Single-Bit Quantization of Binary-Input, Continuous Output Channels

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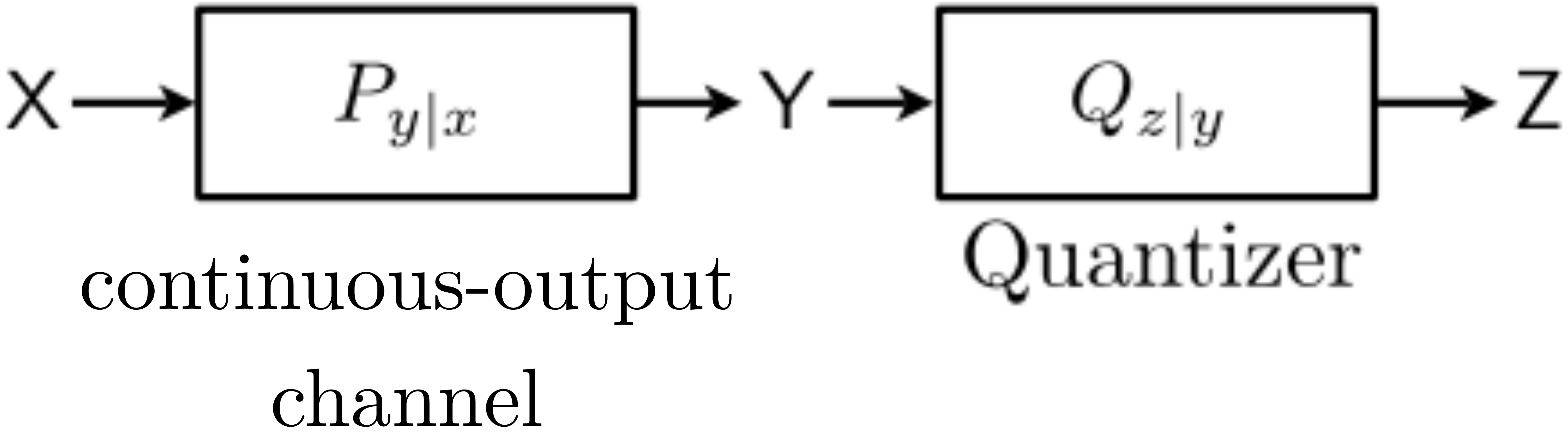
Quantized Channel Capacity



Given an arbitrary continuous-output channel $\Pr(\mathsf{Y}|\mathsf{X})$, find *quantized channel capacity*

$$C = \max_{p_{\mathsf{X}}(x)} \max_{Q} I(\mathsf{X};\mathsf{Z})$$

This is a very difficult problem, in general.

Sample of Previous Work

Massey (1974): Algorithm to quantize channels to maximize the cutoff rate

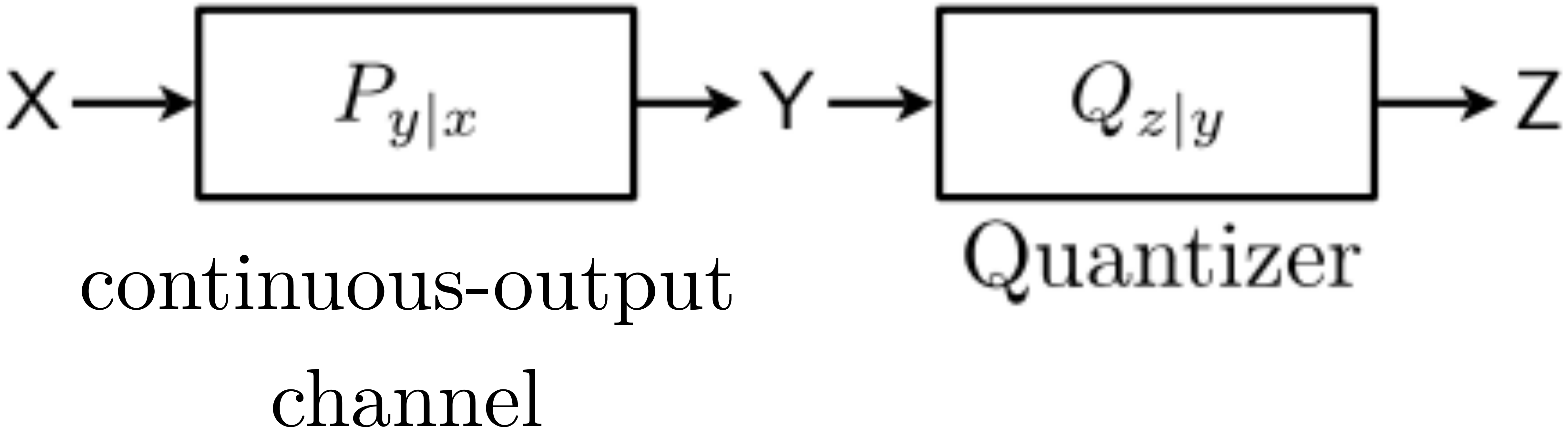
Ma et al (2001) considered channel quantization with a mutual information metric

Singh et al (2009) quantized channel capacity with noise symmetry.

Koch and Lapidoth (2013): low SNR asymmetric quantizers eliminate quantization penalty

