Shaping High-Dimensional Lattice Codes with Group Structure

Brian M. Kurkoski
Japan Advanced Institute of Science and Technology

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

joint work with Nuwan S. Ferdinand (University of Oulu)

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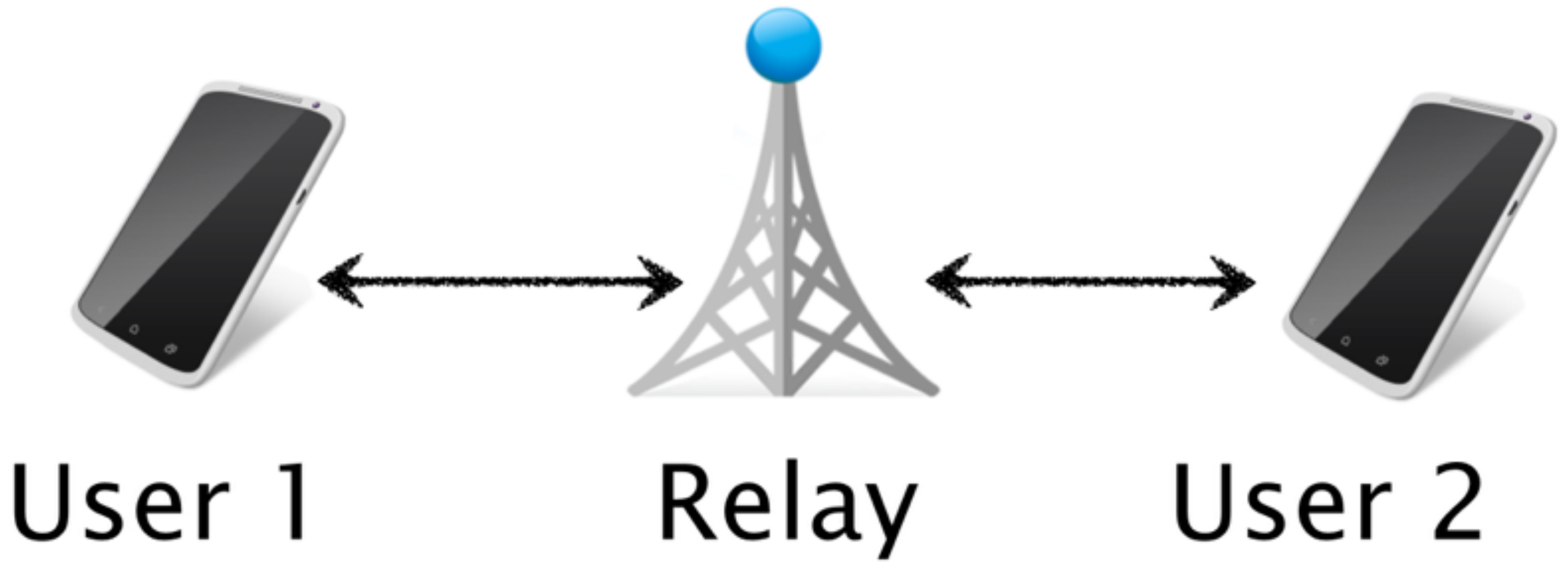
Usefulness of Nested Lattice Codes



Lattice codes can achieve the capacity of AWGN channel [Erez and Zamir '04]

Nested lattice codes: $\Lambda/M\Lambda$

Other *information theoretic* results using lattices:

- Lattices for relay channel e.g. [Song-Devroye '13]
- Two-way (Bidirectional) relay channel e.g. [Wilson et al.]
- Compute-forward relaying [Nazer-Gastpar '11]

How to construct practical, capacity-approaching lattices?

