Single-Gaussian Messages and Noise Thresholds for Low-Density Lattice Codes

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Tokyo, Japan

International Symposium on Information Theory
June 30, 2009
Seoul, Korea
**Background**

- Low-density lattice codes (LDLC) are lattices [Sommer, Feder and Shalvi, IT 2008]
- Decoded using belief-propagation, like low-density parity check codes.
- Decoding complexity is linear in the dimension. Dimension $n=10^5$ possible.
  - “Classical” lattices can be decoded in dimension $n=2\sim200$
- Within 0.6 dB of unconstrained power capacity
- Messages (beliefs) are functions, rather than numbers. Existing implementations
  - Quantize the functions
  - Gaussian mixture approximation

**In this talk**

- Propose a LDLC decoder: belief-propagation messages approximated by a single Gaussian,
- Single-Gaussian decoding has a noise threshold within 0.1 dB of the quantized-message implementation (that is, 0.7 dB from capacity)
- But, single-Gaussian decoding has much lower complexity than other methods
- Design of regular LDLC lattices $\alpha=0.7$ is better than $\alpha$ approaching 1
Low-Density Lattice Codes (LDLCs)

Sommer, Feder and Shalvi gave a lattice construction and decoding algorithm based upon low-density parity-check codes. Extensive convergence analysis in IT Trans, April 2008.

**Low-Density Parity-Check Codes**

- Code over a finite field (binary)
- Sparse parity check matrix

**LDLC**

- Lattice: Code over the real numbers
- Lattice point \( x = G \cdot b \)
- \( n \)-by-\( n \) inverse generator \( H = G^{-1} \) is sparse
- \( H \) has row and column weight \( d \)
- Non-zero entries in rows and columns:
  \[ \pm \{ h_1, h_2, \ldots, h_d \} \]
Low-Density Parity-Check Codes

So, messages are scalars (LLRs)

# check nodes < # variable nodes
\[ x_1 + x_2 + x_3 = 0 \] (over field)

Approaches BIAWGN channel capacity

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Low-Density Lattice Codes (LDLC)

Variables are from the field
\[ x_i \in \{0, 1\} \]

So, messages are functions (!)

# check nodes = # variable nodes
\[ x_1 + x_2 + x_3 = b \] (over real numbers)
\[ b \text{ is an integer} \]

Comes within 0.6 dB of unconstrained power AWGN channel capacity.
Channel Model:
Unconstrained Power Transmission

- Use entire lattice for transmission
  - The code is linear!
- Reliable “communication” [Poltyrev, etc.] if and only if
  \[
  \text{SNR} \geq 2\pi e, \quad \text{SNR} = \frac{\sqrt{V^2}}{\sigma^2}
  \]
  - Noise power \( \sigma^2 \)
- Transmit power is unconstrained; no shaping region.
Implementations of LDLC Decoders

How to represent messages, which are functions?

0. **Exact** True message is a mixture of Gaussians, but number is exponential in iterations.

1. **Quantization** of the messages [Sommer et al.]
   - About 1024 quantization points gets good performance

2. **Mixture of Gaussians** — Approximate the message with a mixture of Gaussians [joint work with Justin Dauwels]
   - Uses a "Gaussian Mixture Reduction Algorithm"
   - Preserves Gaussian nature of messages, low memory requirements.
   - Further improvements: Yona and Feder, ISIT 2009

3. **Single Gaussian (this talk)** Approximate the message with a single Gaussian
**Single Gaussian Decoder:**

**Check Node (Easy Part)**

Message is two scalars:

\[ x_1 \sim \mathcal{N}(m_1, v_1) \quad x_2 \sim \mathcal{N}(m_2, v_2) \]

Output: Single Gaussian (wait for shift-and-repeat)

\[ h_1 x_1 + h_2 x_2 + h_3 x_3 = b \]

