Lattice-Based WOM Codebooks that Allow Two Writes

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Flash Memories

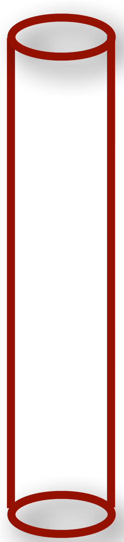
Flash memories store charge on transistors called “cells”

Increasing the number of bits/cell increases data storage density:

SLC

1 bit/cell

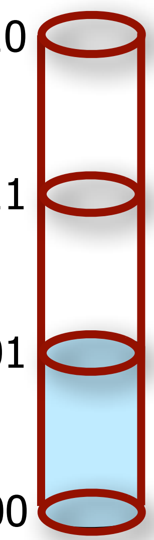
$q = 2$



MLC

2 bit/cell

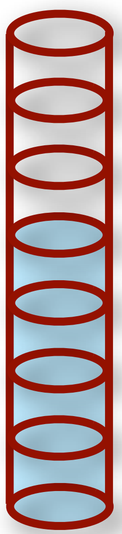
$q = 4$



TLC

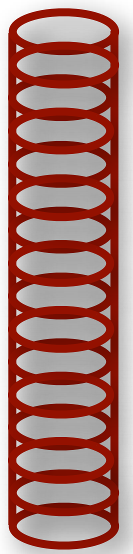
3 bit/cell

$q = 8$



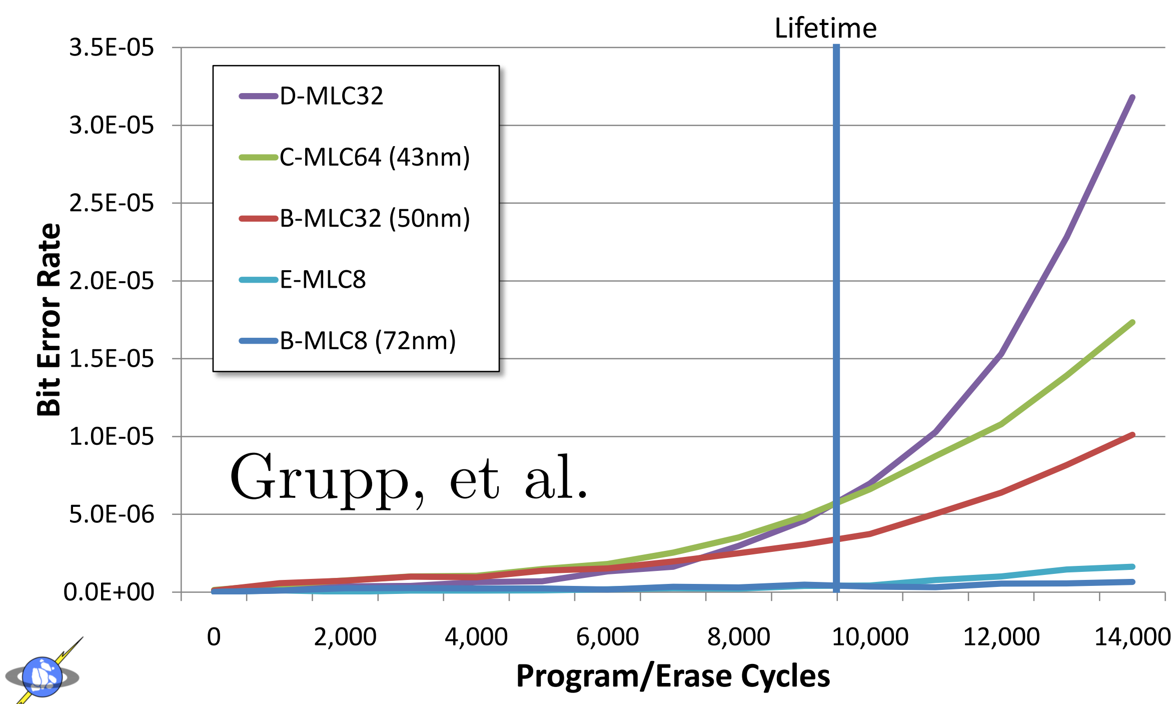
The future

many bits/cell?



Flash Memories Wear Out

- To re-write a memory, must first erase it
- Each time the flash memory is erased, the error rate increases



WOM Codes for Non-Binary Flash

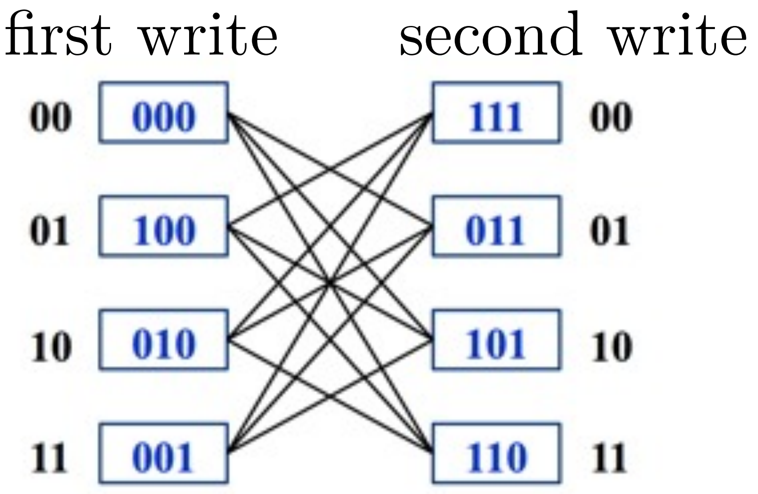


image thanks: Paul Siegel

Codes for "Write Once Memories"

- Pioneered by Rivest and Shamir [1982]
- Memory can change from 0→1 state but not 1→0
  - non-binary case: $0 \to 1 \to 2 \to \ldots \to q-1$
- Remarkable! Possible to re-use a "write once" memory!
- Application: Increase flash write endurance

WOM Codes Rates

- WOM codes allowing $t = 2$ writes, but q-levels
- n is the number of cells
- Code rate for write i is R_i, normalized rate $\widetilde{R}_i$
- Questions. How to:
  - maximize $R_1 + R_2$?
  - maximize $R_1 + R_2$ subject to $R_1 = R_2$?

Conventional Rate Definition:

$$R = \frac{\log_2 \#\text{ of messages}}{n} \text{ bits/cell}$$

Normalized rate:

$$\widetilde{R} = \frac{1}{\log_2 q} \frac{\log_2 \#\text{ of messages}}{n}$$

Capacity of Non-Binary WOM Codes

Fu and Han Vink [1999] gave capacity of a t-write code into q-ary cells.

For t=2:

$$R_1 + R_2 \quad \leq \quad \log_2 \binom{q+1}{q-1}$$

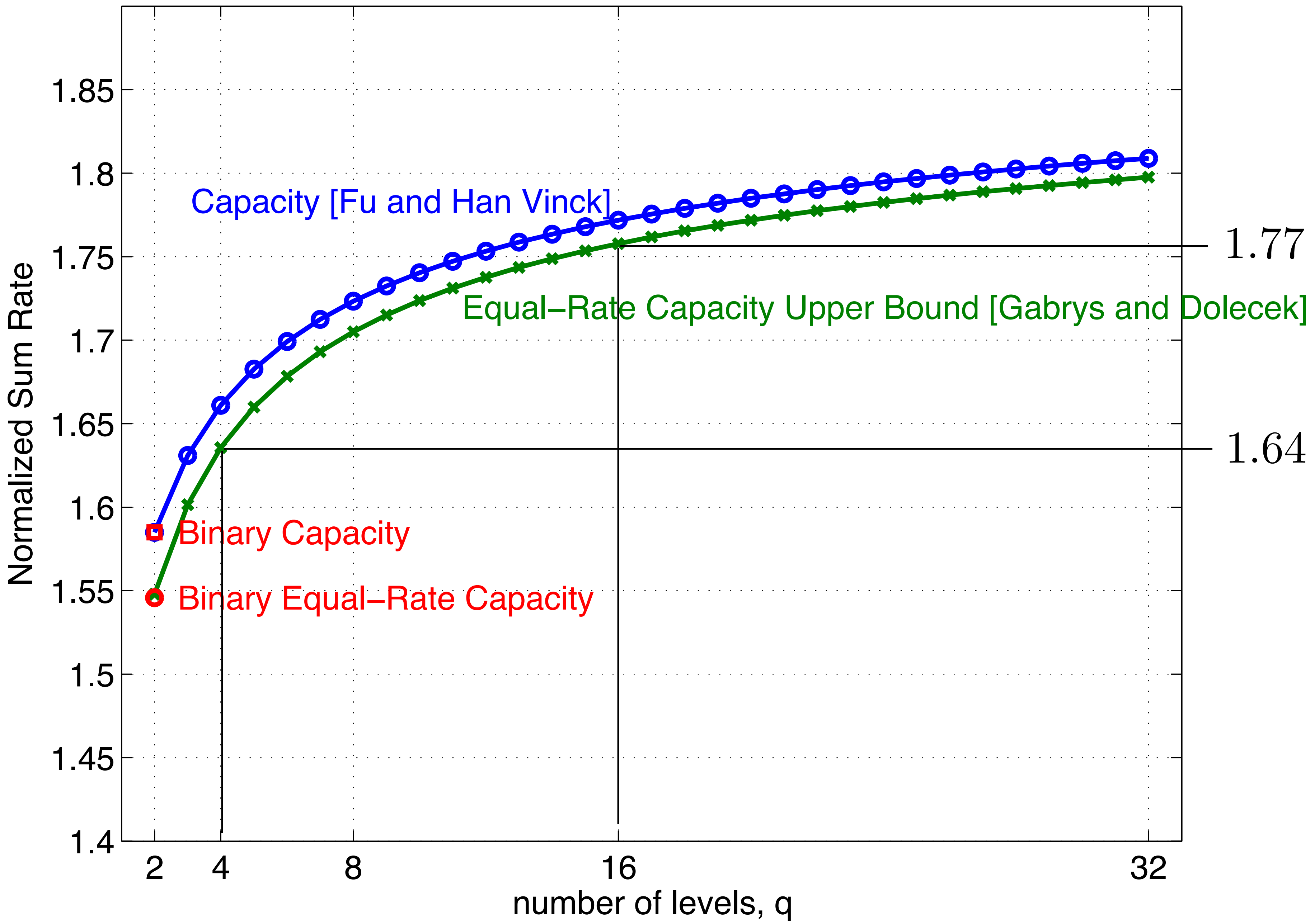
But, capacity-achieving rates are **not equal in general**, $R_1 \neq R_2$

