Power-Constrained Communications Using LDLC Lattices

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Background

The capacity of the constrained-power AWGN channel (Shannon):

$$R \leq \frac{1}{2}\log_2\left(1+\frac{S}{N}\right)$$

can be achieved using lattices

- de Buda, Loeliger, Richardson & Urbanke, Erez & Zamir

Belief-propagation: Codes and Lattices

- Low-density parity-check codes are highly successful, but these are finite-field codes, rather than lattices.
- "LDPC lattices" are lattices based upon LDPC codes. Non-binary LDPC were shown to be better than binary.
- LDLC — Low density lattice codes
  - Within 0.6 dB of capacity of **unconstrained-power** channel.

In This Talk...

Consider LDLC lattices on the constrained-power AWGN channel.

- Limited attention so far.

Use “nested lattices” for encoding, requires quantization

"Continuous approximation" separate shaping gain from coding gain

Propose a simple modification to BP to improve shaping loss (gain)

- Standard belief-propagation does not work well for quantization

Design LDLC lattices which minimize (maximize) the sum of the shaping loss (gain) and coding loss (gain).

Numerical results M=8 (3 bits per dimension), dimension n=100

- Individually: shaping loss of 1.45 dB and coding loss of 2.2 dB = 3.65 dB
- Communication system: Loss of 3.6 dB

LDLC Lattices

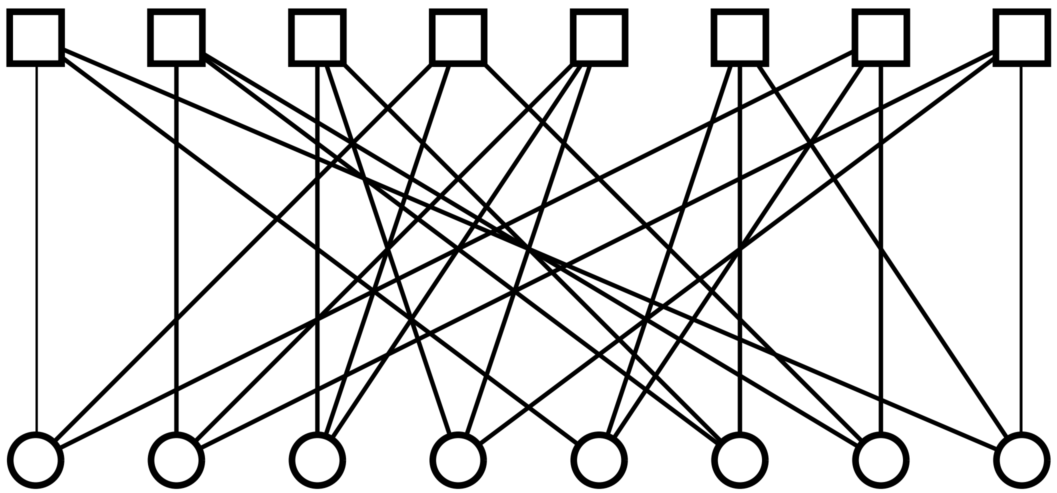
Low-density lattice codes (LDLC) introduced by Sommer, Shalvi and Feder.

LDLC have a sparse inverse generator matrix:

$$H = G^{-1}$$

H has constant row and column weight d. Dominant 1, other positions $w < 1$

$$H = \begin{bmatrix} h_2 & 0 & 0 & 0 & h_1 & 0 & 0 & -h_3 \\ 0 & -h_1 & 0 & 0 & 0 & h_3 & h_2 & 0 \\ 0 & 0 & -h_1 & h_3 & 0 & -h_2 & 0 & 0 \\ h_3 & 0 & -h_2 & 0 & 0 & 0 & h_1 & 0 \\ 0 & -h_2 & h_3 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_1 & 0 & -h_2 \\ -h_1 & 0 & 0 & 0 & h_2 & 0 & h_3 & 0 \\ 0 & h_3 & 0 & -h_2 & 0 & 0 & 0 & -h_1 \end{bmatrix}$$



Each row and each column has:

$$h_1 \geq h_2 \geq \cdots \geq h_d \qquad \text{Define} \quad \alpha = \frac{h_2^2 + h_3^2 + \cdots + h_d^2}{h_1^2} \geq 0$$

Theorem [Sommer et al.] If $\alpha \leq 1$, Gaussian variances converge exponentially fast. $\Longrightarrow$ Use α as LDLC design parameter.

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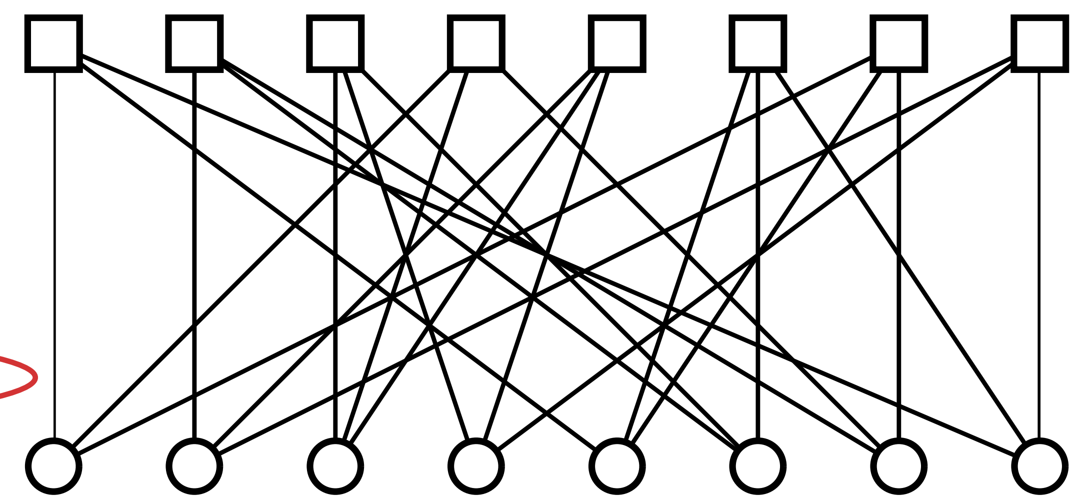