Unknown integer

Check node function is quite simple:

\[
\begin{align*}
    m_3 &= -\frac{h_1 m_1 + h_2 m_2}{h_3} \\
    v_3 &= \frac{h_1^2 v_1 + h_2^2 v_2}{h_3^2}
\end{align*}
\]
Let $X$ be the mixture of two Gaussians $X_1$ and $X_2$:

$$X = X_1 + X_2 = c_1 \mathcal{N}(m_1, v_1) + c_2 \mathcal{N}(m_2, v_2).$$

Let $Y$ be the single Gaussian (mean $m_Y$, variance $v_Y$) which approximates $X$. Then, $\text{KL}(X||Y)$ is minimized when:

$$E[Y] = E[X] \quad \text{(first moment)}$$
$$E[Y^2] = E[X^2] \quad \text{(second moment)}$$

Then,

$$m_Y = E[Y] = c_1 m_1 + c_2 m_2$$
$$E[Y^2] = \sum_i c_i \cdot (v_i + m_i^2)$$

Of course:

$$v_Y = E[Y^2] - E[Y]^2$$
Single Gaussian Decoder: Variable Node (Hard Part)

Forward-backward algorithm at variable node

Output: Single Gaussian

Inputs from check node: Single Gaussians (implied periodicity)
Variable Node One-Step Function

Belief propagation: Multiply

Moment Matching Approximation

Single Gaussian Input (from recursion)

Input from check node (must be shift-and-repeated)

Next recursion step or output
Complexity Measure: Can you write the decoding rule on one slide?

<table>
<thead>
<tr>
<th>Variable Node</th>
<th>Check Node</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantized message</td>
<td>128-point DFT</td>
</tr>
</tbody>
</table>

**Multiple Gaussians**
- Gaussian Mixture Reduction:
  - Compute pair-wise distance between $M^2$ Gaussians
  - Repeatedly combine Gaussians

**Single Gaussian**

\[
m = v' \sum_{b \in B} \left( \frac{b}{v_c h} + \frac{m_c}{v_c} + \frac{m_a}{v_a} \right) \exp \left( -\frac{1}{2} \frac{(b/h + m_c - m_a)^2}{v_c + v_a} \right)
\]

\[
v = v' - m^2 + v^2 \sum_{b \in B} \left( \frac{b}{v_c h} + \frac{m_c}{v_c} + \frac{m_a}{v_a} \right)^2 \exp \left( -\frac{1}{2} \frac{(b/h + m_c - m_a)^2}{v_c + v_a} \right)
\]

- Single-Gaussian decoder is considerably simpler than previous methods
- Single-Gaussian decoder is more complicated than binary LDPC, but at least the decoding rule fits on the page.

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Single-Gaussian decoding noise threshold within 0.1 dB of the quantized-message implementation.

Single-Gaussian decoder has a slight performance loss but is significantly simpler.

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**Noise Thresholds — Monte Carlo Density Evolution**

- Single-Gaussian decoding noise threshold within 0.1 dB of the quantized-message implementation.
- Single-Gaussian decoder has a slight performance loss but is significantly simpler.
Noise Thresholds — Regular Code Design

LDLC sparse matrix non-zero entries:

\[ h_1 \geq h_2 \geq \cdots \geq h_d \]

Characterized by parameter \( \alpha \):

\[ \alpha = \frac{h_2^2 + h_3^2 + \cdots + h_d^2}{h_1^2} \geq 0 \]

Theorem [Sommer, et al.]: Convergence is exponentially fast for \( \alpha \leq 1 \).

It was noted,

\[ \alpha \rightarrow 1 \]

gave good performance.

- Even alpha =0.7 is slightly better that other higher values.
- Increasing d beyond 7 gives little benefit (same as Sommer et al.)
Conclusion

- **Proposed** a single-Gaussian decoder for LDLC lattices

- **Complexity** is much lower than previous methods — we could write the decoding rule on one slide

- **Performance:** Minimal loss, noise threshold 0.1 dB away from more complicated decoder.

- **Code Design** Regular codes found that alpha->1 wasn’t always best, slight gain by picking alpha = 0.7
What about finite or smaller dimension?

"Low" dimension lattices (dimension 100-1000)
  • Single Gaussian numerical results poor.
  • Multiple Gaussian decoder works OK at low dimension.

High dimension (10000 to infinity)  
  Single Gaussian OK

Low Dimension (100-1000)  
  More Gaussians  
  (Performance loss with single Gaussian)

• Suggestion: modify BP decoder to improve convergence at low dimensions