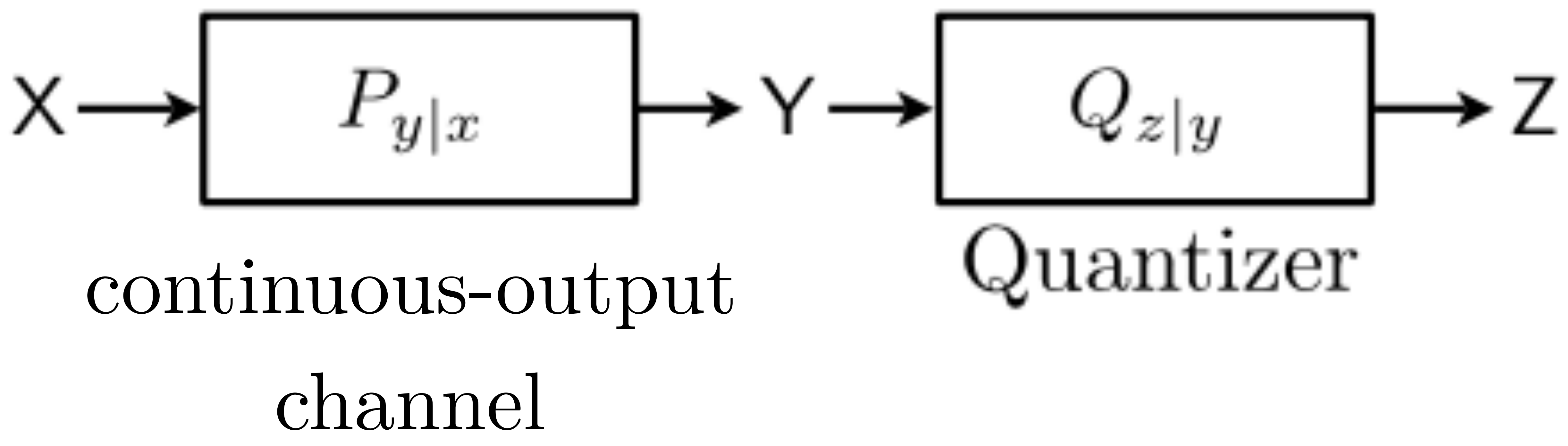
Alirezaei and Mathar (2015): One-bit quantized capacity achieved with two support points.

Problem Setup



- Binary input X channel, fixed input distribution $\Pr(X)$
- Real-valued channel output Y
- Arbitrary and data-dependent noise $\Pr(Y|X{=}0)$, $\Pr(Y|X{=}1)$ (not necessarily additive)
- Quantize the channel output to **one bit** $Z \in \{0,1\}$
- Maximize mutual information $\mathrm{I}(X;Z)$

Contribution



By concentrating on the channel quantization aspect, we give two contributions:

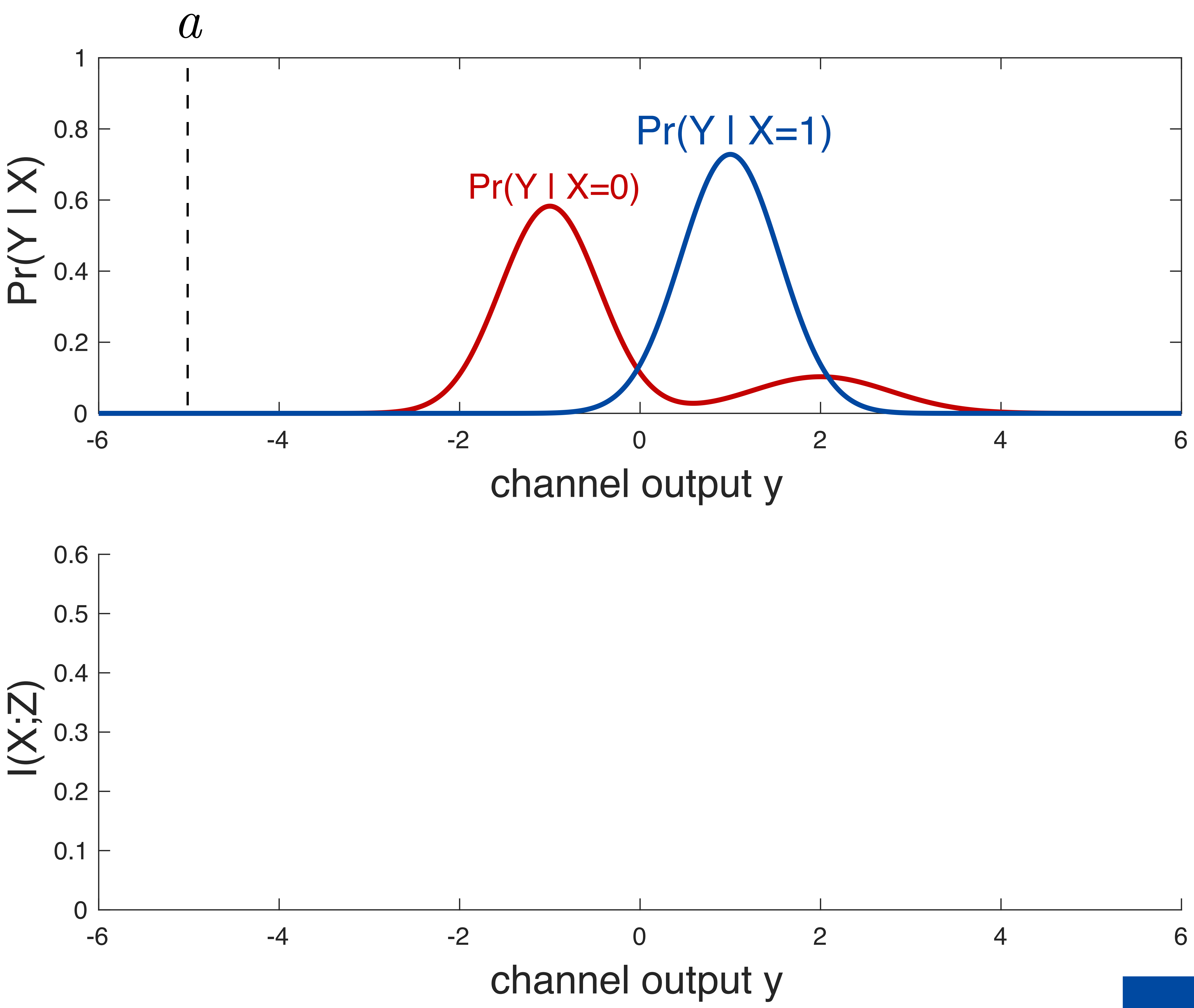
- Optimal quantizer: the preimage of the *backward* channel quantizer are convex
- If the channel satisfies a “sorting condition” then the preimage of the optimal *forward* channel quantizer is convex.
- Even this simple problem is difficult (nonconvex in general).