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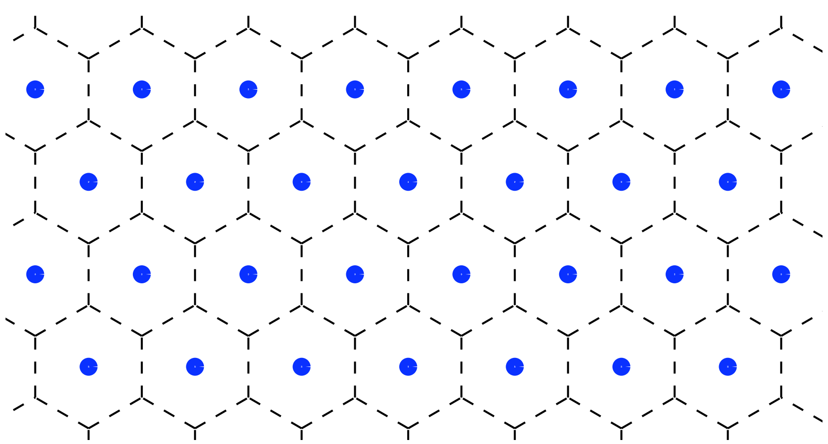
Unconstrained Power Channel

Transmit an arbitrary lattice point over AWGN channel.



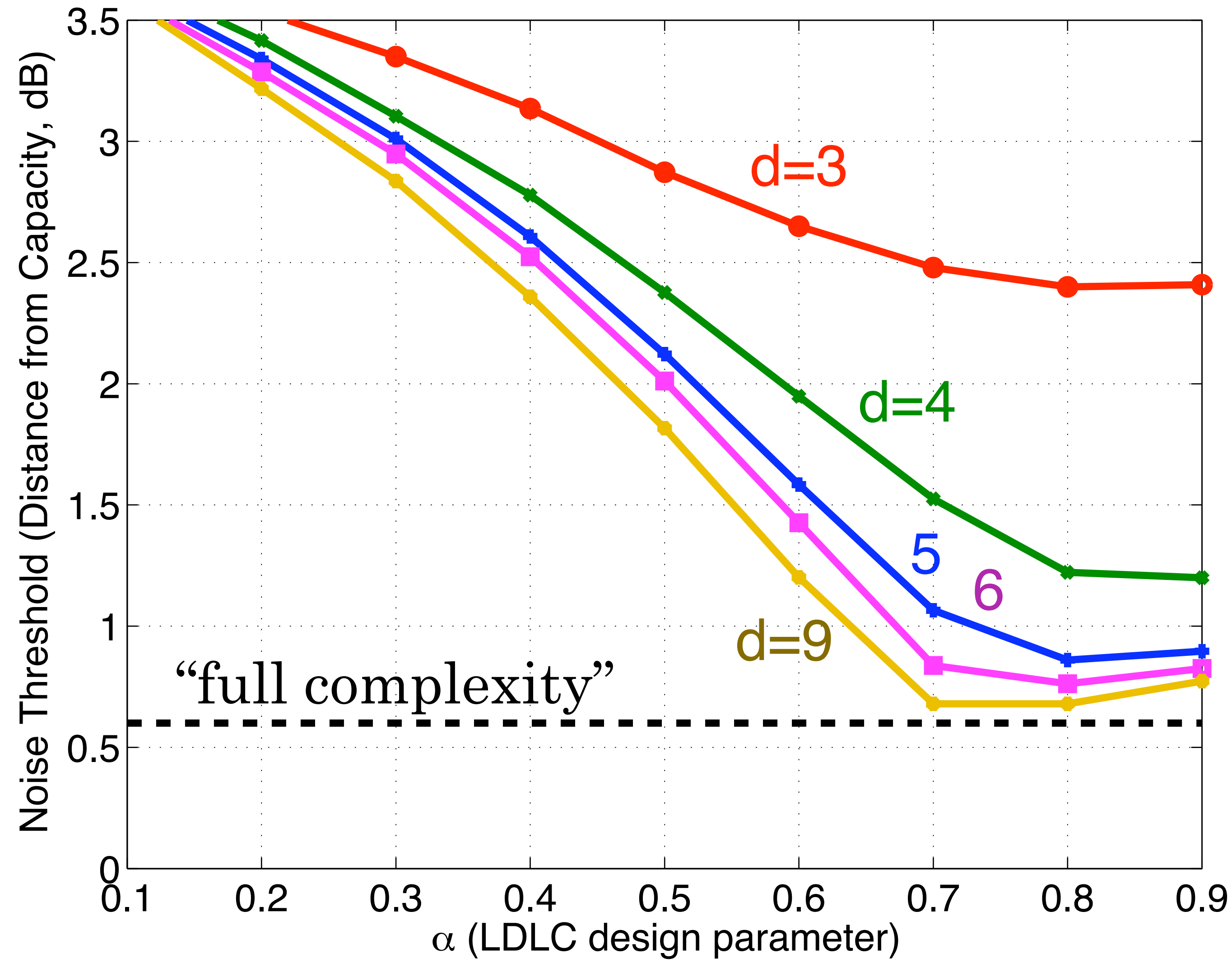
No power constraint, but the lattice density is constrained.

Study coding gain, no shaping gain.