Equal rates $R_1 = R_2$ is of practical concern.

Gabrys and Dolecek [2011] found an upper bound on equal-rate capacity:

$$2R_1 = 2R_2 \;\leq\; \frac{2}{3} \log \big(\frac{q\,(q+1)\,(2q+1)}{6}\big)$$

Capacity for t=2 Writes: Normalized Sum Rate vs. Levels q



$$1.77 = \frac{2 \cdot 2.65 \text{ bits/cell}}{3 \text{ bits/cell}}$$

$$1.64 = \frac{2 \cdot 1.64 \text{ bits/cell}}{2 \text{ bits/cell}}$$

The normalized sum rate is a measure of **efficiency**

- Efficiency increases as q increases

In This Talk: WOM Properties of Lattices

Lattices have an inherent error-correction property

What about the WOM properties of lattices?

Outline of this talk:

- Sphere packings and lattices
- Lattice Codes = intersection of a **shaping region** and a lattice
- WOM properties of lattices
  - Using **continuous approximation**,
    code rate is from the **volume of the shaping region**
  - The ideal shaping region is **hyperbolic**
  - Give an expression for normalized sum rate under equal rate assumption,

$$\widetilde{R} \geq 1 - \frac{1}{e \ln 2} \frac{1}{\log_2(q-1)}$$

close to Gabrys-Dolecek upper bound

A **Sphere Packing** is an arrangement of non-overlapping spheres in space

A Lattice Is A Linear Sphere Packing

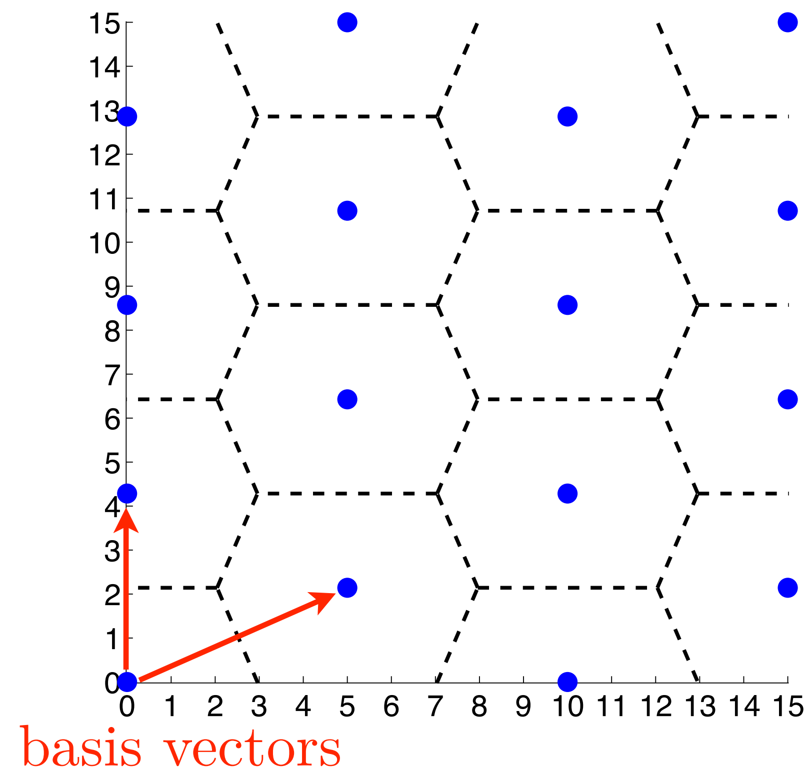
A lattice is a linear subgroup of $\mathbb{R}^n$

G: n-by-n generator matrix

$$\mathbf{x} = \mathrm{G} \cdot \mathbf{b}$$

$\mathbf{b} = (b_1, b_2, \ldots, b_n)^{\mathrm{t}}$: n-by-1 vector of integers

$\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{t}}$: n-by-1 vector, lattice point



Lattice (linear)

$$G = \begin{bmatrix} 5 & 0 \\ 2.15 & 4.3 \end{bmatrix}$$

Hexagonal Lattice

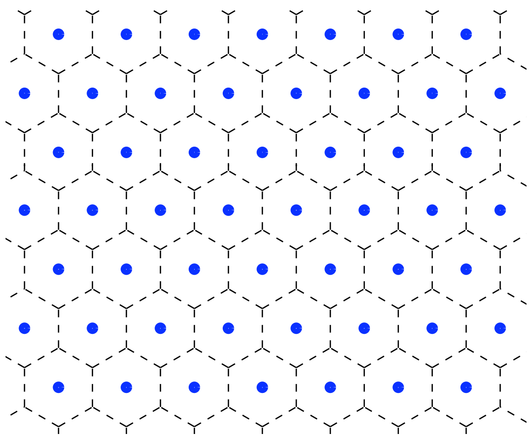
16 codewords, $d_{\min} = 4.29$

Lattices:

- Have a rich theory
- Can correct errors, achieve AWGN capacity

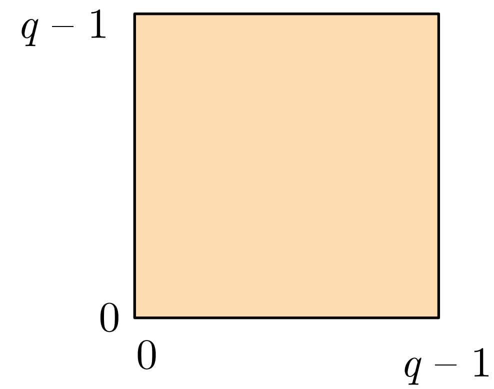
What about the WOM properties of lattices?