Quantization By Threshold Search

Quantizer:

$$z = \begin{cases} 0 & y < a \\ 1 & y \geq a \end{cases}$$

Example: data-dependent Gaussian noise mixtures



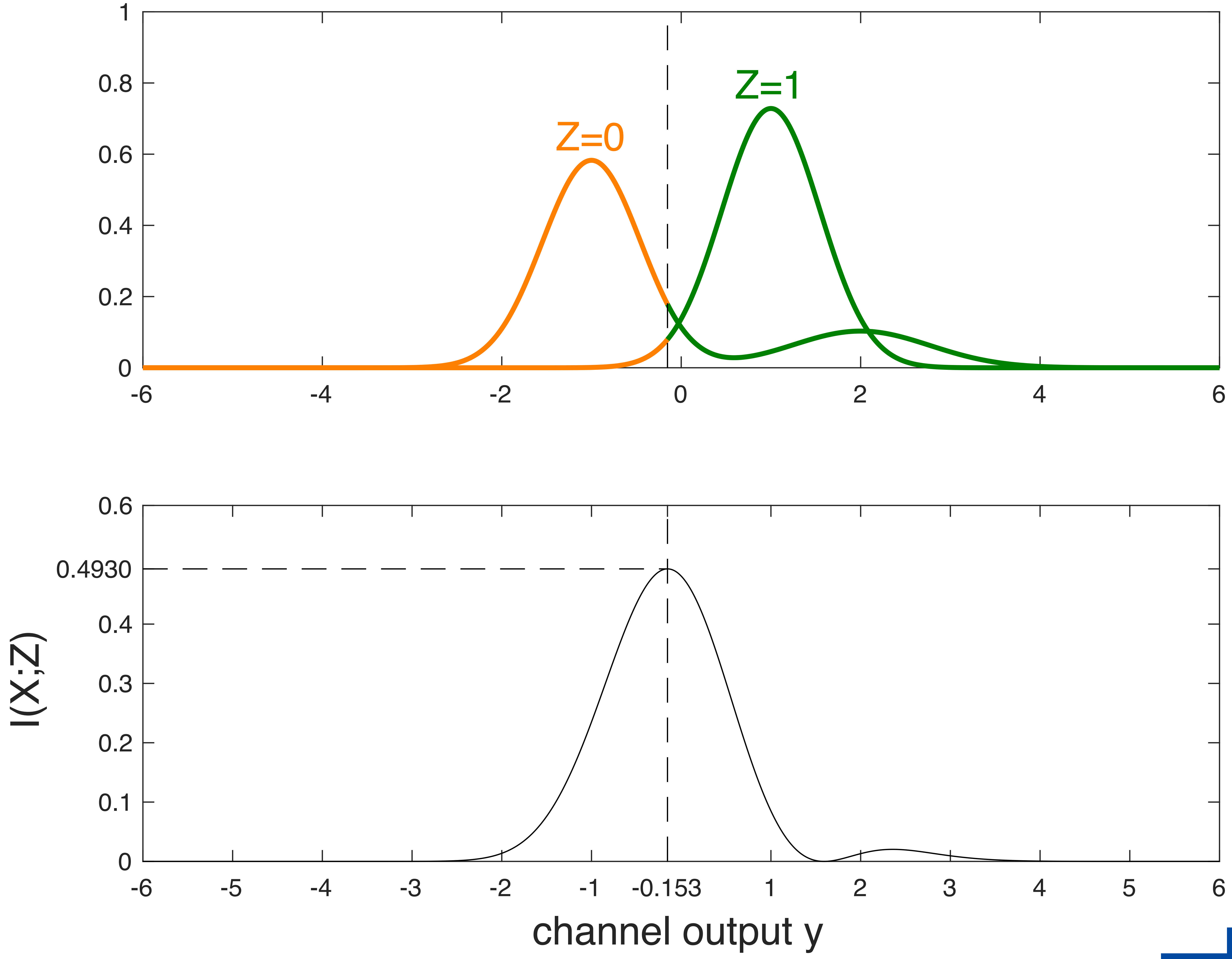
Quantization By Threshold Search

Quantizer:

$$z = \begin{cases} 0 & y < a \\ 1 & y \geq a \end{cases}$$

Example: data-dependent Gaussian noise mixtures

max I(X;Z) = 0.493 at
a* = -0.153



Two Thresholds Has Greater I(X;Z)