"Capacity"

$$N \leq \frac{V(\Lambda)^{2/n}}{2\pi e}$$

(see Poltyrev 1994)



Noise Thresholds

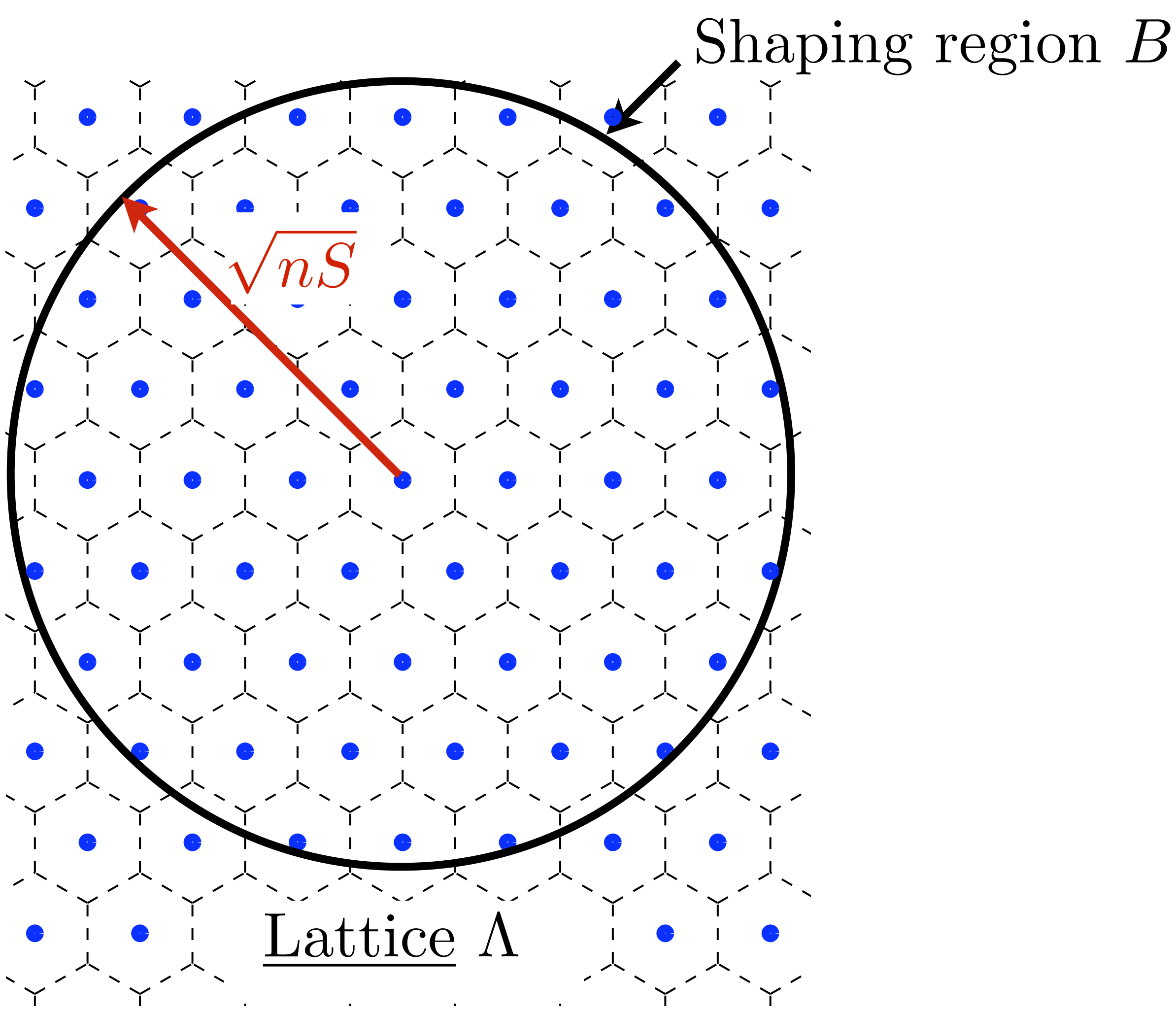
simplified (single Gaussian) decoder

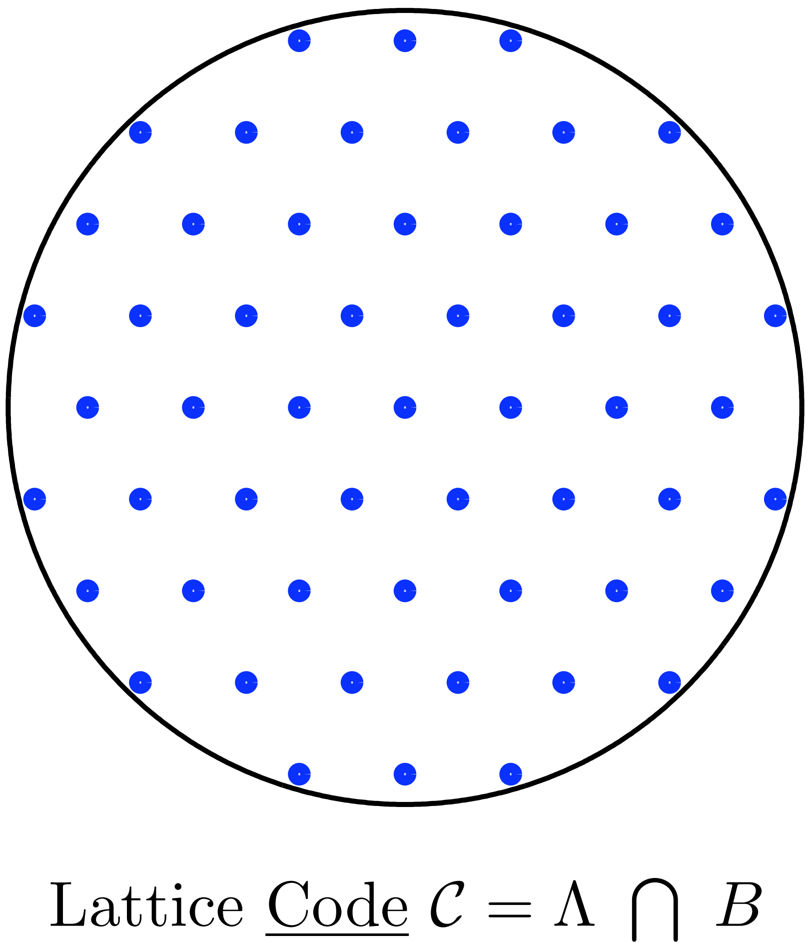
increasing alpha improves threshold

increasing d increases complexity

Lattices Codes for the AWGN Channel

- *Lattice* is an infinite number of points.
- *Lattice code* (finite) is the intersection of a shaping region and a lattice.
- Shaping region satisfies the power constraint.
- Lattices have elegant structure, are “easy” to decode.





Continuous Approximation

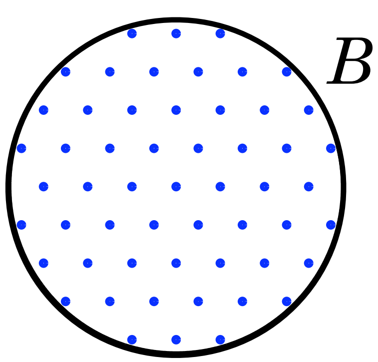
Separate lattice Λ and shaping region B contribution to signal power:

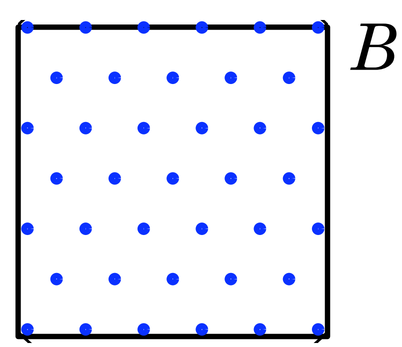
