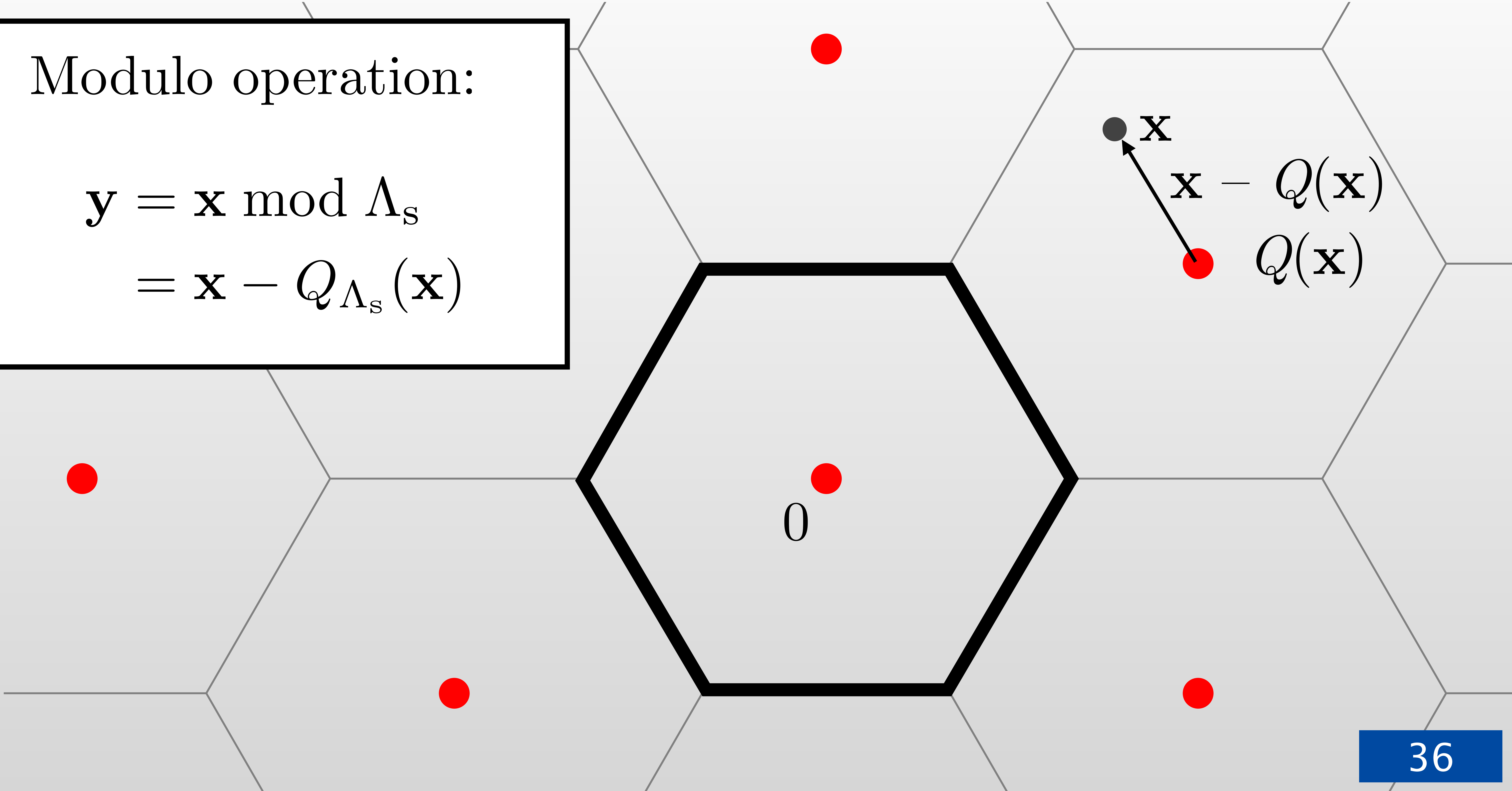

Quantization and Modulo

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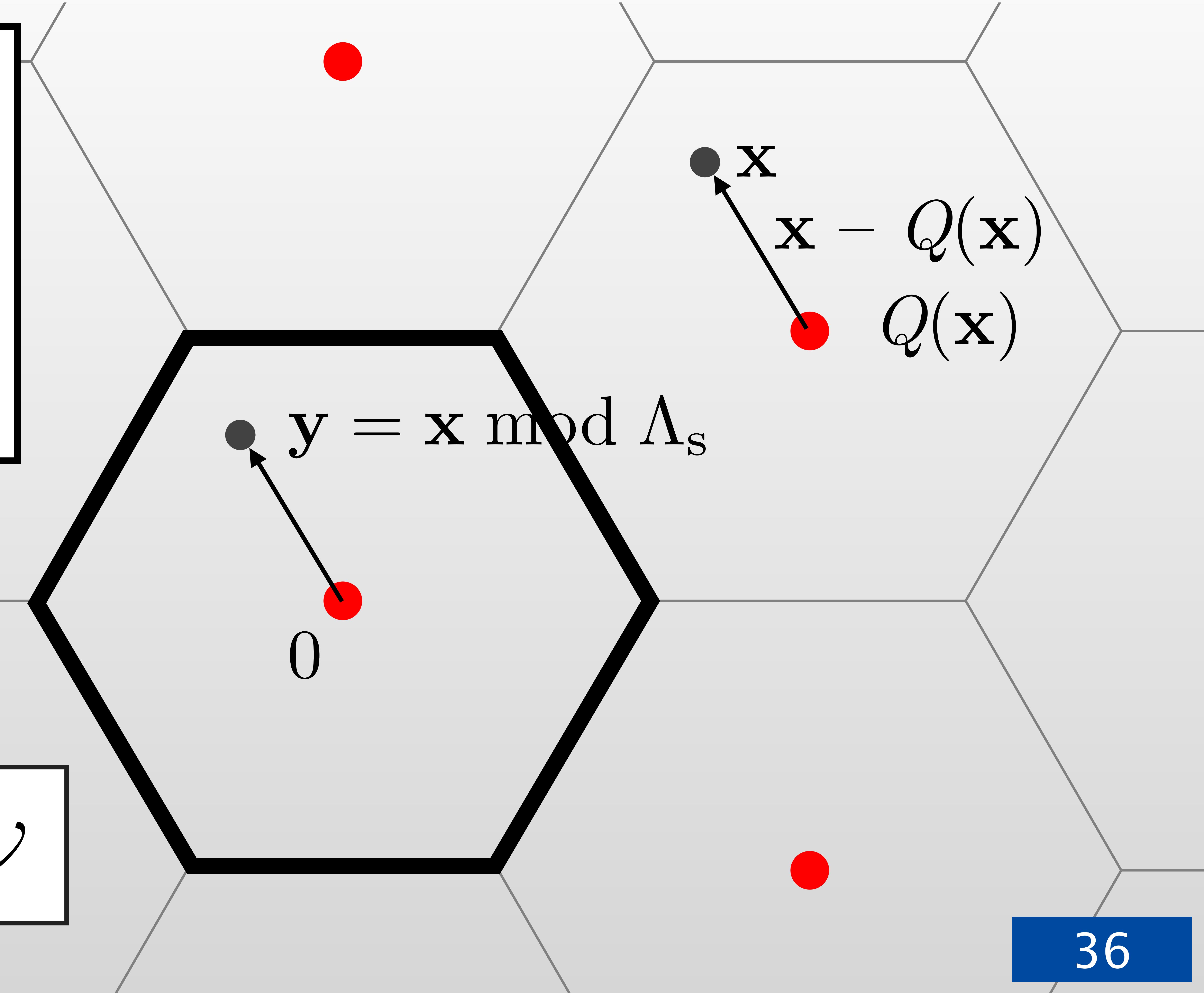
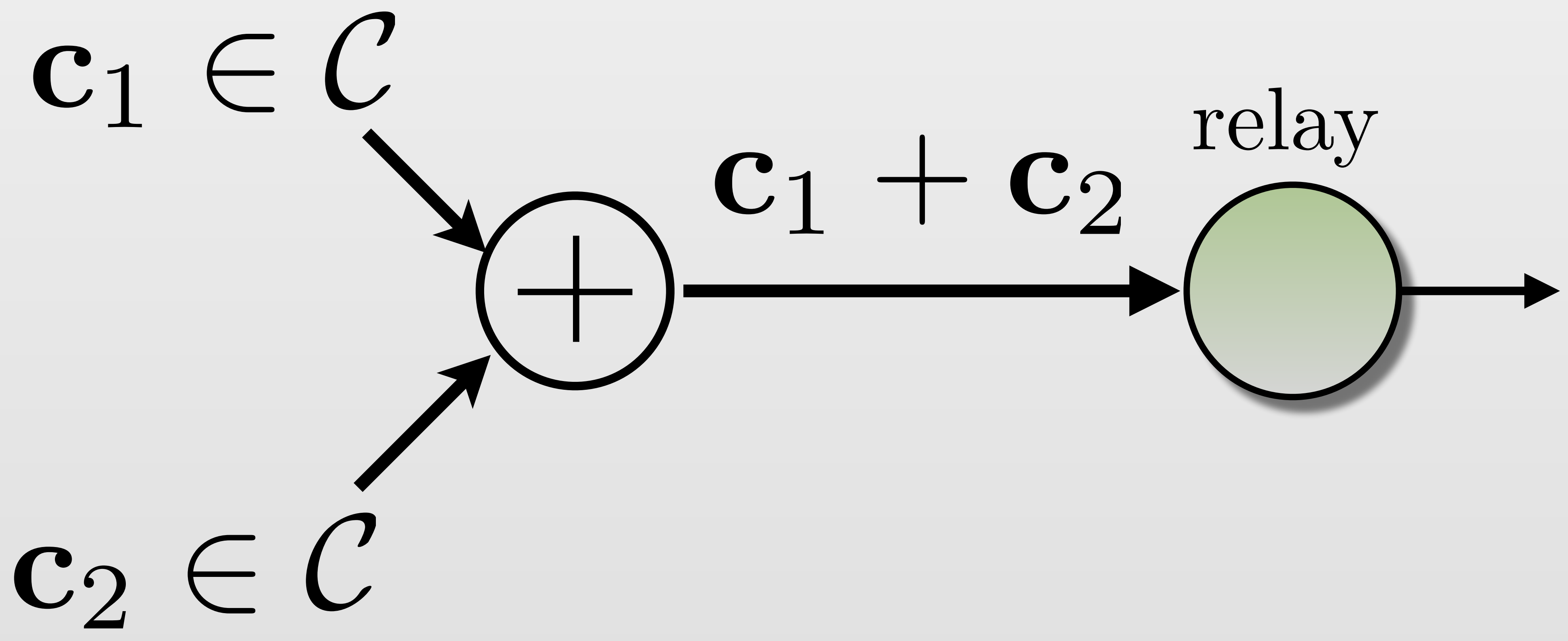


find the image of $\mathbf{x}$ in $\mathcal{V}$

Real Addition with Lattice Codes

Recall the multiple-access scenario



- $\mathbf{c}_1,\ \mathbf{c}_2 \in \mathcal{C}$ are finite group elements
- $\mathbf{c}_1 \oplus \mathbf{c}_2 \in \mathcal{C}$ is well defined
- But, real addition in the channel:

$$\mathbf{c}_1 + \mathbf{c}_2 \notin \mathcal{C}$$

- Solution: $\mathbf{c}_1 + \mathbf{c}_2 \bmod \Lambda_{\mathrm{s}} \in \mathcal{C}$