one threshold

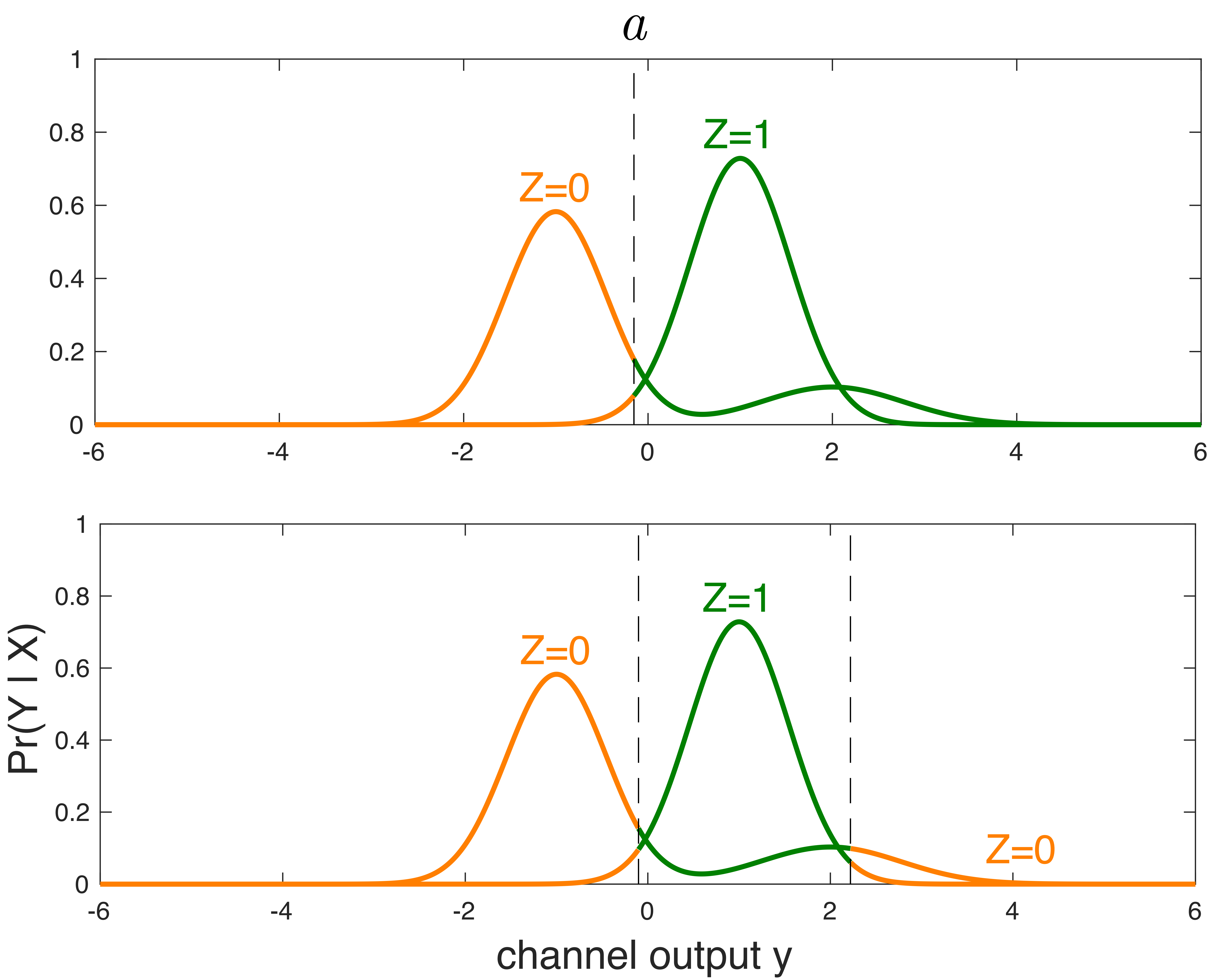
max I(X;Z) = 0.493

at a* = -0.153

two thresholds

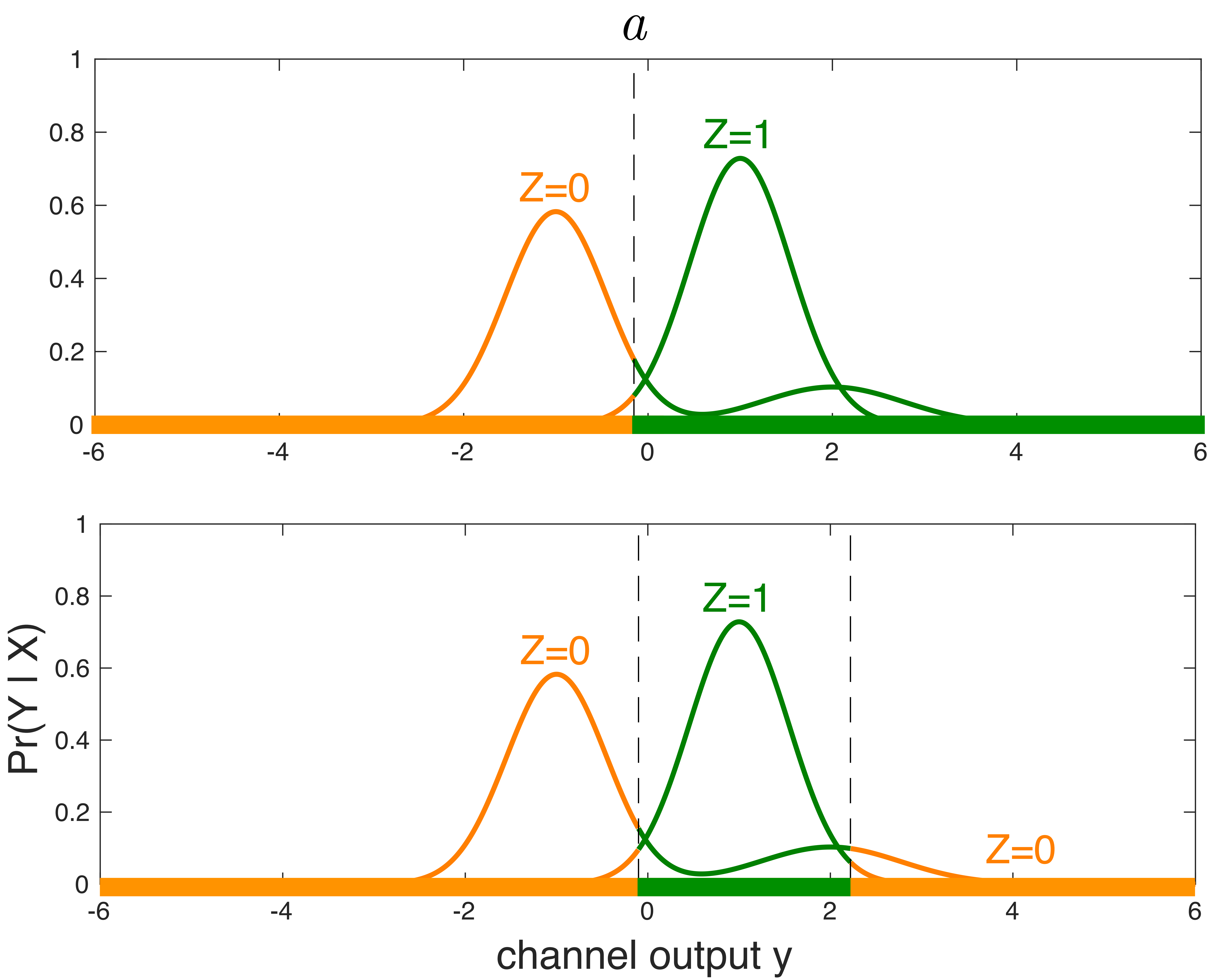
max I(X;Z) = 0.559

a* = -0.102,2.220



Forward Quantizer Convexity

Convex (1-D)