$$\text{Average Power} \approx \underbrace{\frac{\int_B ||\mathbf{x}||^2 d\mathbf{x}}{nV(B)^{\frac{2}{n}+1}}}_{G(B)} \cdot M^n V(\Lambda)$$

Depends only on *shape* of B (normalized second moment)

Depends only on coding lattice $\boldsymbol{\Lambda}$

Shaping Loss (Gain)





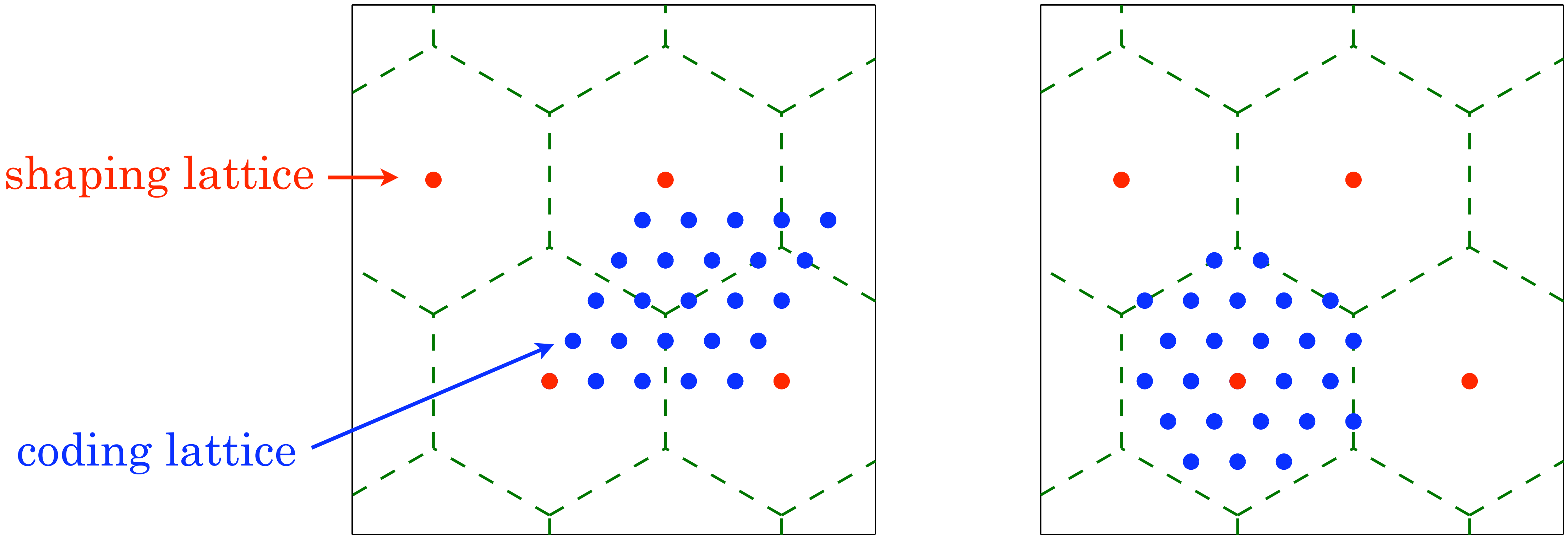
$$\lim_{n\to\infty} \frac{G(\text{cube})}{G(n\text{-sphere})} = \frac{\pi e}{6} = 1.53 \text{ dB}$$

$$\lim_{n\to\infty} G(n\text{-sphere}) = \frac{1}{2\pi e}$$

$$G(\text{cube}) = \frac{1}{12}$$

Practical Encoding:
Nested Lattices (Conway and Sloane 1983)

- How to map information $b \in \{0, 1, ..., M-1\}^n$ to those lattice points inside B.
- B is the Voronoi region of a sublattice.