Lattice Code Construction



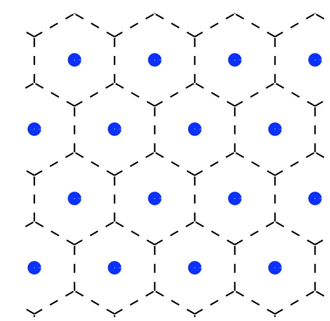
Lattice Λ is infinite
code over reals
"minimum distance"

$\cap$



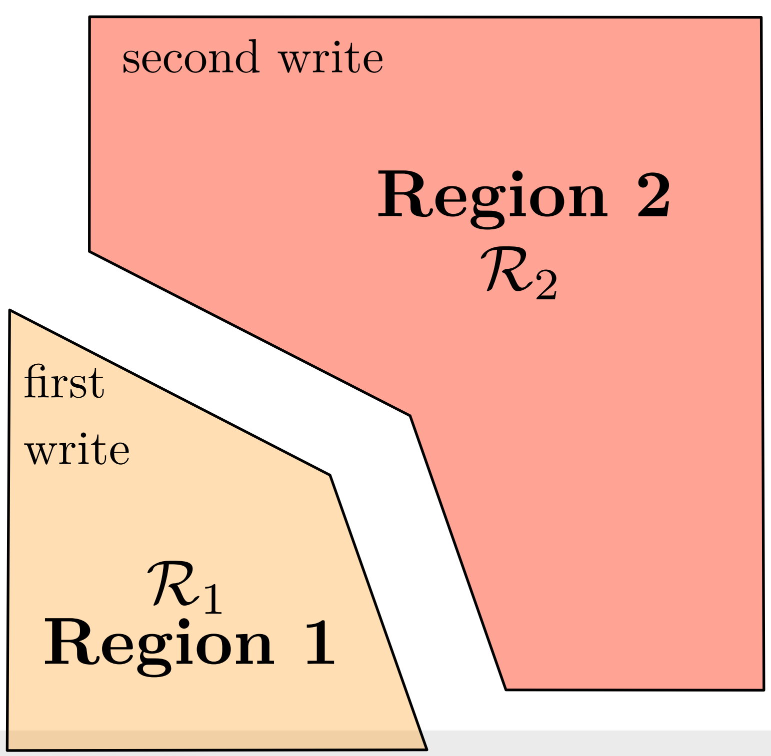
Shaping region $\mathcal{R}$
finite

$=$



codebook $\mathcal{C} = \Lambda \cap \mathcal{R}$
is finite

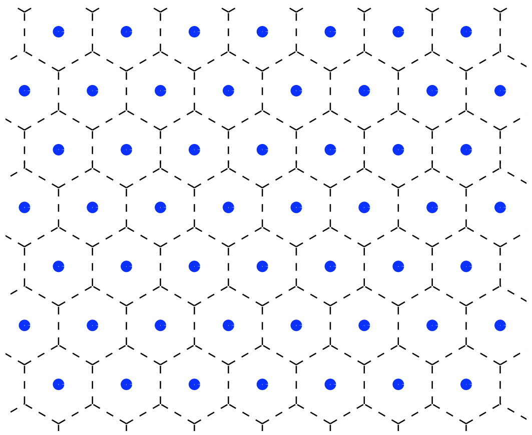
WOM Lattice Code Construction



Construct a code using two regions $\mathcal{R}_1$ and $\mathcal{R}_2$

- Codebook for region 1, $\mathcal{C}_1 = \mathcal{R}_1 \cap \mathcal{C}$
- Codebook for region 2, $\mathcal{C}_2 = \mathcal{R}_2 \cap \mathcal{C}$
- Separated by boundary B

Lattice Code Construction



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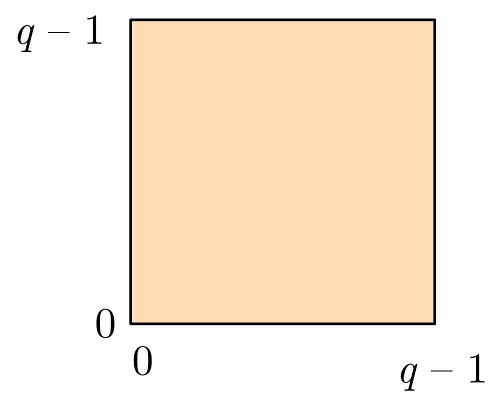
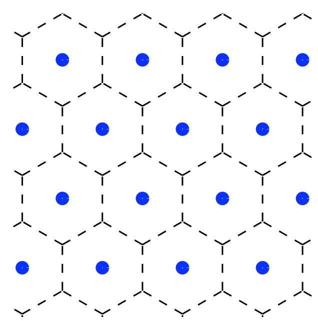


Shaping region $\mathcal{R}$
finite

$=$



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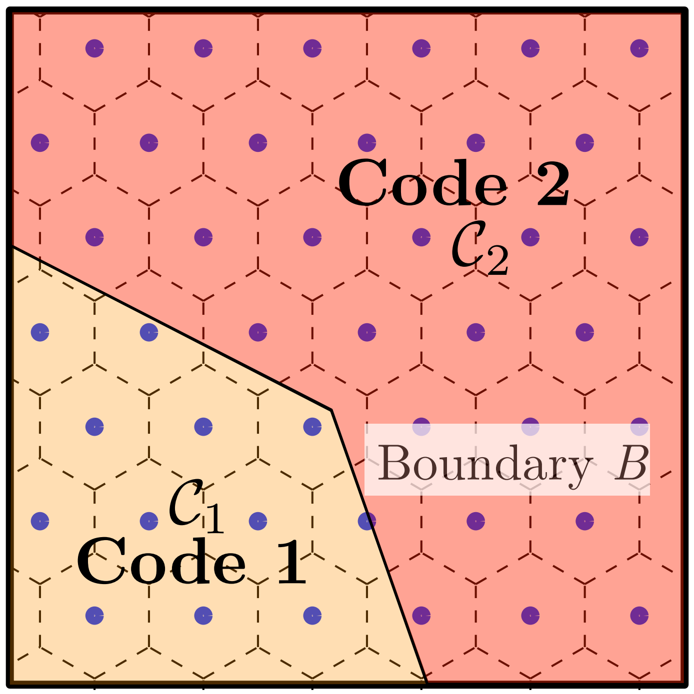
WOM Lattice Code Construction



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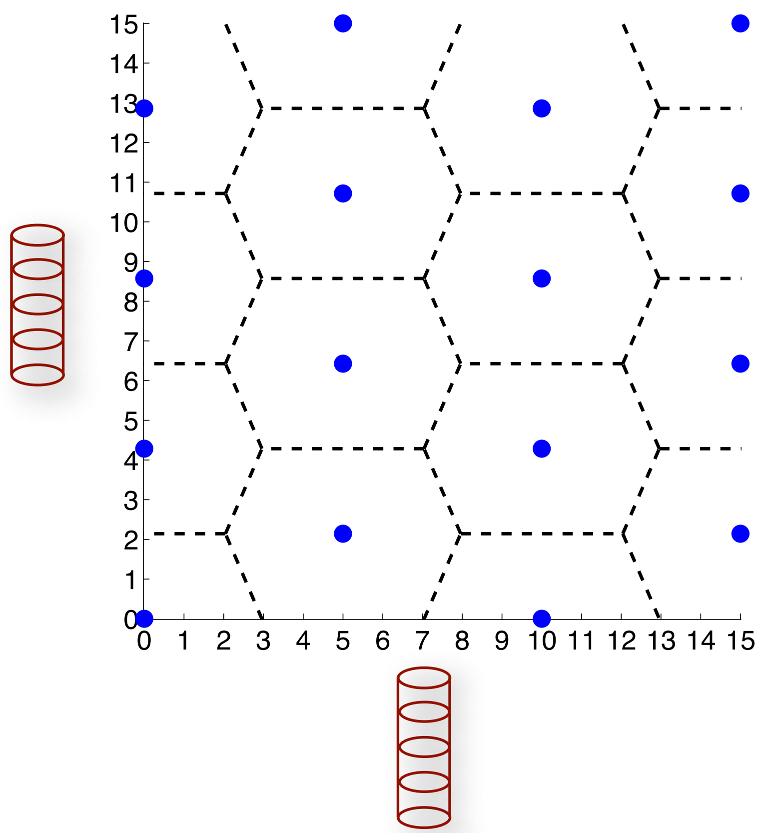
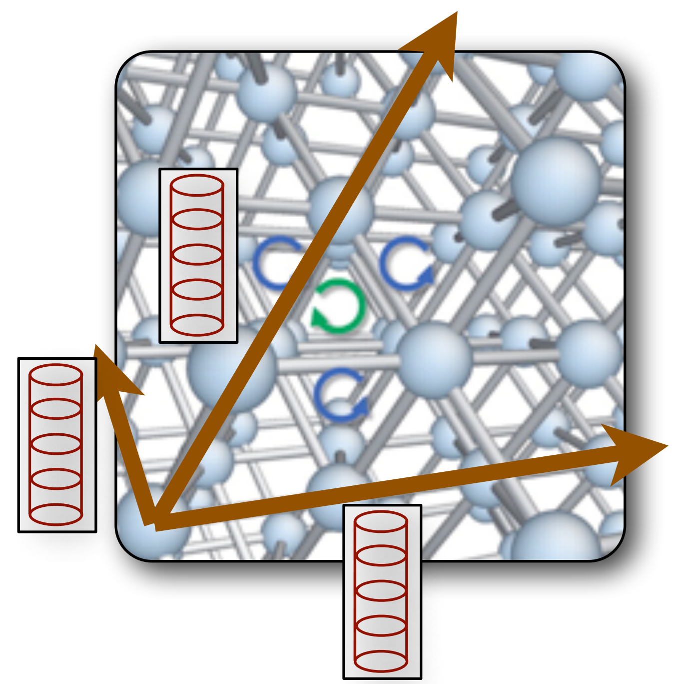
n-Dimensional Lattice in n Flash Cells