z preimage is convex

Not Convex

z preimage is not convex

Preimage of Optimal Classifier (Quantizer) is Convex

The Annals of Statistics
1992, Vol. 20, No. 3, 1637–1646

MINIMUM IMPURITY PARTITIONS

BY DAVID BURSHTEIN, VINCENT DELLA PIETRA, DIMITRI KANEVSKY AND ARTHUR NÁDAS

Tel Aviv University and IBM, T. J. Watson Research Center

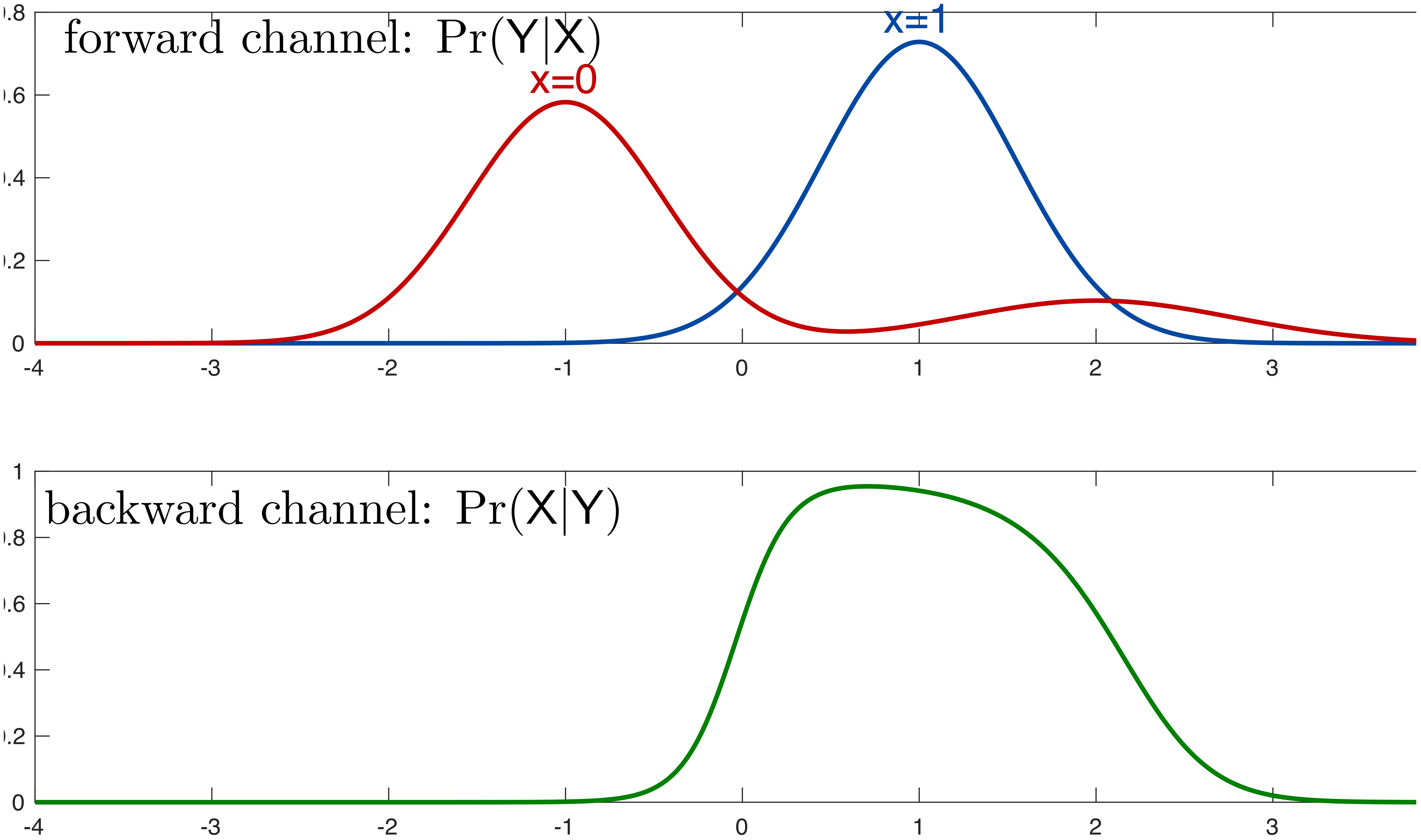
$$I(X;Z) = H(X) - H(X|Z)$$

concave in $\Pr(X|Y)$

Let $\mathcal{U}$ be the probability simplex for $p_{\mathsf{X}|\mathsf{Y}}(x|y)$. A backward channel quantizer $\widetilde{Q}$ is $\widetilde{Q} : \mathcal{U} \to \mathcal{Z}$.

Theorem There exists an optimal backward channel quantizer $\widetilde{Q}^* : \mathcal{U} \to \mathcal{Z}$ for which any two distinct preimages $Q^{*-1}(z)$ and $Q^{*-1}(z')$ are separated by a hyperplane in $\mathcal{U}$.