Real Addition with Lattice Codes

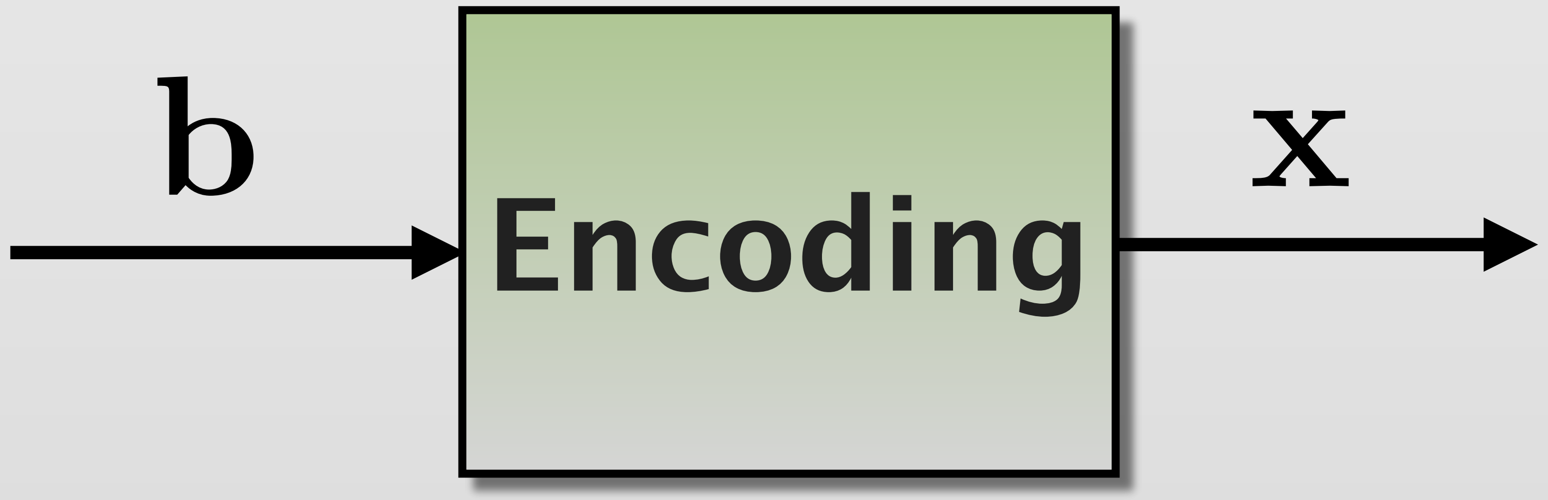
Real Addition with Lattice Codes

Encoding and Indexing

Index $\mathbf{b}$ is information: $\mathbf{b} = [b_1 \ b_2 \ \cdots \ b_n]^t$,

where $b_i \in \{0, 1, \cdots, K-1\}$.

given index $\mathbf{b}$, find $\mathbf{x} \in \mathcal{C}$



$$\mathbf{x} = \mathrm{enc}(\mathbf{b})$$

given $\mathbf{x} \in \mathcal{C}$, find index $\mathbf{b}$

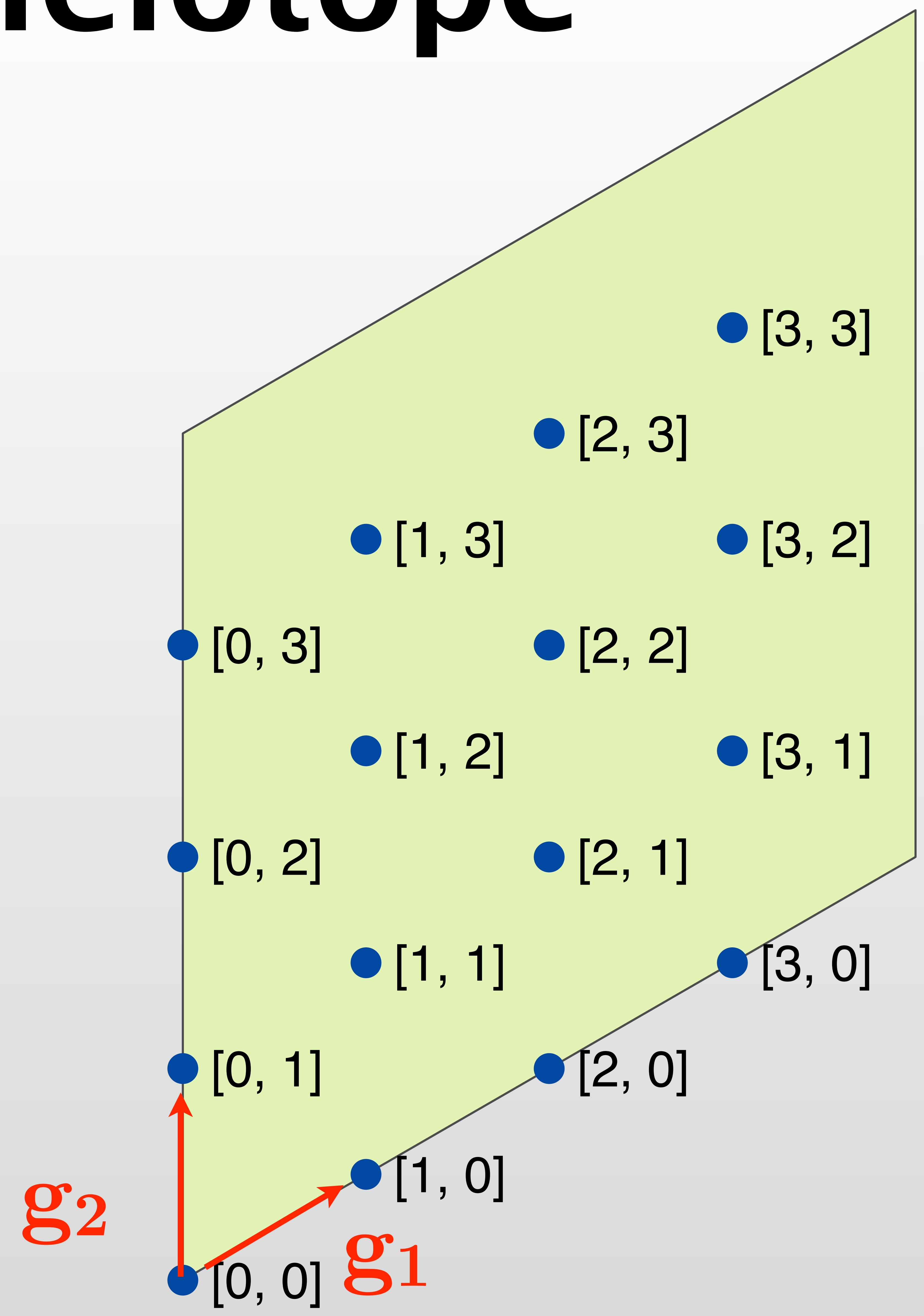
$$\mathbf{b} = \mathrm{enc}^{-1}(\mathbf{x}) = \mathrm{index}(\mathbf{x})$$

Encoding Parallelotope

Example: $A_2/4A_2$

Index $\mathbf{b}$ is information $b_i \in \{0,1,2,3\}$:

$$\mathbf{x} = \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_n \\ | & | & & | \end{bmatrix}}_{\text{generator matrix}} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$