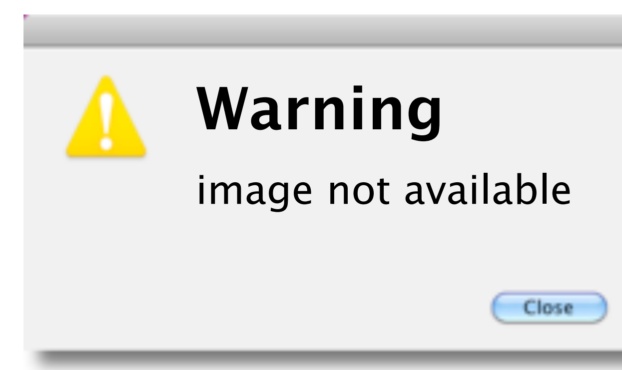
$n = 2$

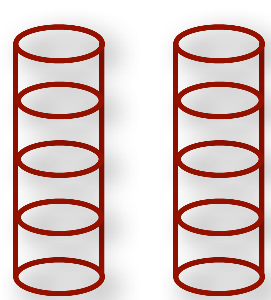


$n = 3$

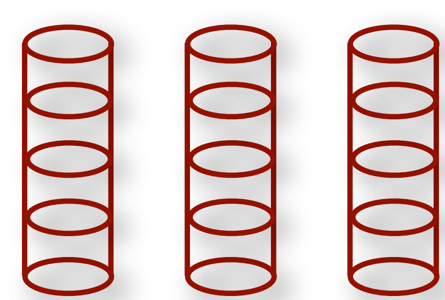


$n = 4,\ 5,\ 6,\ \ldots$

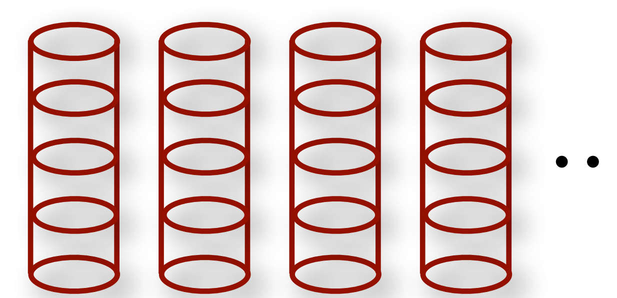




2 flash cells



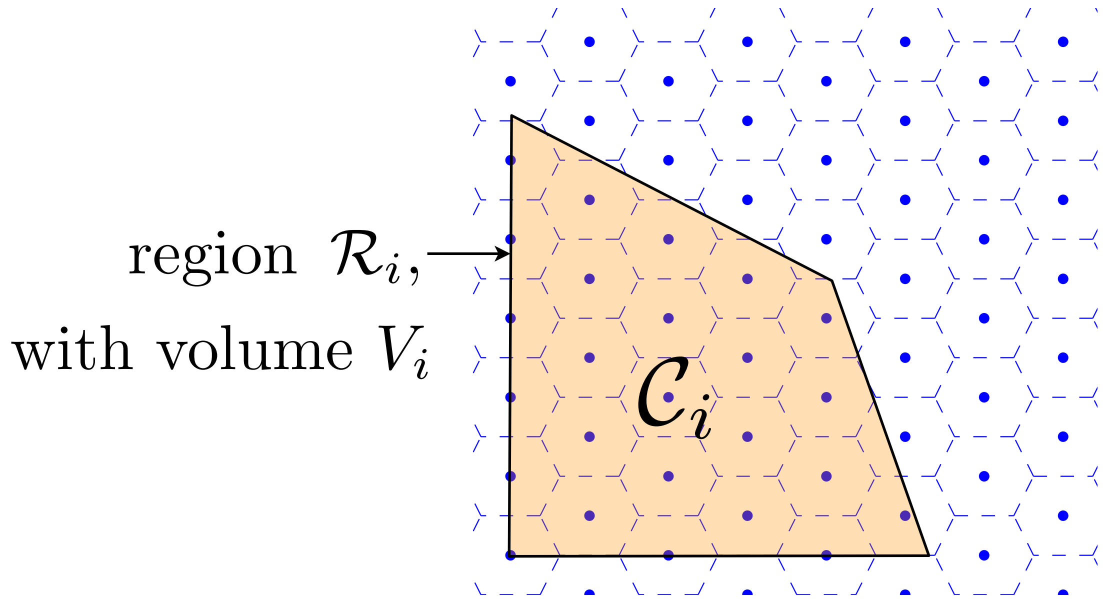
3 flash cells

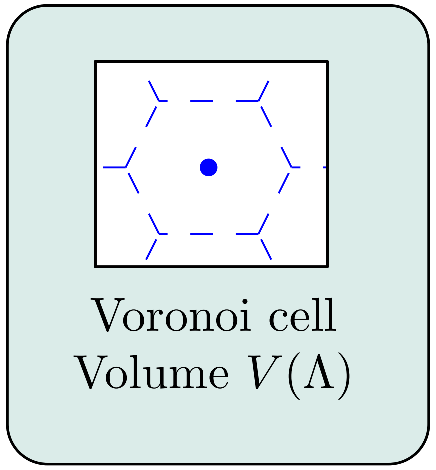


4, 5, 6, … flash cells

Code Rates using Continuous Approximation

$$\text{number of points in } \mathcal{R} \cdot \text{Volume of Voronoi region} \approx \text{Volume of } \mathcal{R}$$
$$|\mathcal{C}| \cdot V(\Lambda) \approx V$$



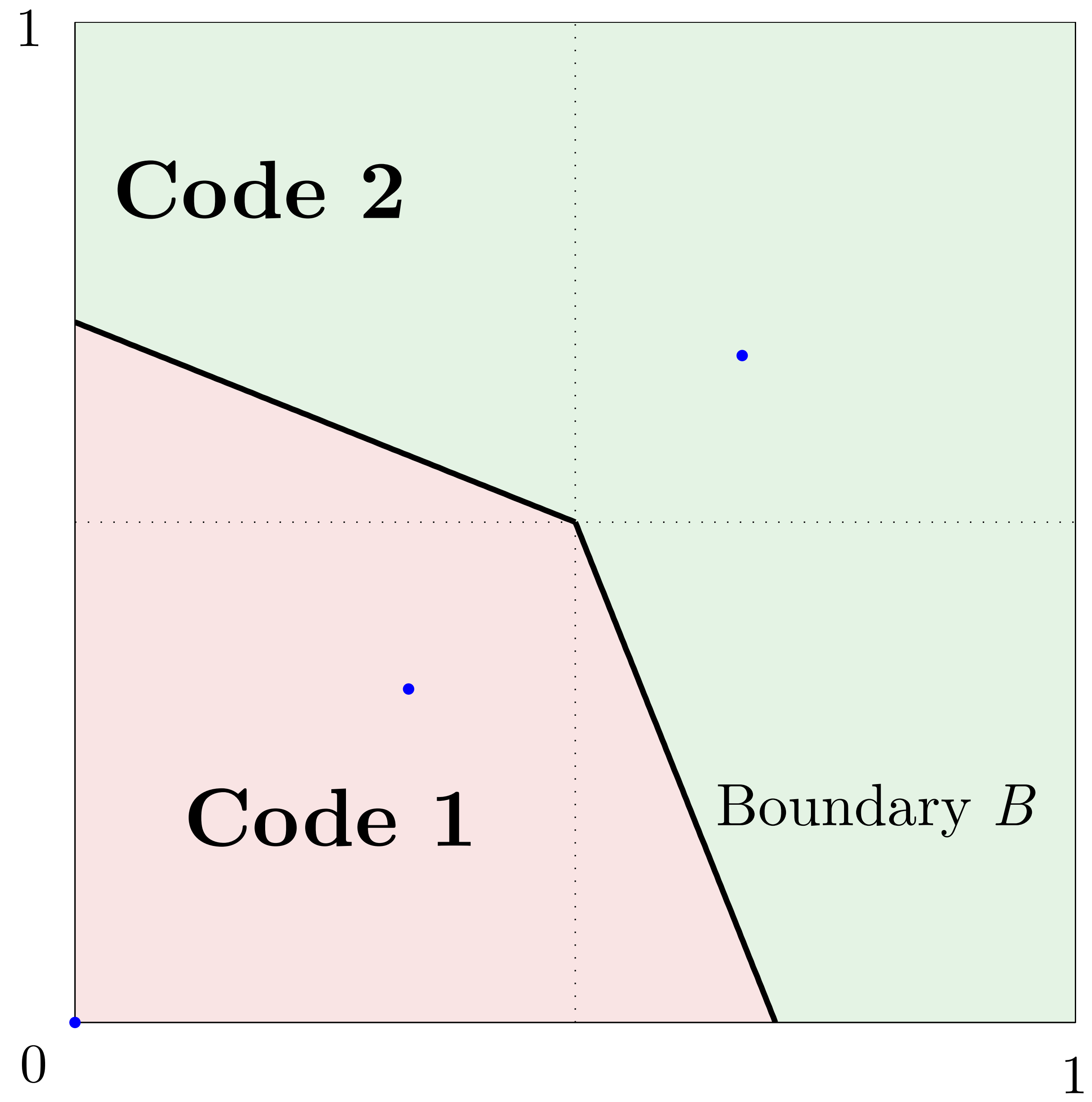


Normalized Rate

$$\widetilde{R}_i = \frac{\log_2 |\mathcal{C}_i|}{\log_2 |\mathcal{C}|}$$
$$\widetilde{R}_i \approx 1 - \frac{\log_2 V_i}{\log_2 V(\Lambda)}$$