Backward Channel $\Pr(X = x|Y = y)$

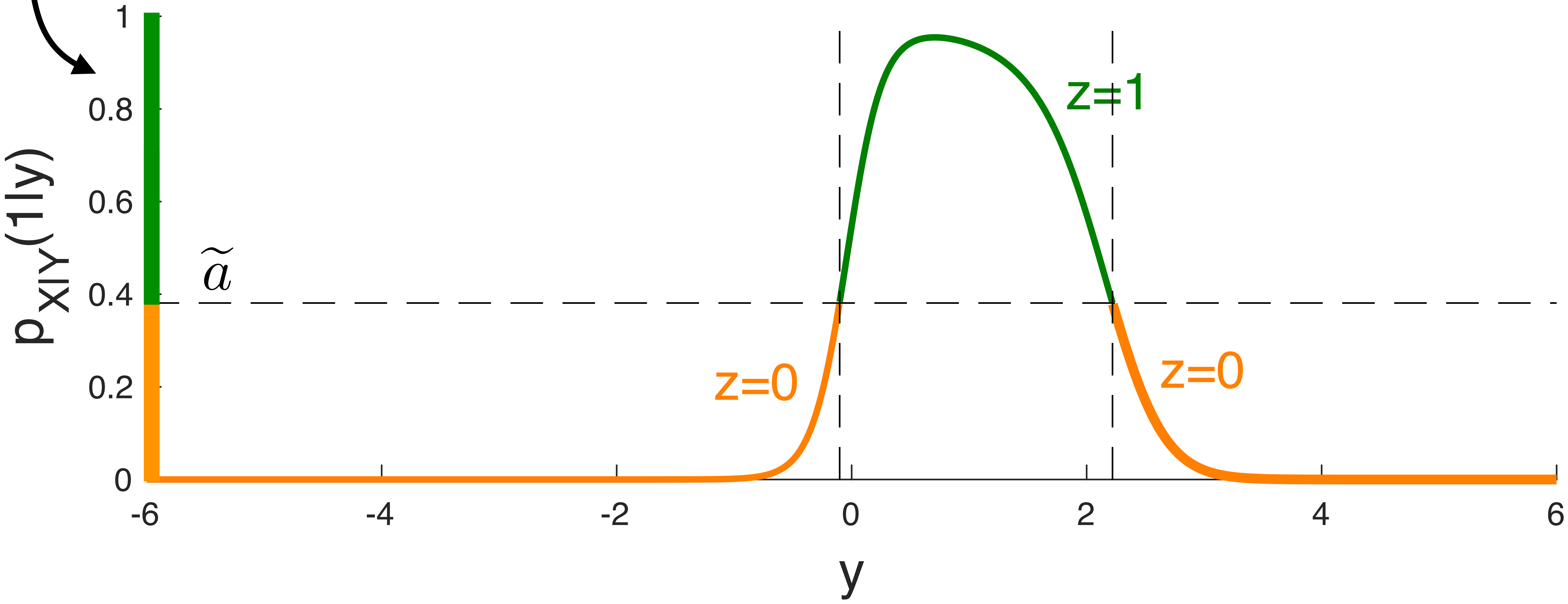


Convex Backward Channel Quantizer

Threshold $\widetilde{a}$. Backward channel quantizer $\widetilde{Q}$:

$$z = \begin{cases} 0 & \Pr(X = 1|y) < \widetilde{a} \\ 1 & \Pr(X = 1|y) \geq \widetilde{a} \end{cases}$$

z preimage
is convex



Optimal Quantizer in Backward Channel

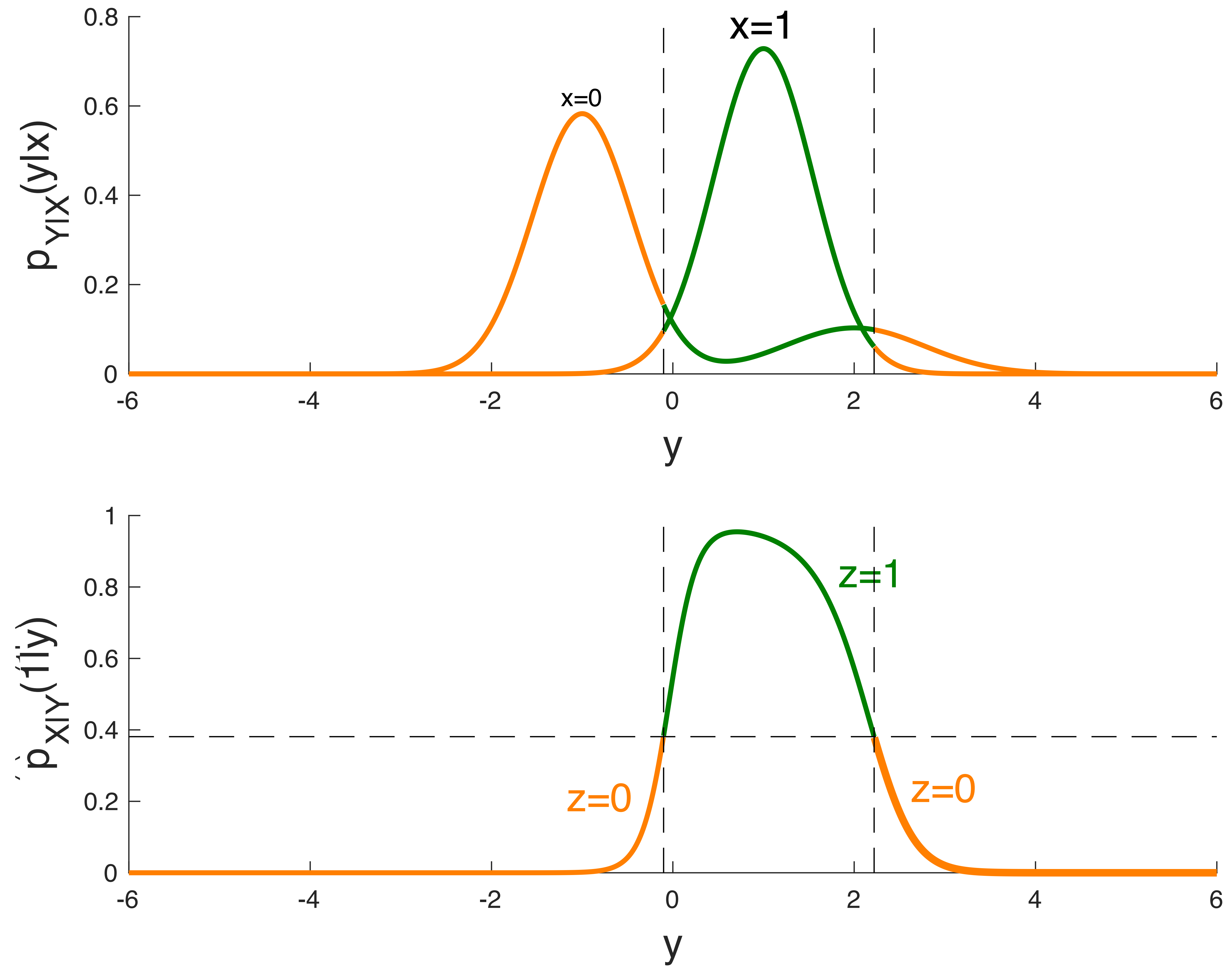
Lemma 1 There exists an optimal backward channel quantizer which is a convex quantizer.