Encoding the Voronoi Region

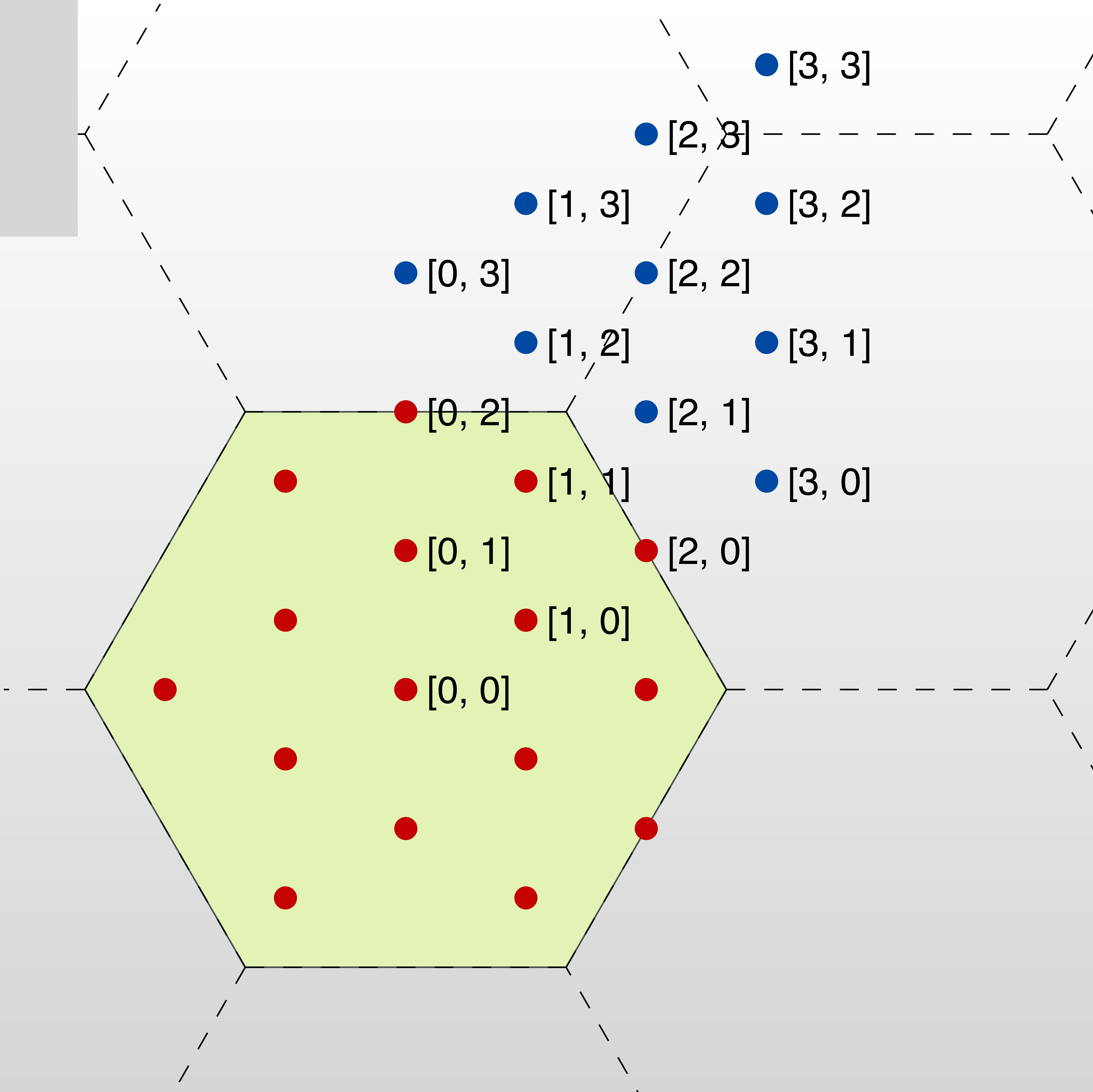
Two steps:

1. Parallelotope encoding

$$\mathbf{x} = G \cdot \mathbf{b}$$

2. Modulo operation

$$\mathbf{x} = G \cdot \mathbf{b} - Q_{\Lambda_s}(G \cdot \mathbf{b})$$



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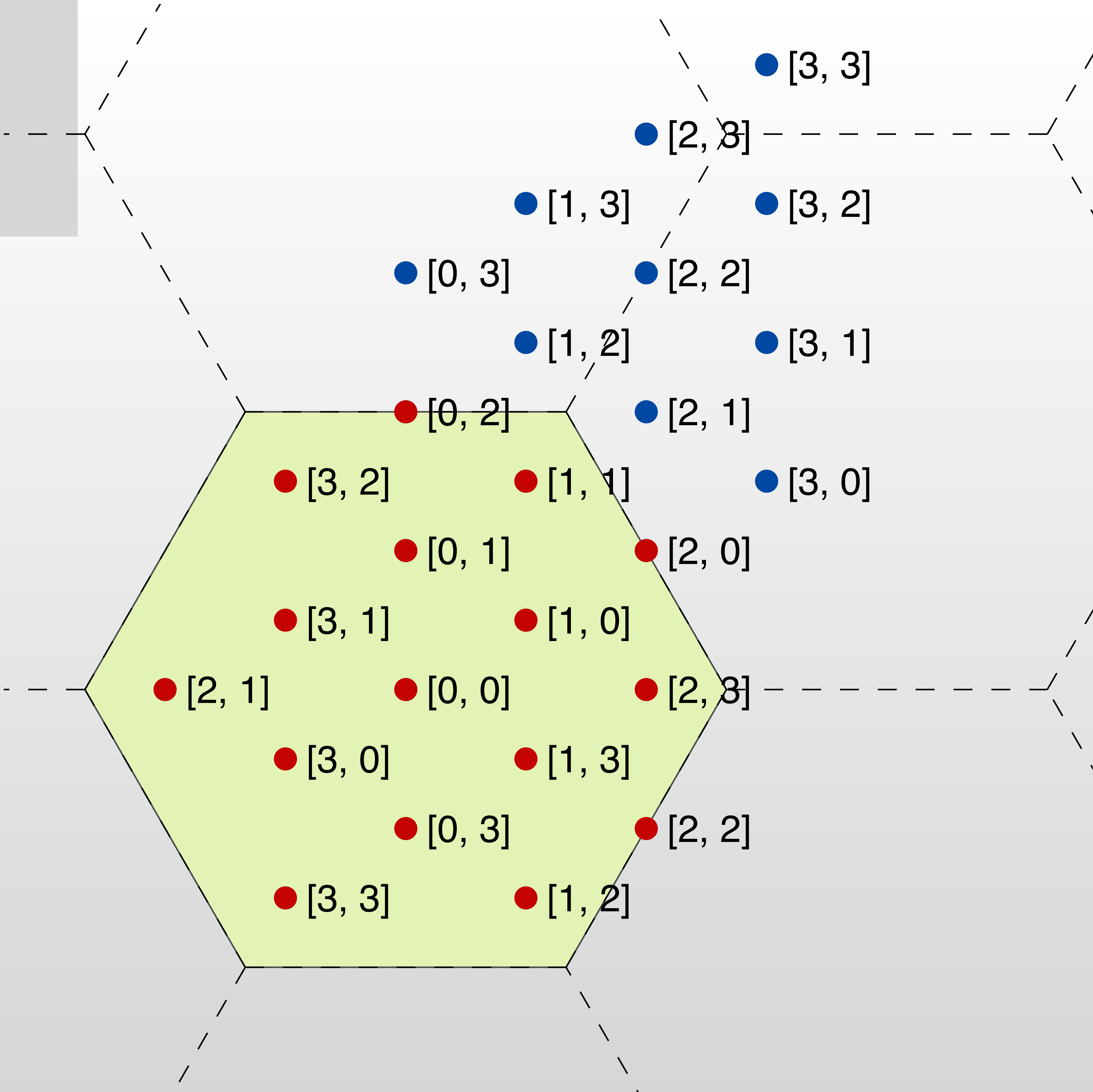

Group Isomorphism for PLNC

Information (indices) $\mathbf{b}_i \in \mathbb{Z}/K\mathbb{Z}$ form a ring with operation $\oplus, \otimes$ (integers modulo K).