Code rates R_i expressed as volume V_i

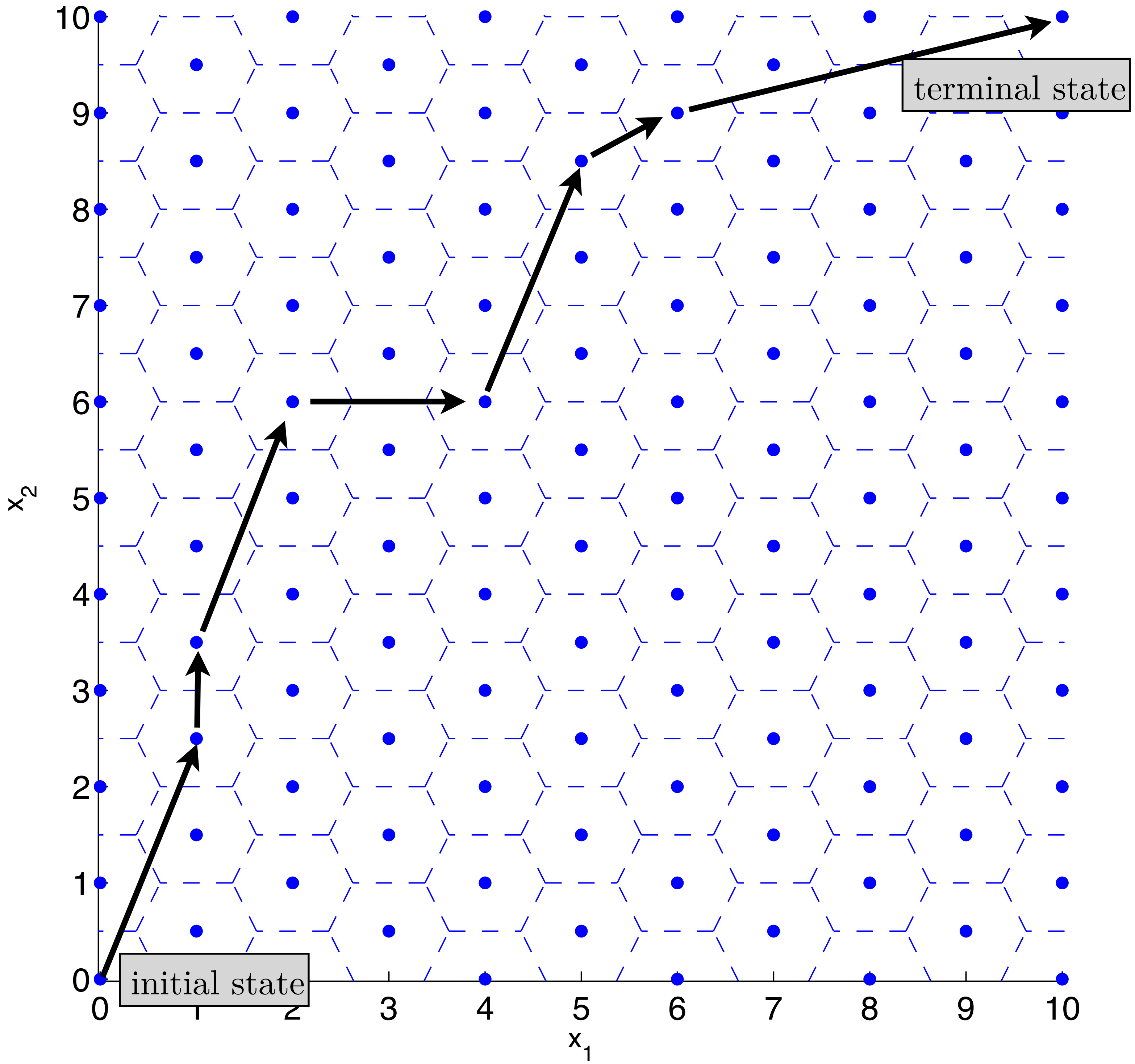
Approximation Improves as q→large



Continuous approximation was used by Forney for AWGN channels

- well-known shaping gain of 1.53 dB

Cell values increase
= rectangular “accessible region”

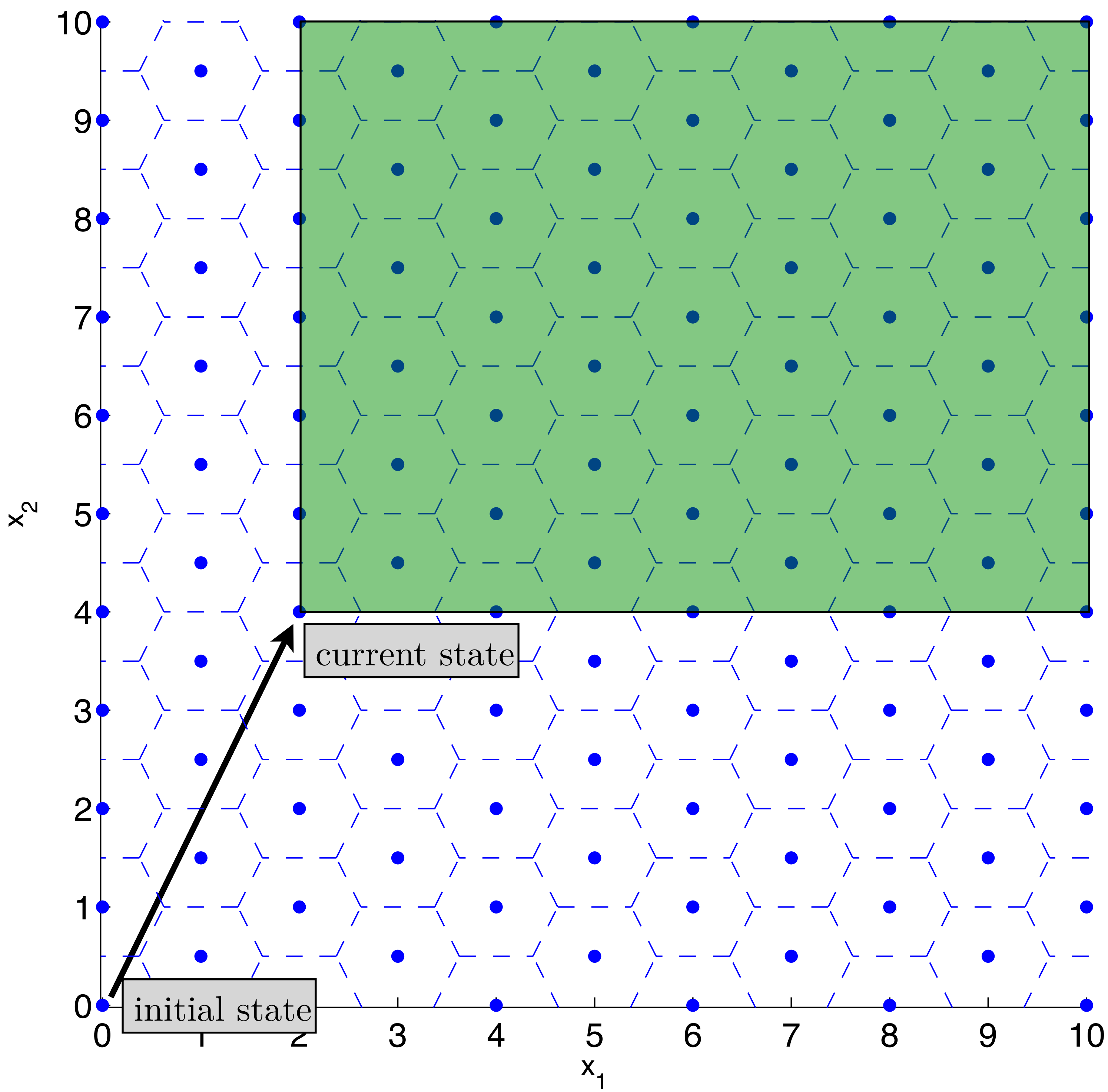


Recall that cell values can only increase

A path from:

- initial state
- terminal state

Cell values increase = rectangular region "accessible points"



Recall that cell values can only increase

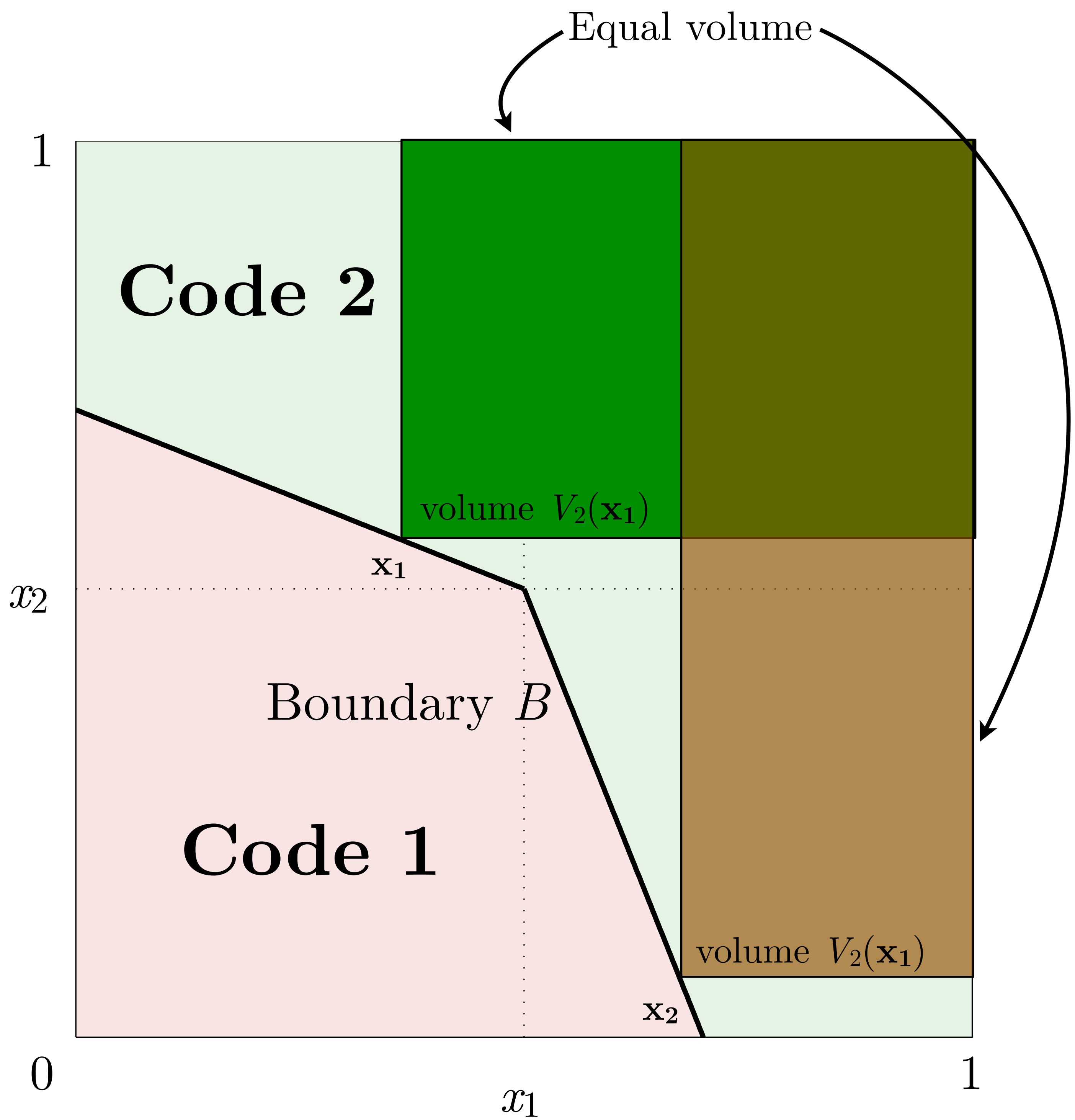
A path from:

- initial state
- terminal state

Consider a **current state**

The "accessible points" are in a rectangular region

Maximizing the Rate: B is a Hyperbola



$V_2(\mathbf{x})$: volume of space from $\mathbf{x}$

Hypothesis For any $\mathbf{x} \in B$, selecting $V_2(\mathbf{x})$ equal to a constant V_2 will maximize the rate.

For any point on B, the volume V_2 should be constant:

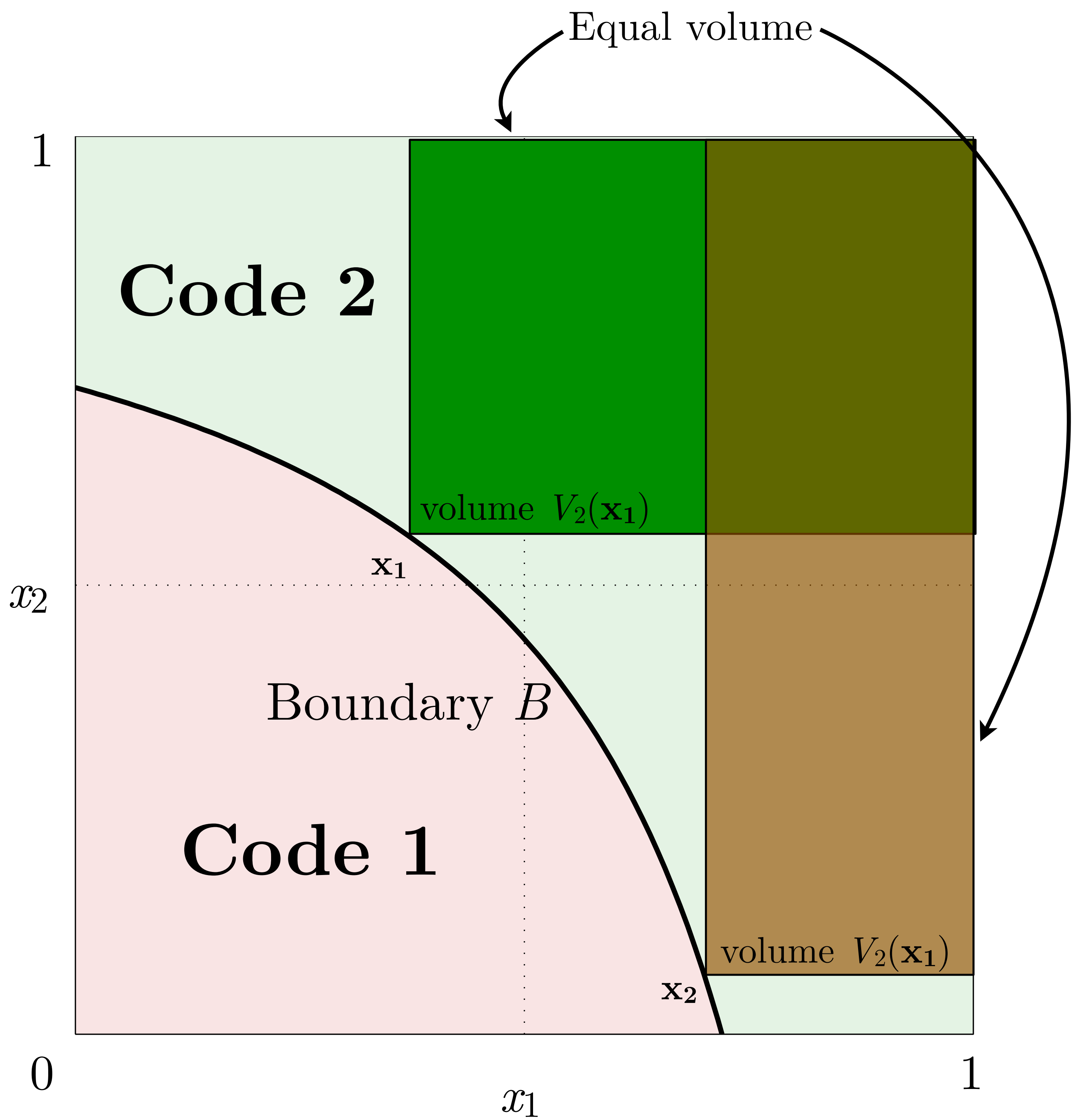
$$V_2 = (1 - x_1)(1 - x_2)$$

and in n dimensions:

$$V_2 = \prod_{i=1}^{n} (1 - x_i)$$

So, B is a **hyperbola**. We have a **hyperbolic shaping region**.