$\mathbf{x} = G \cdot \mathbf{b}$
Easy to encode,
poor shaping gain

$\mathbf{x} = G \cdot \mathbf{b} \mod \Lambda_s$
Shaping by
"self-similar" Voronoi region

Find $\mathbf{x} = G \cdot \mathbf{b} \mod \Lambda_s$ by quantizing $G \cdot \mathbf{b}$ to the nearest point in Λ_s.

Proposed Quantizer
Modification of Belief-Propagation Decoder

Belief-propagation algorithm:

- Is effective when the input is a lattice point plus noise.
- Does not usually converge when input is an arbitrary point.

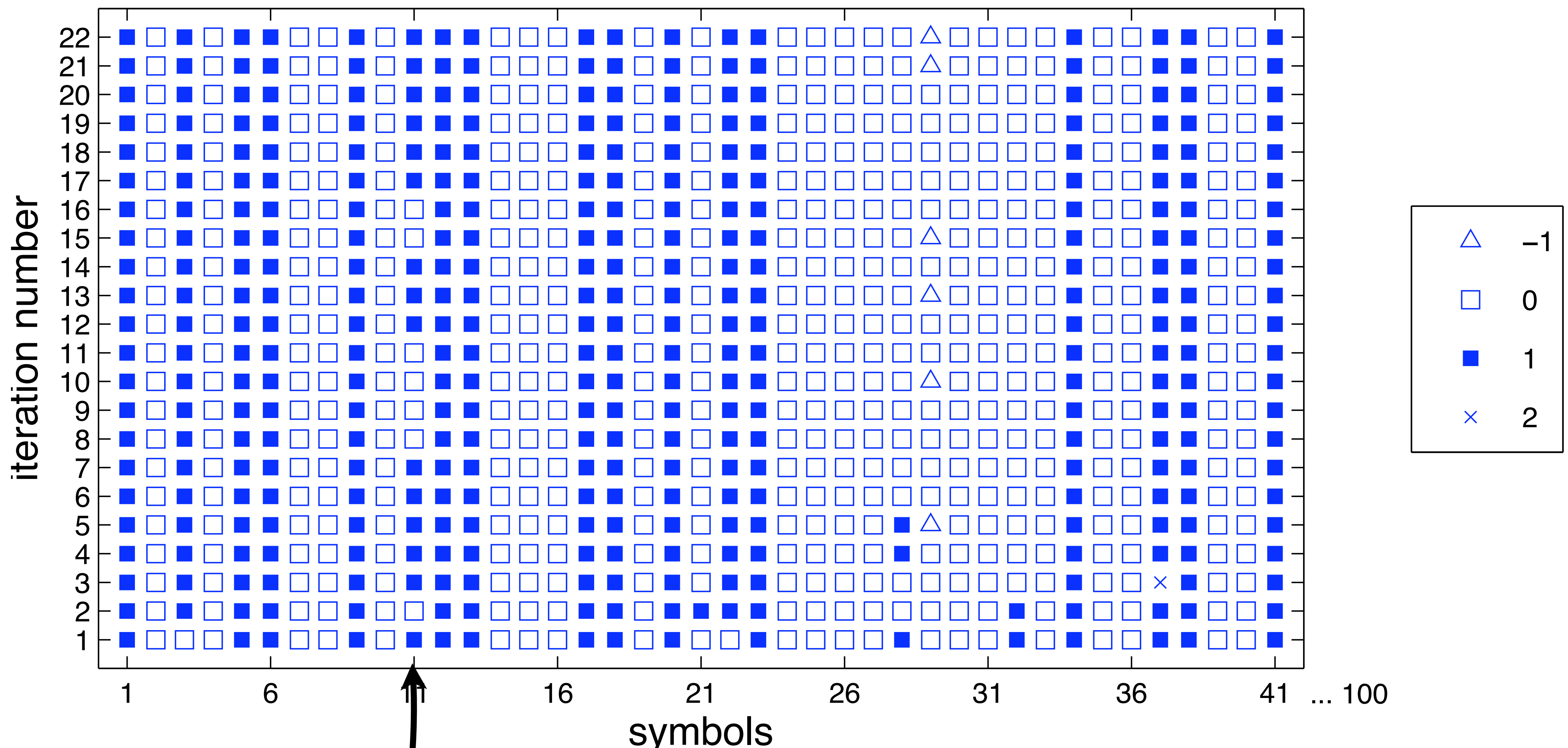
Proposed quantizer

- Let the belief-propagation algorithm run for 100 iterations
- After each iteration, make hard decisions on integers
- Using 100 hard decisions, construct a distribution for each integer
- Identify the Q least reliable symbols (e.g. $Q = 10$)
- Perform a brute-force search over these symbols (exponential in Q)
- Accept the integer sequence with the lowest mean squared error.

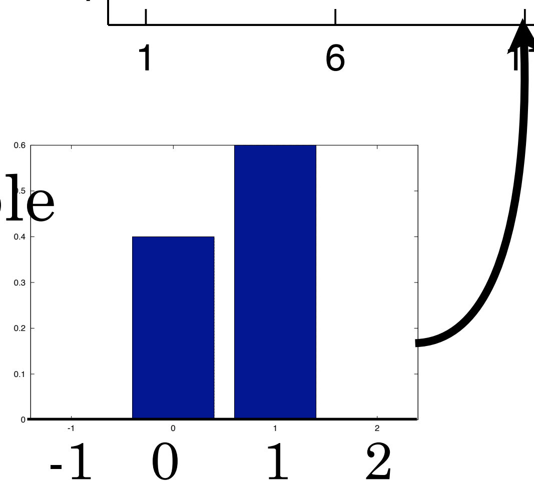
This method is not optimal, but optimality is not required for encoding

- penalty is unnecessary increase in signal power.