Lattice codewords $\mathbf{x} \in \mathcal{C}$ for a group with operation $+$ (vector addition modulo Λ_{s}).

Easy to show there is isomorphism:

$$\mathrm{enc}(\mathbf{b}_1 \oplus \mathbf{b}_2) = \mathrm{enc}(\mathbf{b}_1) + \mathrm{enc}(\mathbf{b}_2) \text{ or}$$
$$\mathrm{index}(\mathbf{x}_1) \oplus \mathrm{index}(\mathbf{x}_1) = \mathrm{index}(\mathbf{x}_1 + \mathbf{x}_2)$$

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Group Isomorphism for PLNC

Simple multiple access channel

Group Isomorphism for PLNC

Simple multiple access channel

Assuming successful decoding:

Decoder produces $\mathbf{x}_1 + \mathbf{x}_2$ (not $\mathbf{x}_1$, $\mathbf{x}_2$ individually)

Indexing produces $\mathbf{b}_1 \oplus \mathbf{b}_2$ (not $\mathbf{b}_1$, $\mathbf{b}_2$ individually)

Highly suitable for network coding

And now for something new...

Nested lattice codes with non-self-similar lattices

- High dimension lattices (LDLC, etc.): excellent coding gain, computationally hard to perform shaping,
- Low dimension lattices (E8, Barnes-Wall): Good shaping gain with efficient algorithms, not very good coding gain.

Nested lattice codes with non-self-similar lattices

Proposed method. Construct a quotient group:

$$\Lambda_{\mathrm{C}} / \Lambda_{\mathrm{S}}$$

High-dimension lattice:
$n = 1{,}000$ to 10^5

E8, Barnes-Wall, etc. lattice
$n = 8,\ 16$

Nested lattice codes with non-self-similar lattices

Proposed method. Construct a quotient group:

$$\Lambda_{\mathbf{c}} / \Lambda_{\mathbf{s}} \times \cdots \times \Lambda_{\mathbf{s}}$$

High-dimension lattice:

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E8, Barnes-Wall, etc. lattice

$n = 8,\ 16$

Sufficient Conditions to form a Group

Given a coding lattice Λ_{c} and a shaping lattice Λ_{s}, we need to test the condition $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$.

Let G_{s} be a $n \times n$ generator matrix for Λ_{s}.

Let $H = G^{-1}$ be the check matrix for Λ_{c}

Lemma $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$ if and only if $H \cdot G_{\mathrm{s}}$ is a matrix of integers.

Easy to design Λ_{c} such that $\Lambda_{\mathrm{c}} \subseteq \Lambda_{\mathrm{s}}$

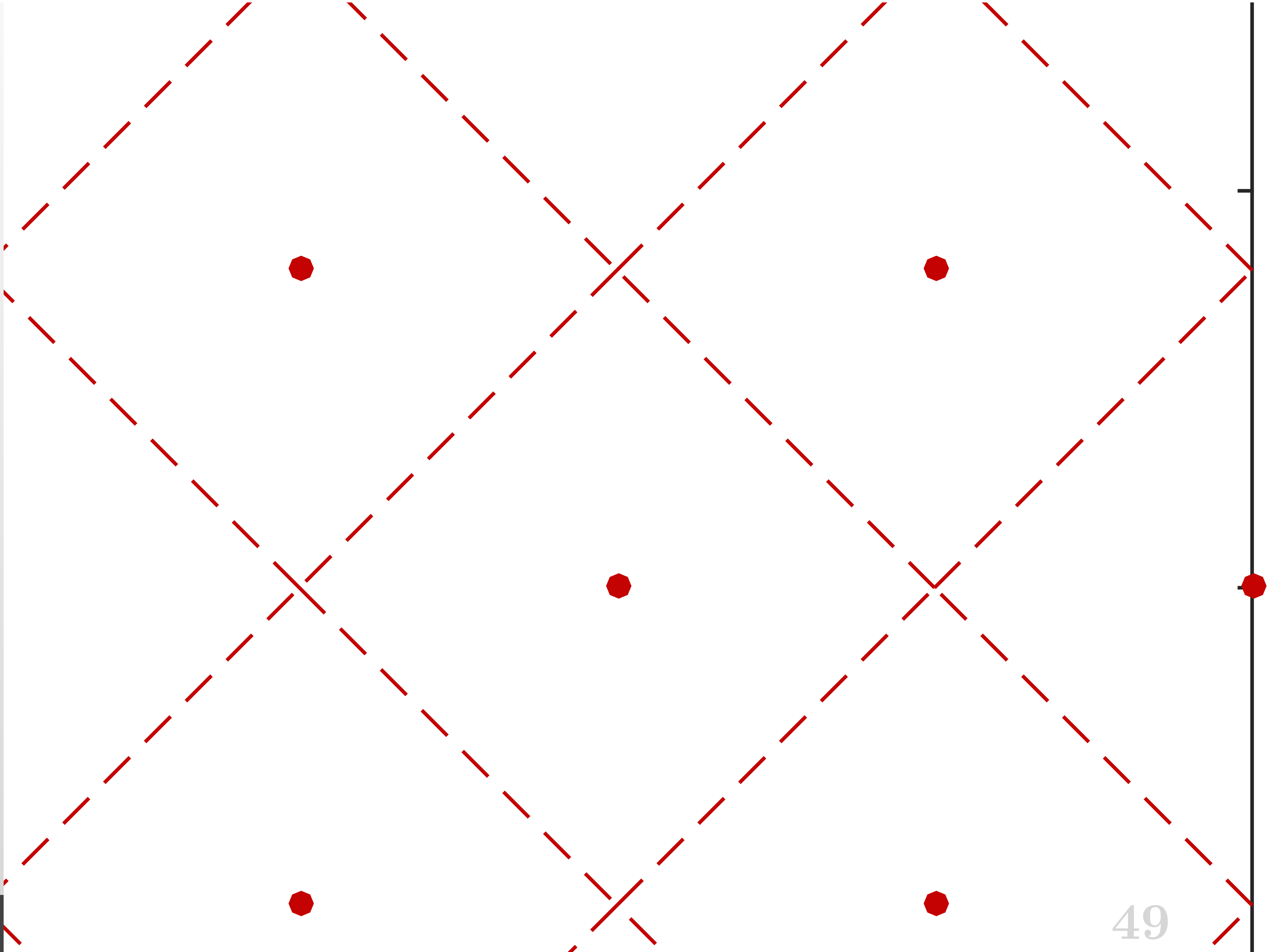
Indexing Non-Nested Lattice Codes

Achieving $\Lambda_\mathrm{s} \subset \Lambda_\mathrm{c}$ is easy.