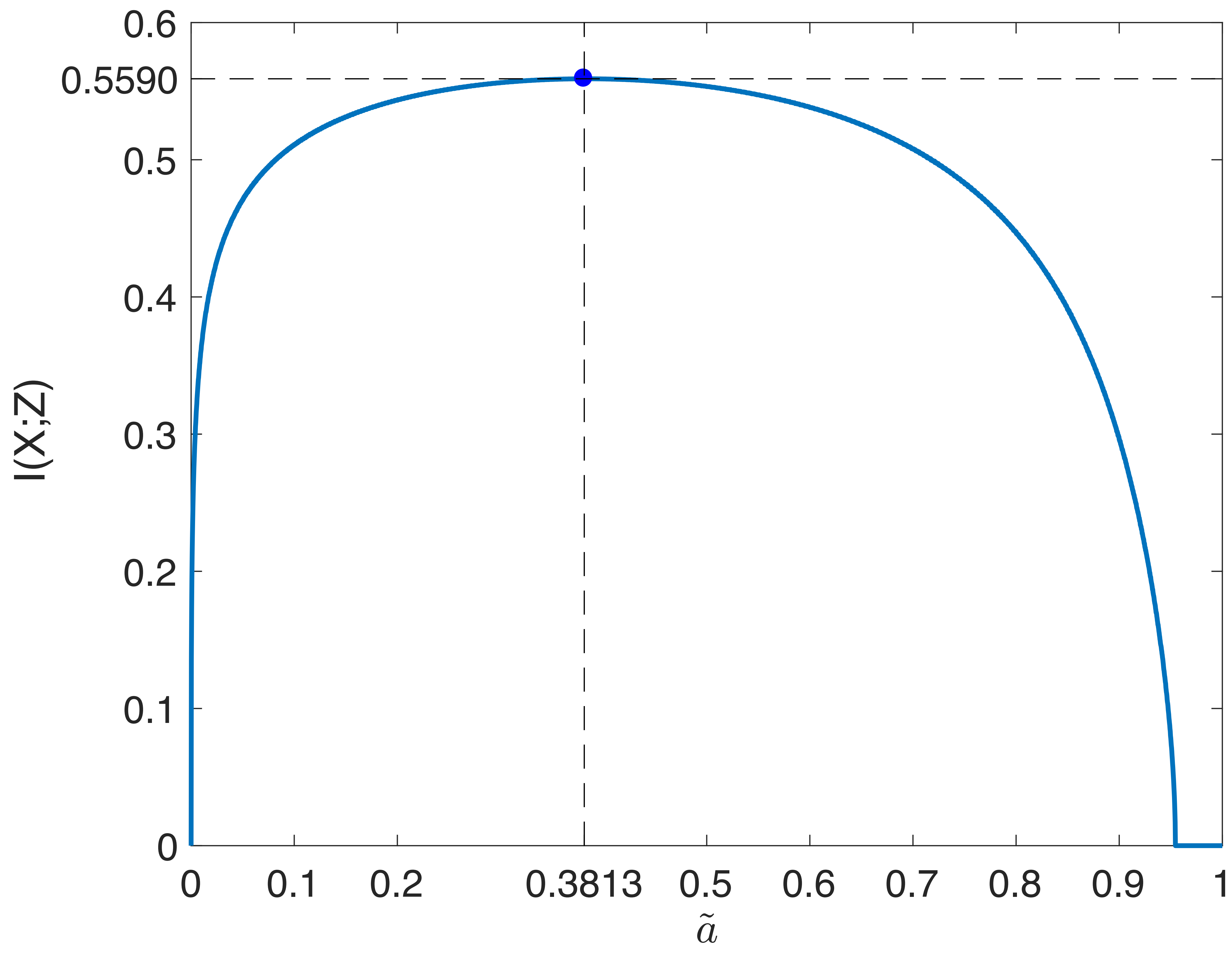
Proof is direct application of Burshtein et al.'s theorem.

Consequences:

- The corresponding forward channel quantizer is not necessarily convex
- It is easier to search over convex quantizers, so better to consider the backward channel

Optimal Quantizer in Backward Channel





strictly concave when non-zero

Optimal Convex Forward Quantizer

Lemma 2 If the channel log-likelihood ratio satisfies:

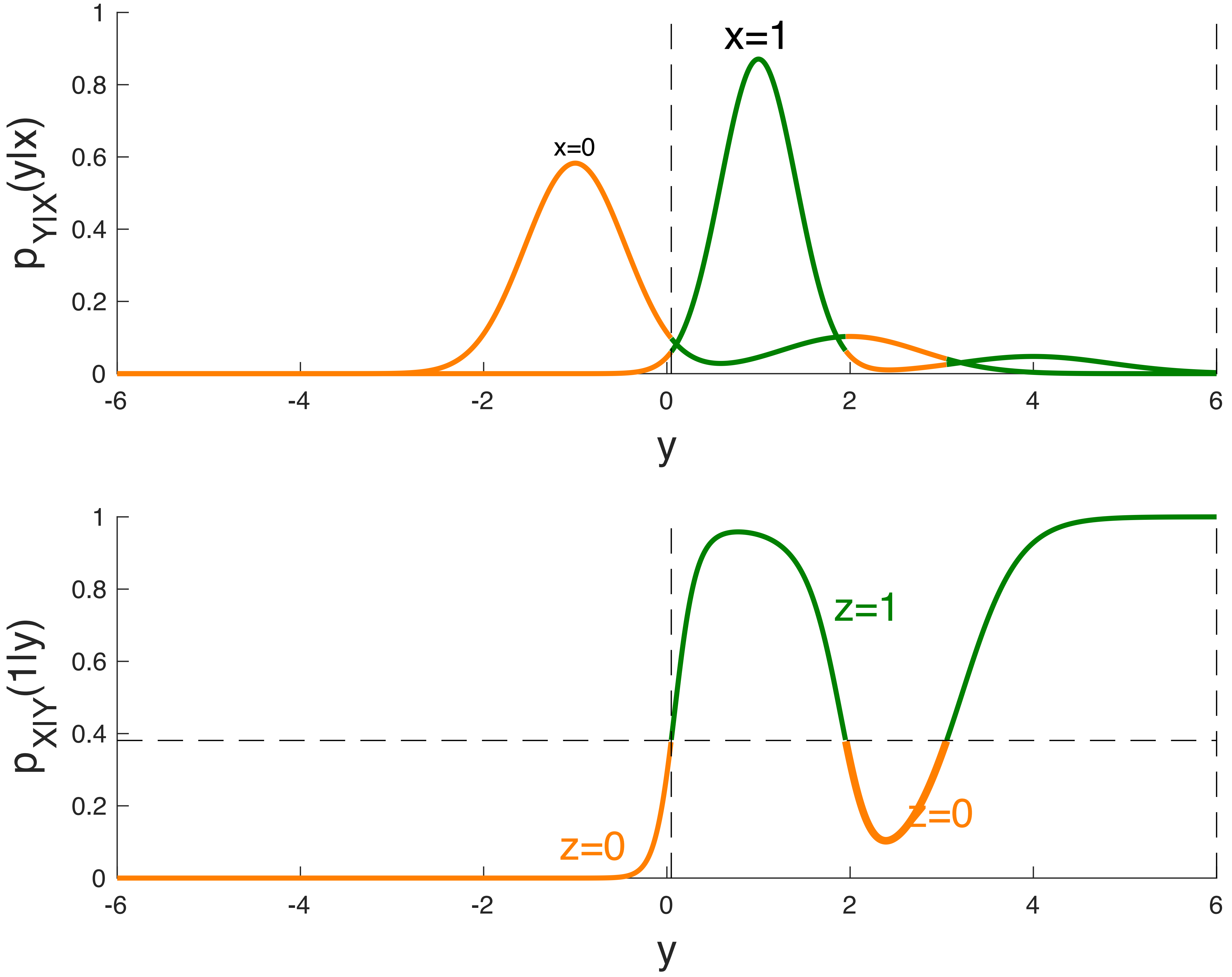
$$\log \frac{\Pr(y|X=1)}{\Pr(y|X=0)} \leq \log \frac{\Pr(y'|X=1)}{\Pr(y'|X=0)}$$