Proposed Quantizer
Hard Decisions After Each Iteration



unreliable position

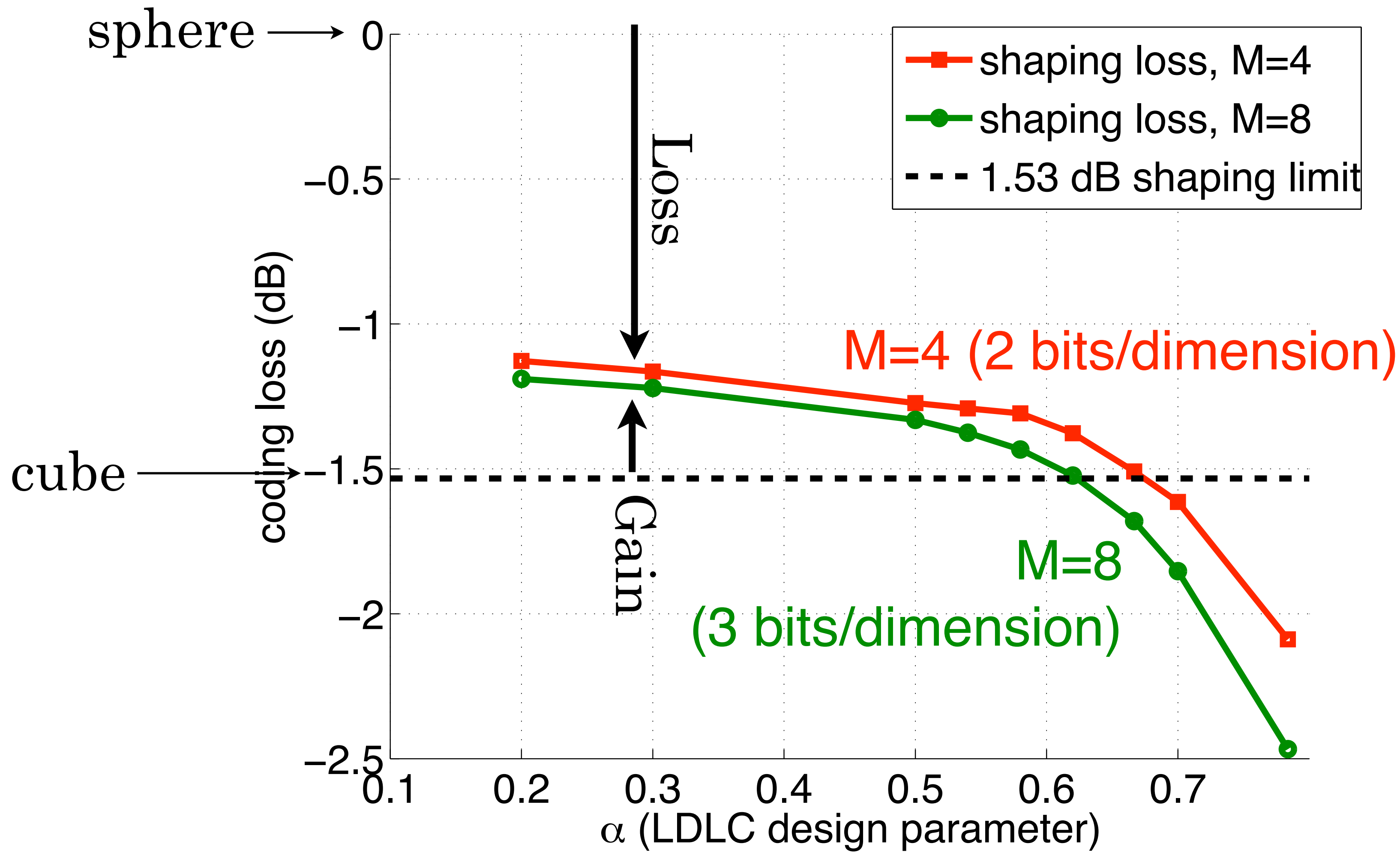


Each position oscillated between two values
Complexity of brute-force search is 2^Q
For n=100 lattice, used Q=10

Shaping Loss (Gain)

Compute $G(B)$ via Monte Carlo integration.