Encoding/indexing is nontrivial.

Example for $n = 2$:

$$G_\mathrm{s} = \begin{bmatrix} 4 & 0 \\ 4 & 8 \end{bmatrix} \quad \longleftarrow \quad \Lambda_\mathrm{s}$$



Indexing Non-Nested Lattice Codes

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Example for $n = 2$:

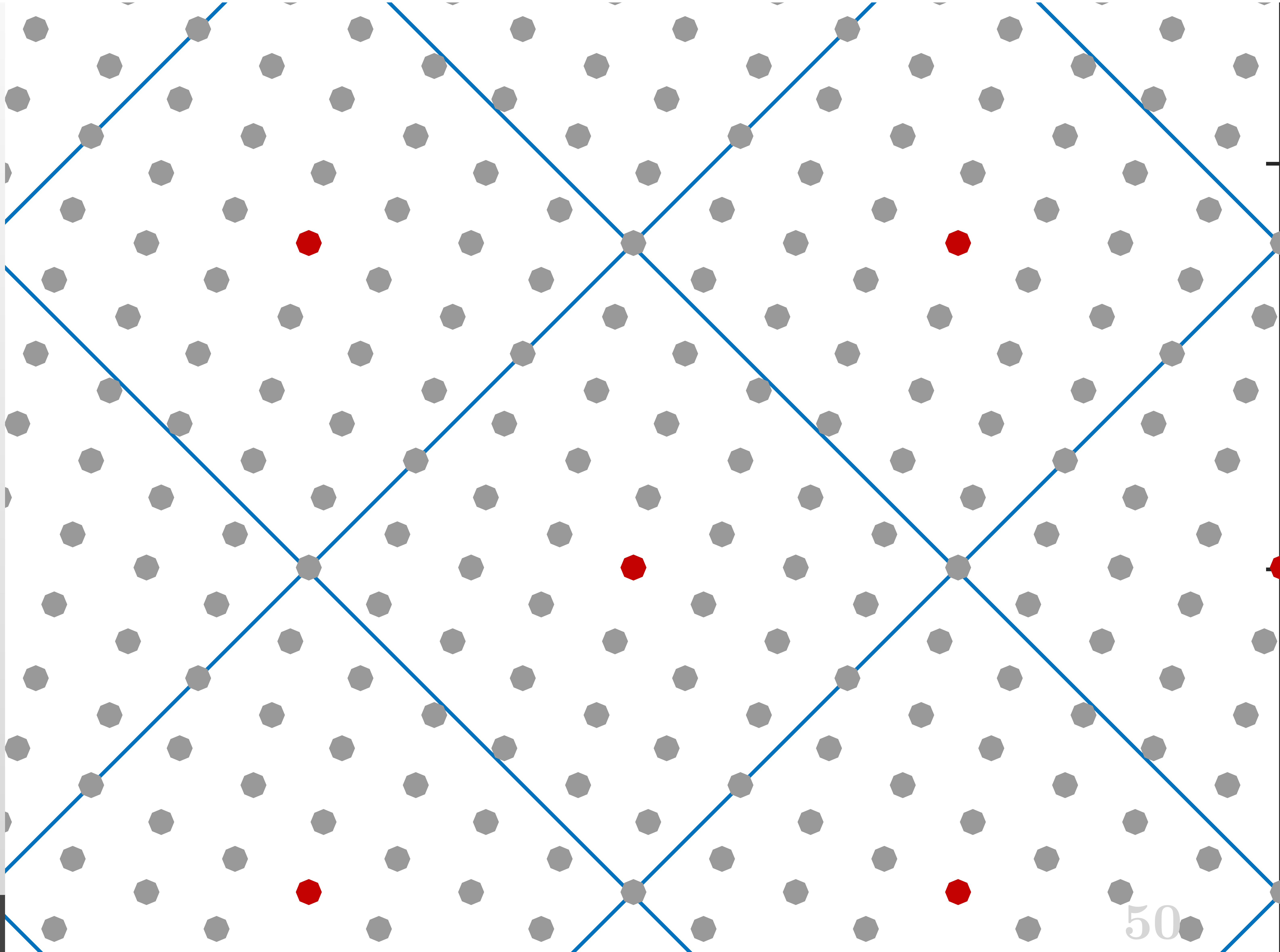
$$G_s = \begin{bmatrix} 4 & 0 \\ 4 & 8 \end{bmatrix} \quad \longleftarrow \quad \Lambda_s$$

$$G_c = \begin{bmatrix} 8/9 & 2/9 \\ -4/9 & 8/9 \end{bmatrix} \quad \longleftarrow \quad \Lambda_s$$

$$\left(G_c^{-1} = \begin{bmatrix} 1 & -1/4 \\ 1/2 & 1 \end{bmatrix} \right)$$

Note:

- $\Lambda_s \neq K\Lambda_c$ not self similar
- but $\Lambda_s \subset \Lambda_c \Rightarrow \Lambda_c/\Lambda_s$



Indexing Non-Nested Lattice Codes

Number of codewords:

$$\frac{\det(G_\mathrm{s})}{\det(G_\mathrm{c})} = 36$$

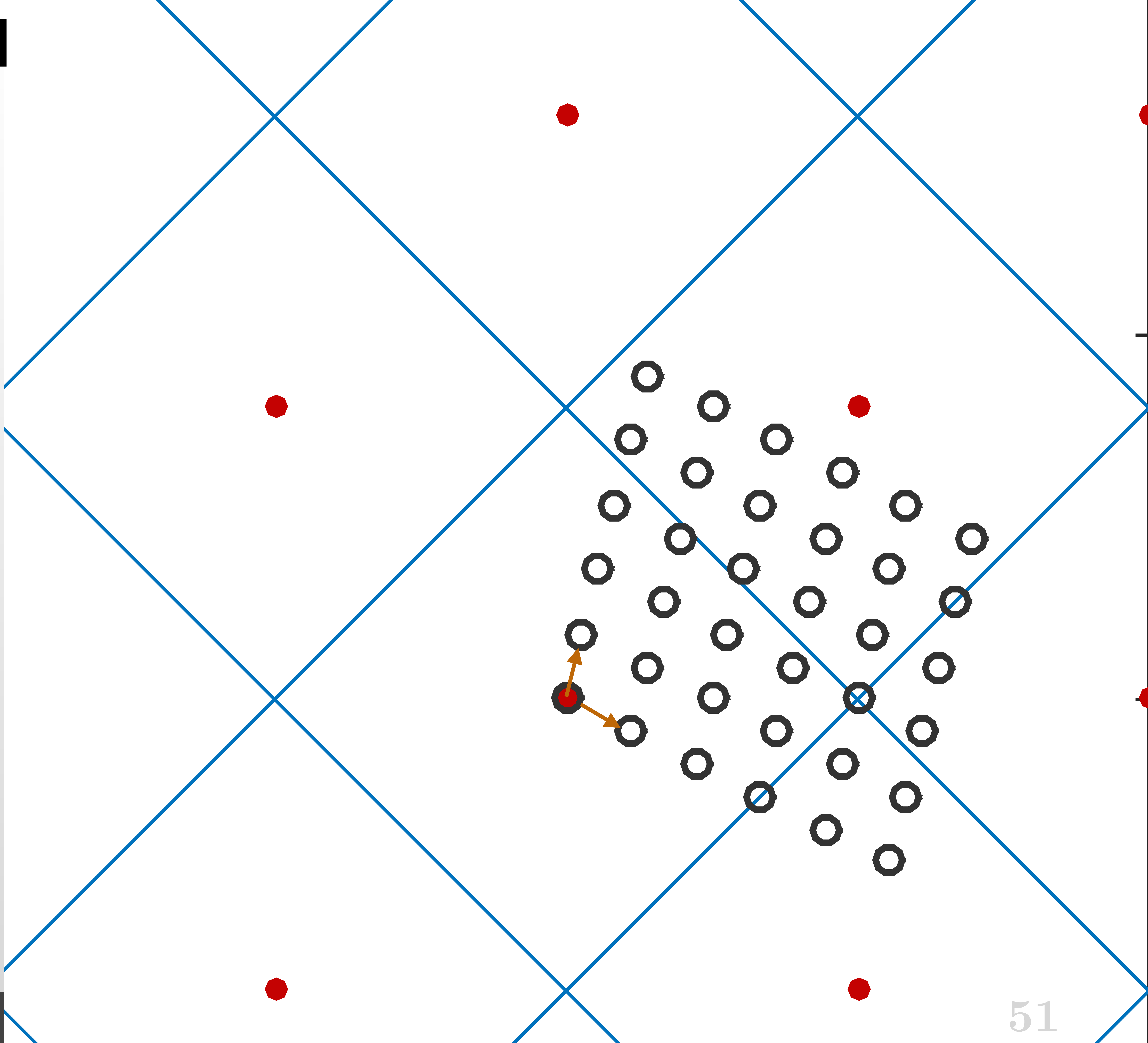
Natural candidate:

$$b_1 \in \{0, 1, 2, 3, 4, 5\}$$
$$b_2 \in \{0, 1, 2, 3, 4, 5\}$$

Parallelotope encoding step:

$$G_\mathrm{c}\mathbf{b} = \begin{bmatrix} 8/9 & 2/9 \\ -4/9 & 8/9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Do these points form coset leaders?



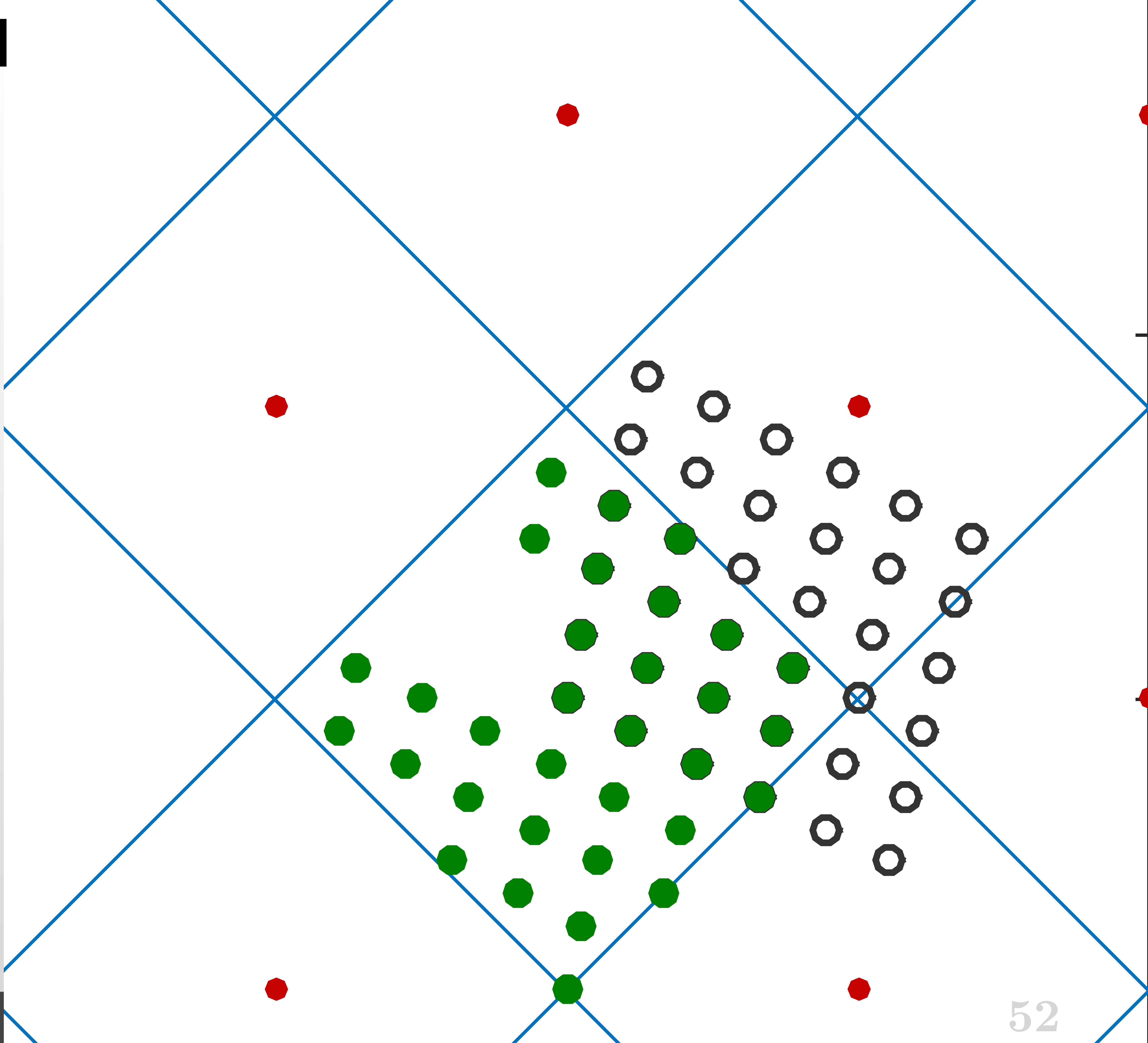
Indexing Non-Nested Lattice Codes

Encoding Step 2:

$$x = G\mathbf{b} - Q_{\Lambda_s}(G\mathbf{b})$$

No! Coset leaders not formed.