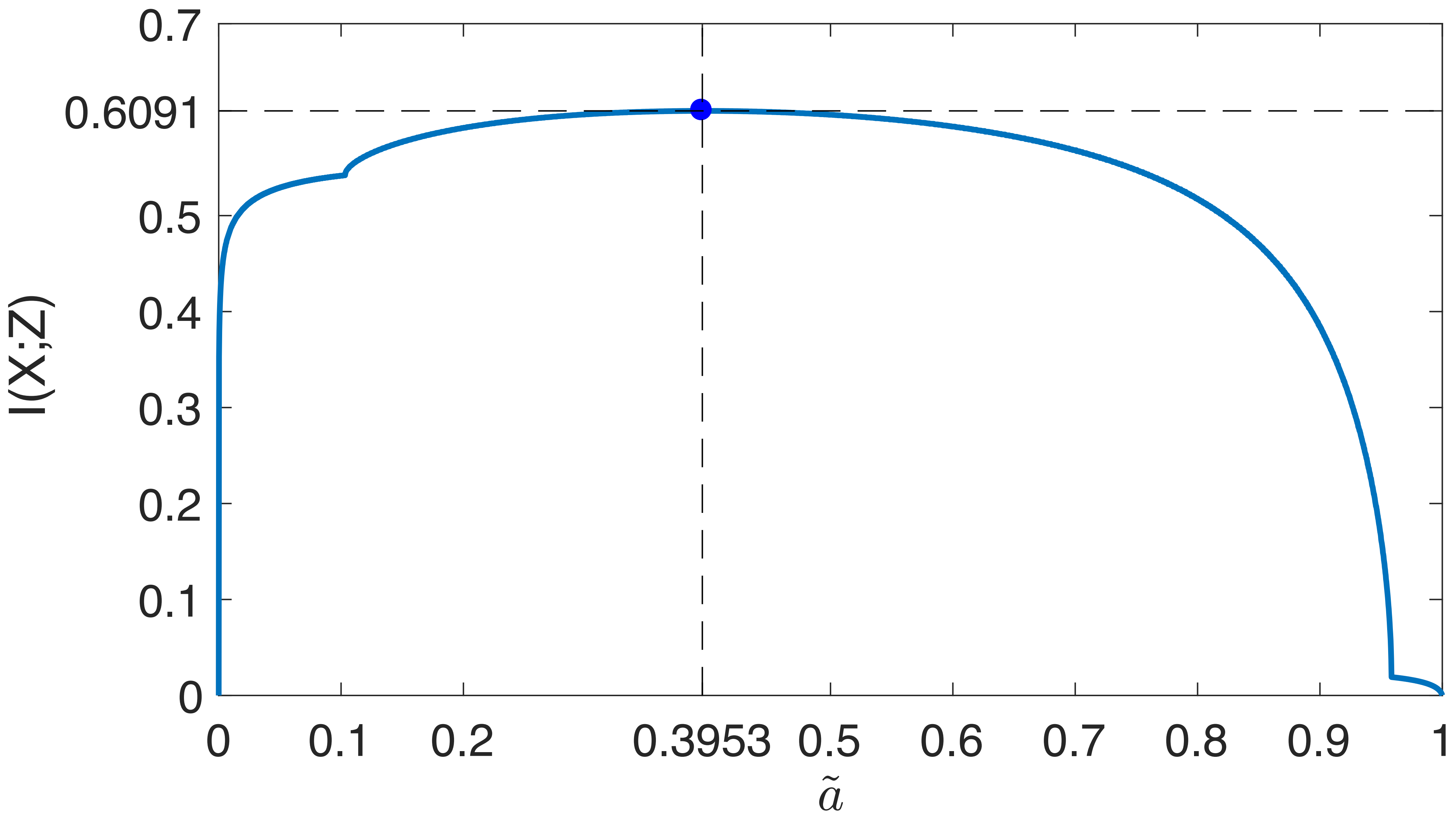
for all $y < y'$, then there exists an optimal forward channel quantizer Q^* which is a convex quantizer.

Consequences:

- For many well-behaved channels, the optimal forward channel quantizer is convex.
- The BI-AWGN channel satisfies this condition

Even Backwards Channel is Hard





Mutual information is not convex in $\widetilde{a}$

Discussion

Using the theorem of Burshtein et al., it is far easier to deal with quantization in the backward channel.

Unfortunately, channel quantization is not convex optimization problem in general.