$$G(B) = \frac{\int_B ||\mathbf{x}||^2 d\mathbf{x}}{nV(B)^{\frac{2}{n}+1}}$$



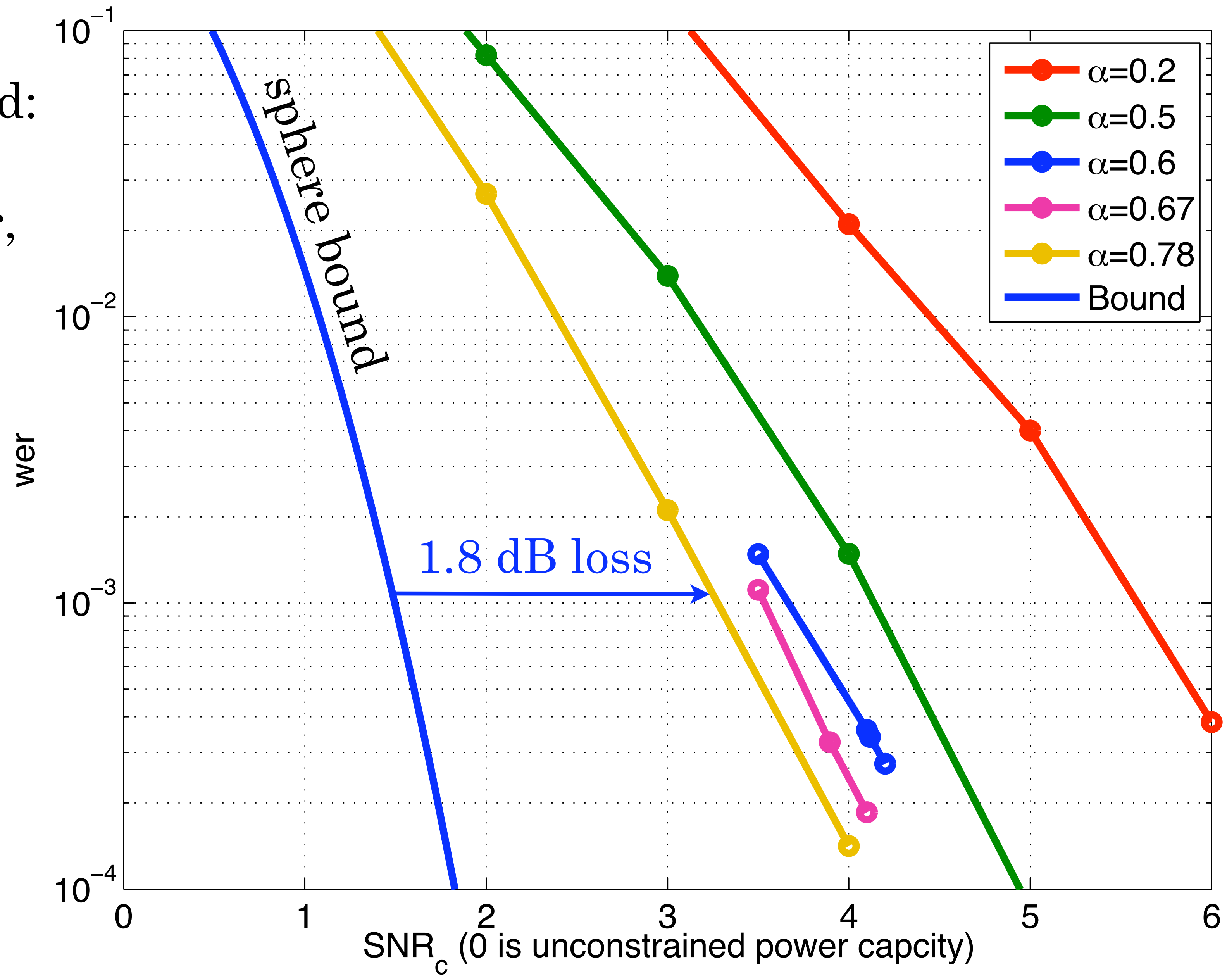
LDLC lattice:
n=100
d=3

Increasing alpha increases shaping loss.

Both lattice design and suboptimal quantizer contribute to shaping loss.

Coding Loss (Unconstrained Power)