What about a change of basis?



Finding a Basis Suitable for Encoding

We want to transform the basis of G_c:

$$G'_\mathrm{c} = G_\mathrm{c} W$$

where W is has integer entires and $\det W = 1$. New basis is:

$$G'_\mathrm{c} = \begin{bmatrix} \frac{\mathbf{g}_1}{M_1} & \frac{\mathbf{g}_2}{M_2} & \cdots & \frac{\mathbf{g}_{n-1}}{M_{n-1}} & \mathbf{q} \end{bmatrix}$$

where $\mathbf{q}$ is some vector to be found. Find W:

$$\begin{aligned} (G_\mathrm{c})^{-1} \cdot G'_\mathrm{c} &= W \\ &= \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1,n-1} & z_1 \\ w_{21} & w_{22} & \cdots & w_{2,n-1} & z_2 \\ & \vdots & & & \\ w_{n,1} & w_{n,2} & \cdots & w_{n,n-1} & z_n \end{bmatrix} \end{aligned}$$

Then $\det W = 1$ is a **linear diophantine equation** in $z_1, z_2, \ldots, z_n$.

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$\mathbf{g}_i$ from shaping lattice

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linearly dependent

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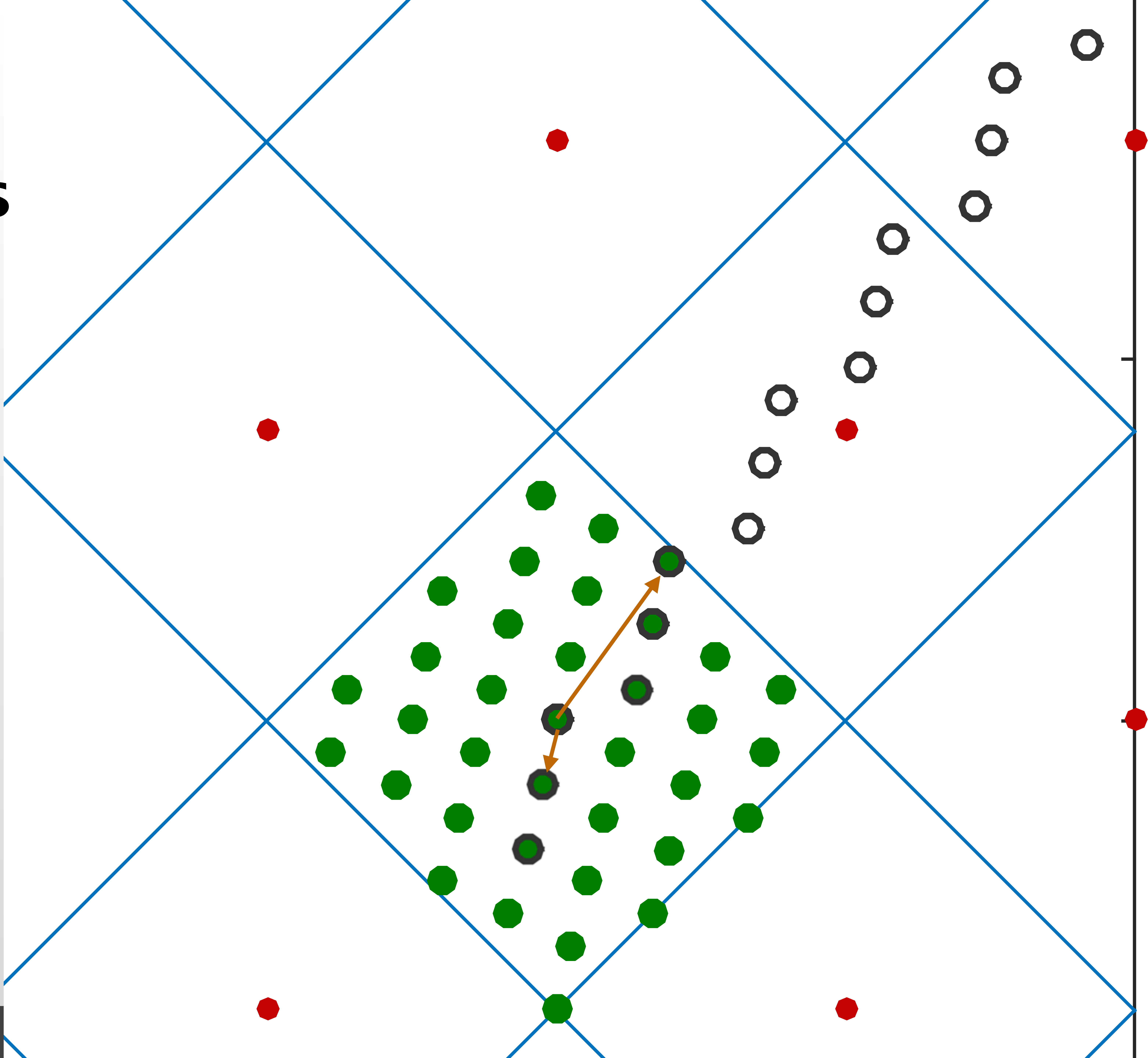
Indexing Non-Nested Lattice Codes Using a Suitable Basis

$$\begin{bmatrix} 1 & -1/4 \\ 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 & q_1 \\ 4/3 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 2 & z_2 \end{bmatrix}$$

$\det W = 1 \Rightarrow 1z_2 - 2z_1 = 1$ has numerous solutions.



Brian Kurkoski, JAIST

Summary – Physical Layer Network Coding

PLNC:

- Technique for cooperative wireless networks
- Exploit network coding to increase capacity
- Lattices: real codes to correct errors, shaping gain
- Remove noise first, and interference later
- Compute-and-Forward relaying also deals with fading

Recommended Reading

John Conway and Neil Sloane, *Sphere Packings, Lattices and Groups*, Springer, Third Edition, 1999

G. David Forney, Lecture notes for Principles of Digital Communications II Course at MIT http://dspace.mit.edu/

Ram Zamir, *Lattice Coding for Signals and Networks*, Cambridge Univ Press, September 2014

Bobak Nazer and Michael Gastpar, “Reliable Physical Layer Network Coding,” *Proceedings of the IEEE*, March 2011