Maximizing the Rate: B is a Hyperbola



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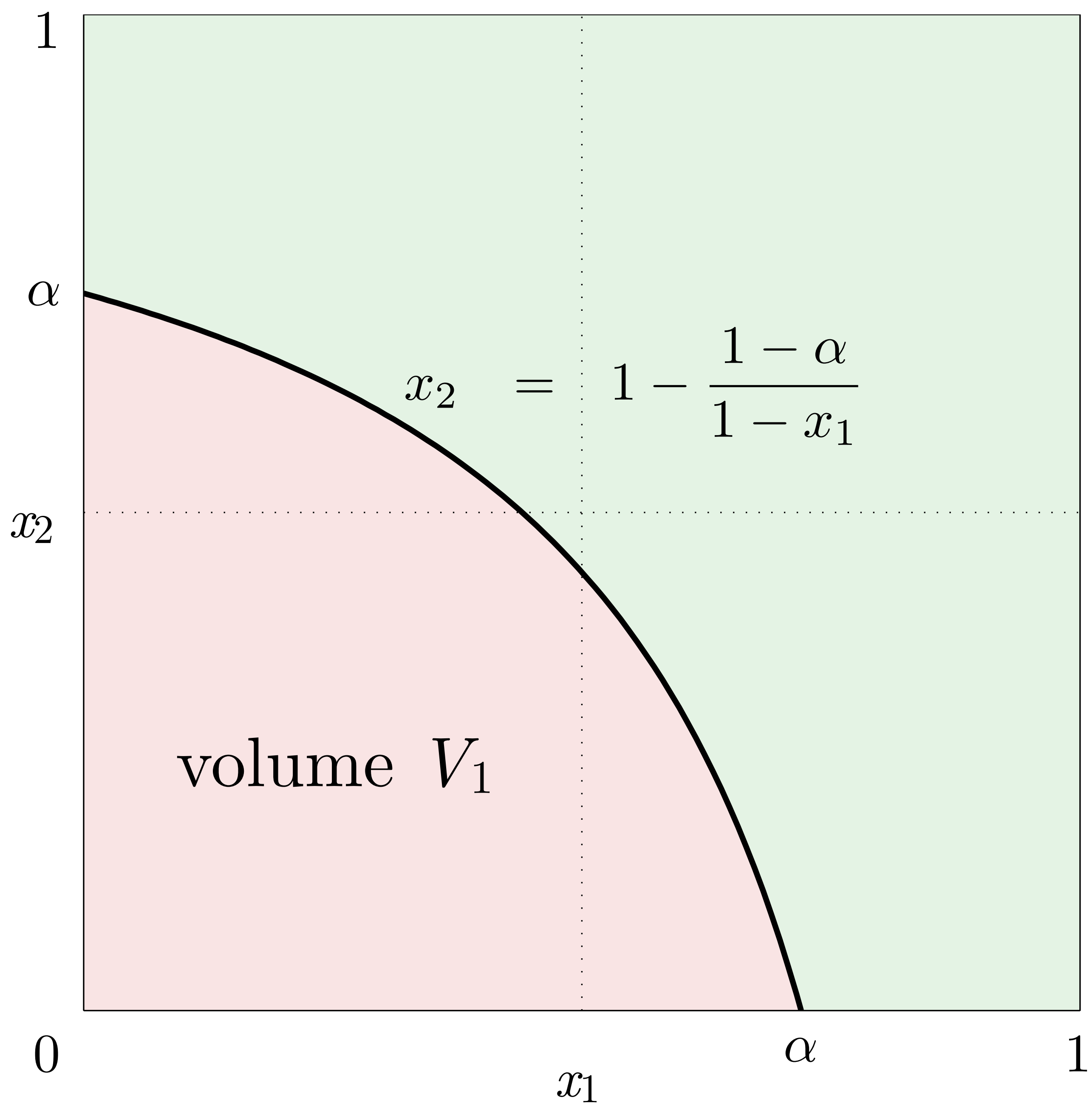
Computation of Volume V1



We can calculate the volume (and thus the rate).

For $n = 2$:

$$V_1 = 1 - (1-\alpha) + (1-\alpha)\log(1-\alpha)$$

For arbitrary n:

$$V_1 = 1 - e^{-z} \sum_{m=0}^{n-1} \frac{z^m}{m!},$$

where $z = -\log(1-\alpha)$

Assume equal rates for first and second writes:

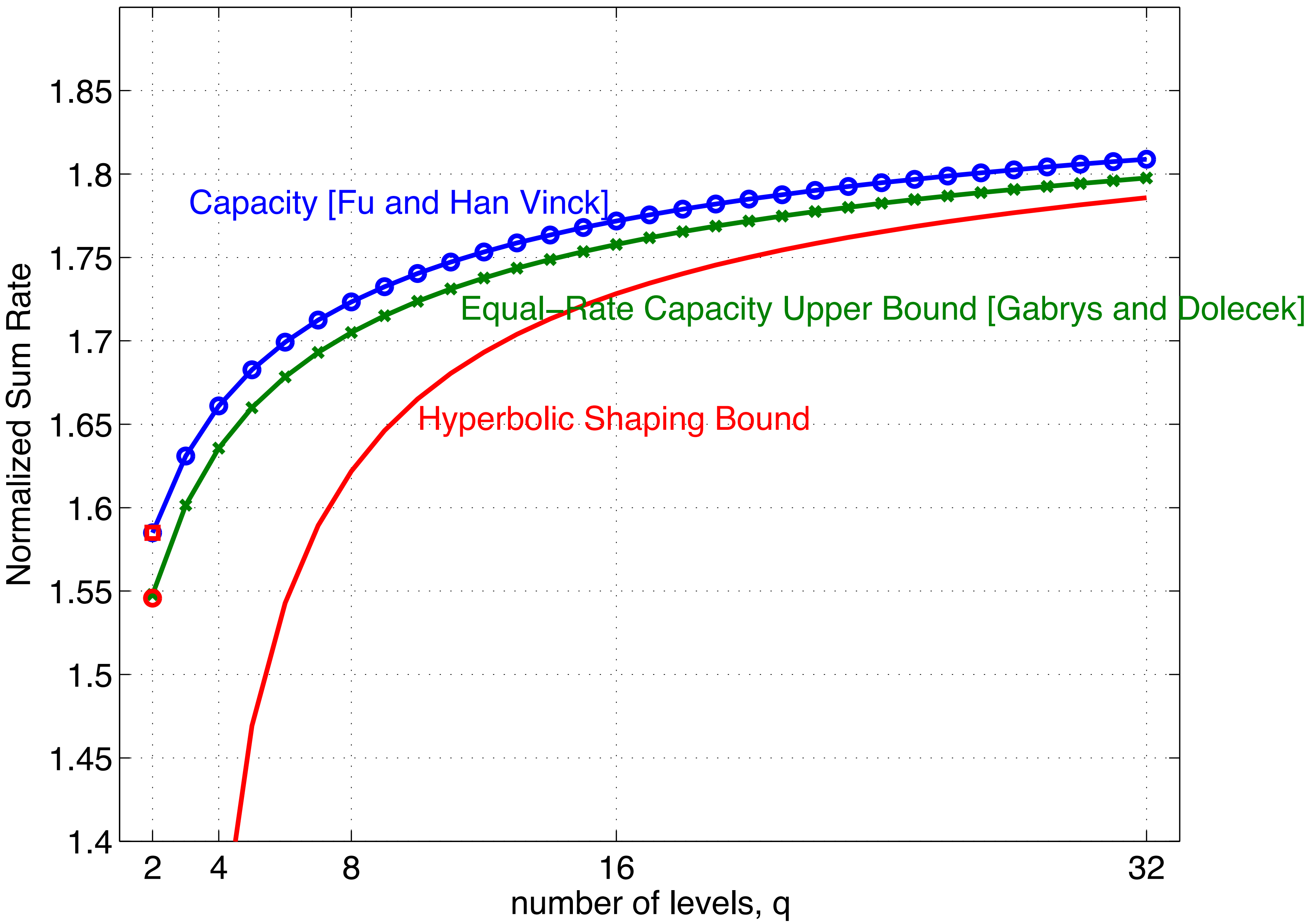
$$\begin{aligned} V_1 &= V_2 \\ 1 - e^{-z} \sum_{m=0}^{n} \frac{z^m}{m!} &= e^z - 1 \end{aligned}$$

The solution z^* can only be found numerically. But can form an upper bound:

$$\widetilde{R} \geq 1 - \frac{1}{e \ln 2} \frac{1}{\log_2(q-1)}$$

for the cubic lattice

Hyperbolic shaping approaches capacity as n $\to \infty$



Conditions:

- t=2 writes
- Capacity: non-equal rates (equal rates cannot achieve capacity)
- Hyperbolic lower bound: equal rates

lower bound approaches capacity as $q \to \infty$

If lower bound is tight, then gap to capacity for equal rates is small

What about encoding?