Sphere bound: Tarokh, Vardy, Zeger, 1999



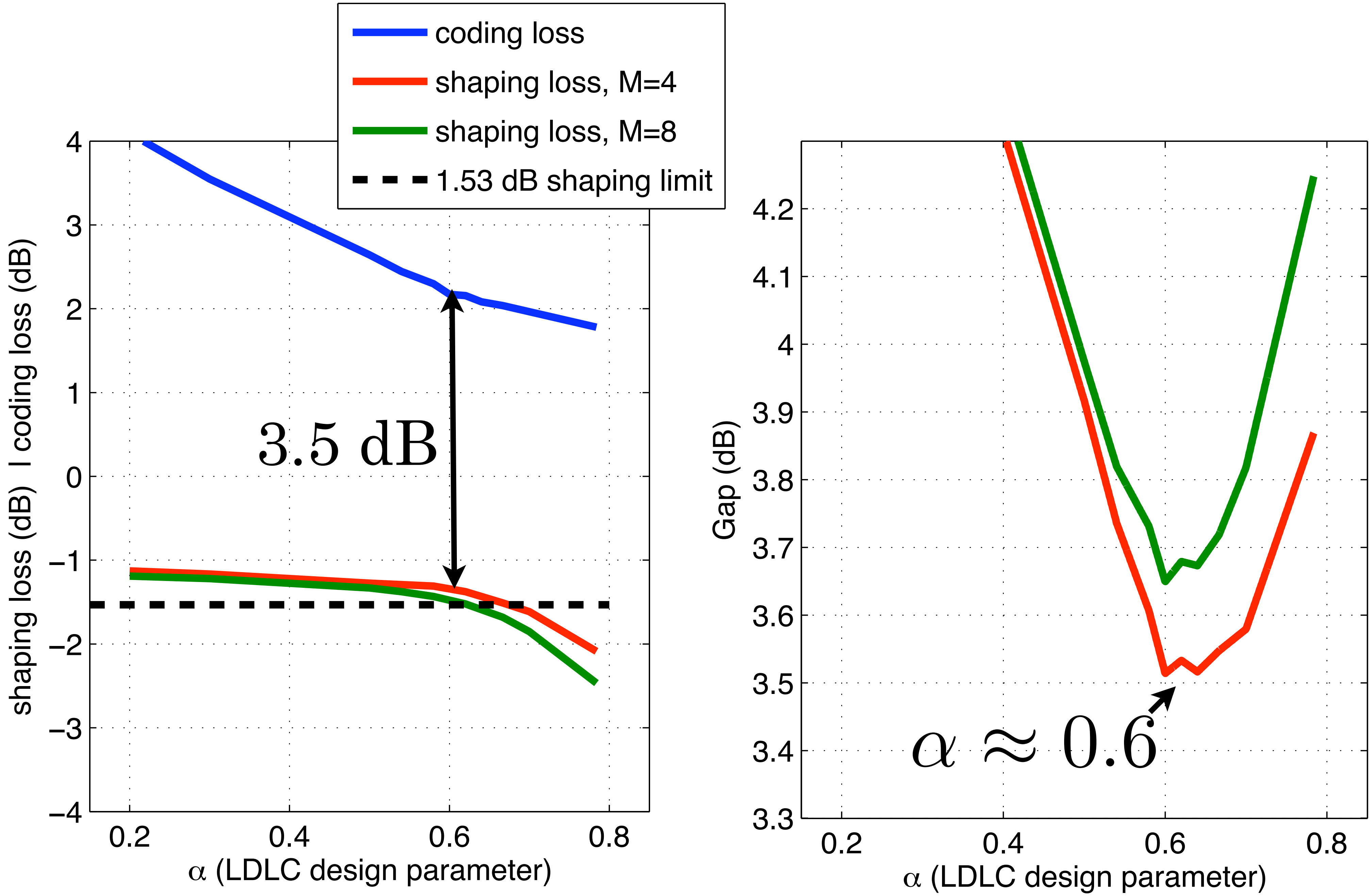
LDLC lattice:
n=100
d=3

Dimension 100 lattice has 1.8 dB loss from sphere bound

This loss decreases for increasing dimension (0.7 dB at dimension 10,000)

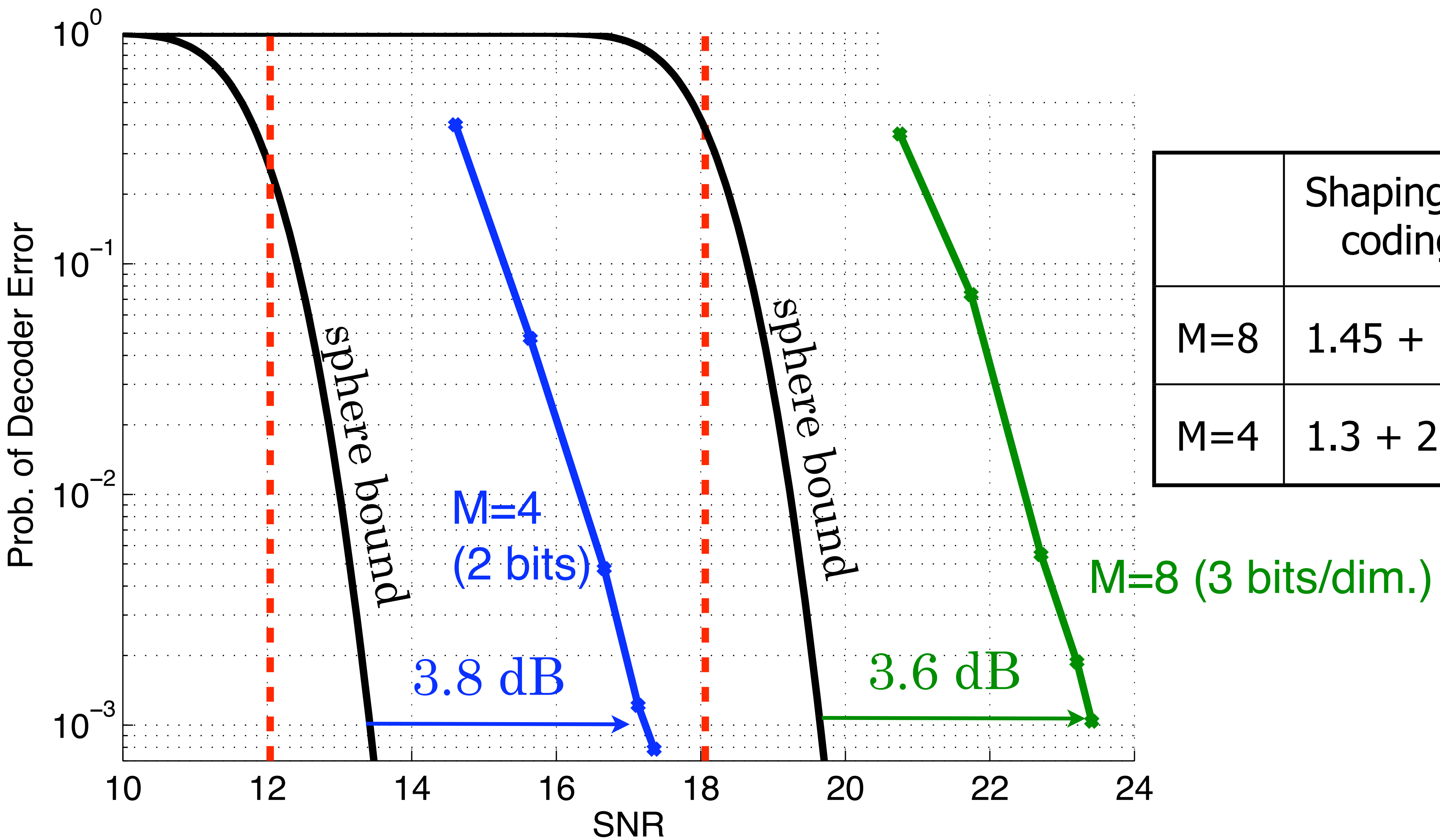
Lattice Design

Continuous approximation allows separation of shaping loss and coding loss
Find the value of alpha which minimizes sum losses.



Power-Constrained Channel, n=100

End-to-end simulation: proposed encoder, standard BP lattice decoding



	Shaping loss + coding loss	Simu.
M=8	1.45 + 2.2 = 3.65	3.6
M=4	1.3 + 2.2 = 3.5	3.8

Continuous approximation is better at higher dimensions.

Conclusion

Considered LDLC lattices on the power-constrained AWGN channel.

Coding loss (gain) and shaping loss (gain) can be separated,

- But with nested lattices, the shaping lattice is a sublattice of the coding lattice.
- Designed LDLC lattices to minimize the signal power

Encoding for the power-constrained channel requires quantization

- Made a simple modification to BP decoder to improve shaping loss
- The modification has complexity exponential in lattice dimension