Bhatia, Iyengar and Siegel considered $n = 2$ [ITW 2012]

- $n = 2$ Easy to label all points $\to$ encode at the promised rate

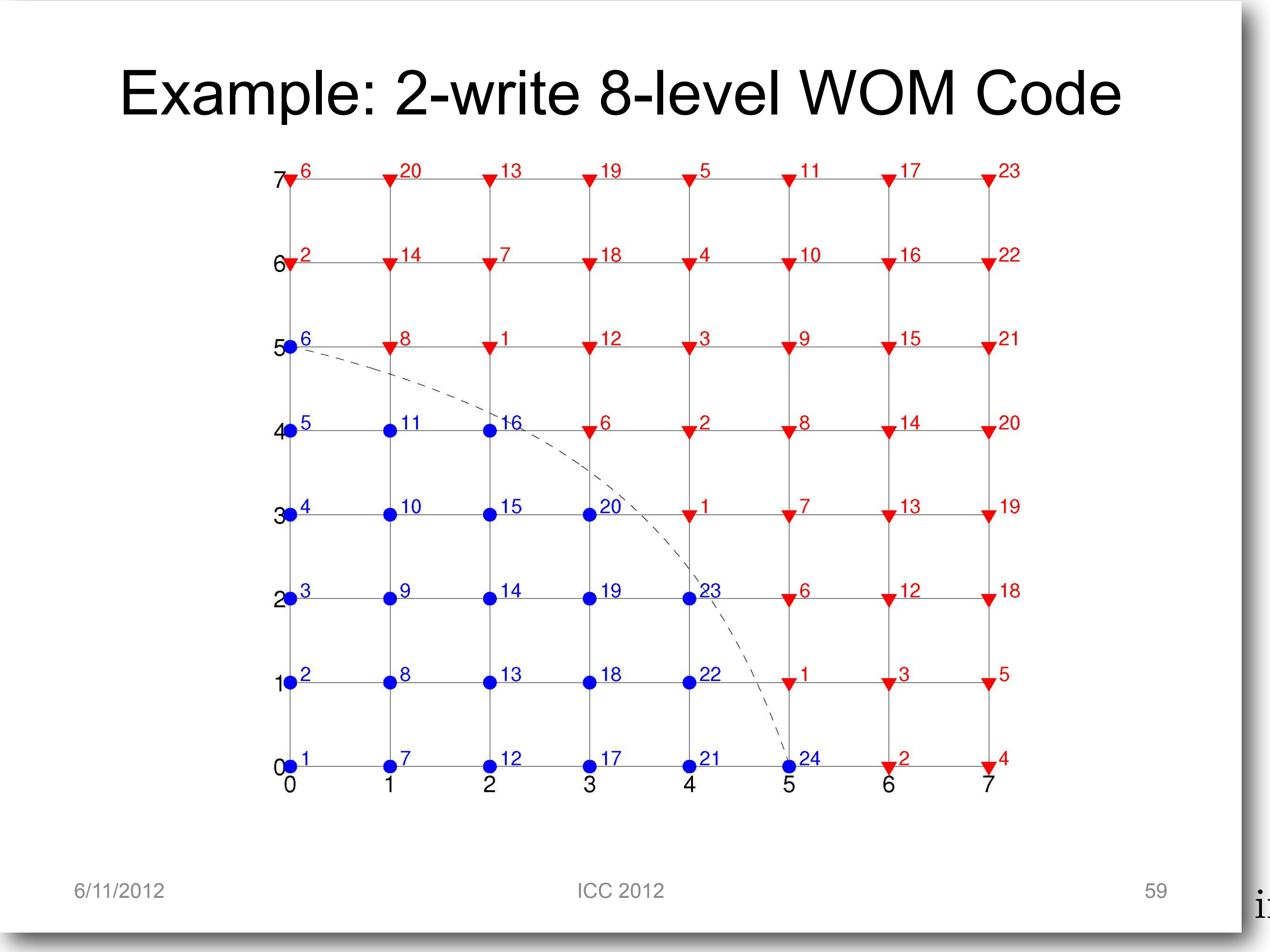
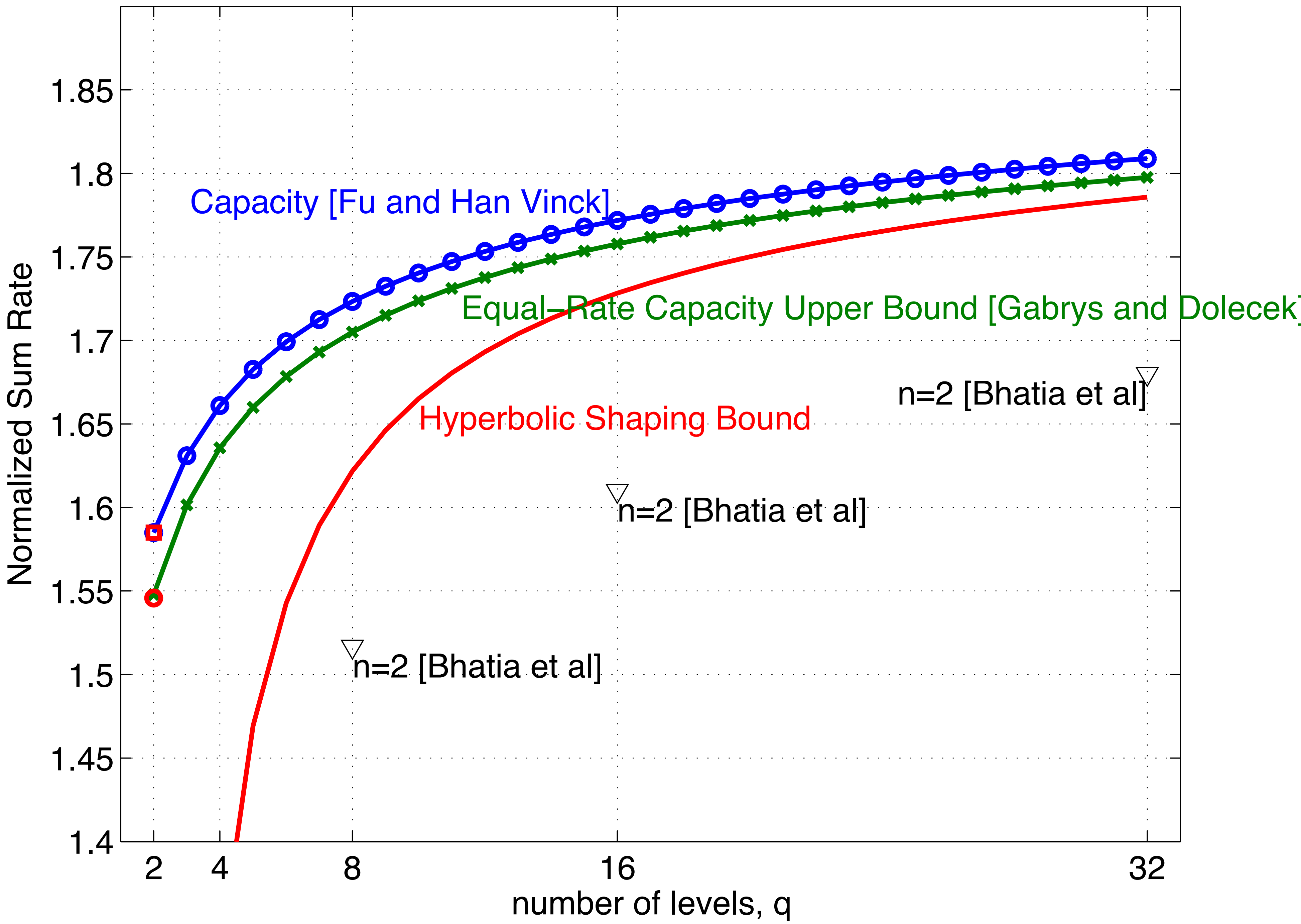


image thanks: Paul Siegel

Problem:

- for $n > 2$ some points are not "consistent" — there may be a rate penalty
- Hyperbolic shaping bound may not be tight (still unknown!)

Bhatia et al., Constructions with n=2
A Large Gap Remains!



Summary: WOM Properties of Lattices

WOM-properties of lattices

- Used a "continuous approximation"
  - Convert a discrete problem to a continuous problem
- Shaping region is a **hyperbola** in n dimensions
- Compute a lower bound on the code rate
- Much work to do on achievability of coding schemes