Capacity-Approaching Lattice Constructions

Recent high-dimension lattice constructions approach capacity

- Construction A with LDPC codes
- Construction D with turbo codes, spatially coupled LDPC
- Lattices based on polar codes
- Low-Density Lattice Codes [Sommer et al. 2008]

Common claim: within few tenth of dB of **unconstrained capacity:**

$$\frac{V(\Lambda)^{2/n}}{\sigma^2} \geq 2\pi e$$

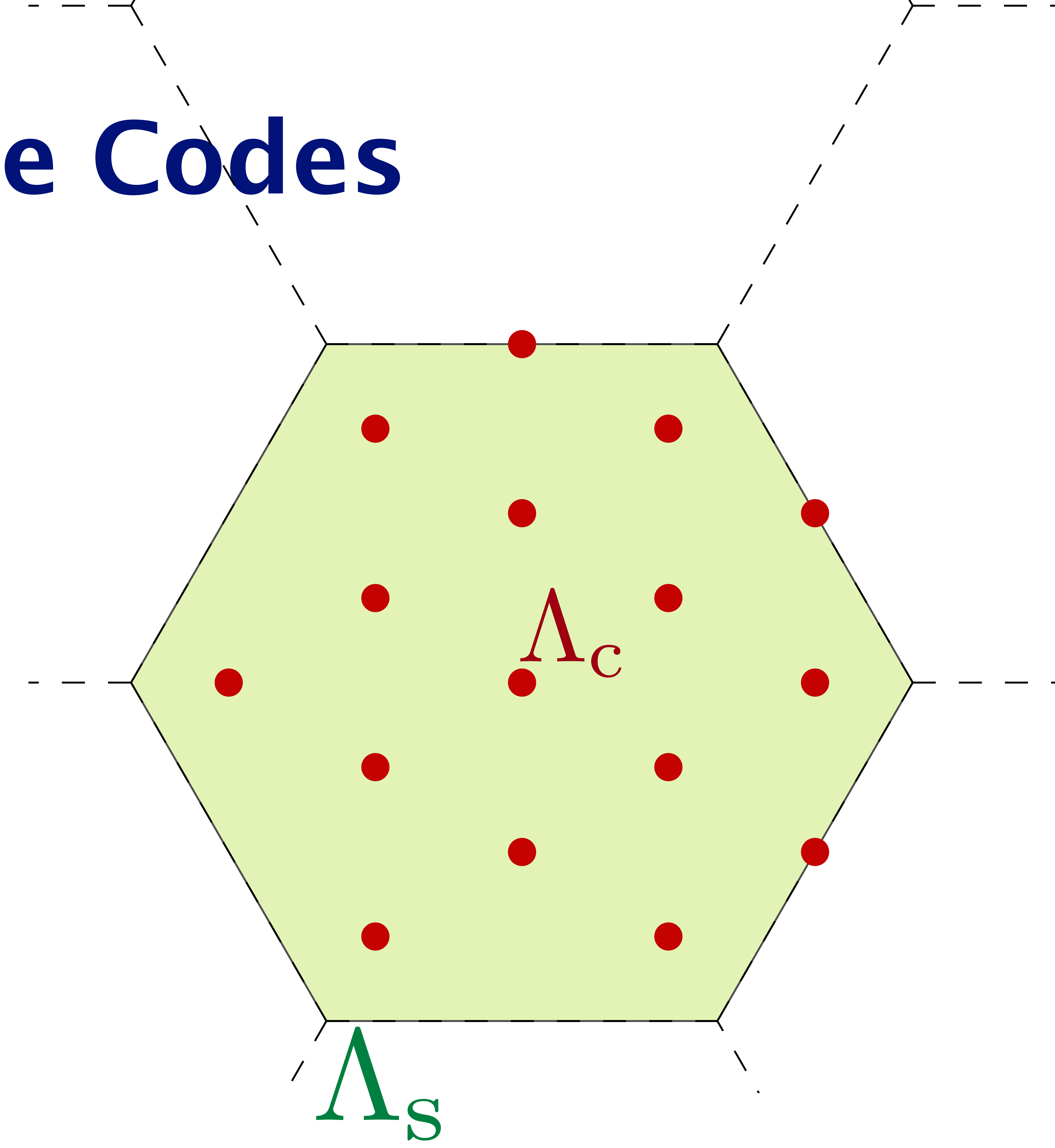
But the AWGN channel has a power constraint....

$$\Lambda_{\mathrm{c}} / \Lambda_{\mathrm{s}}$$

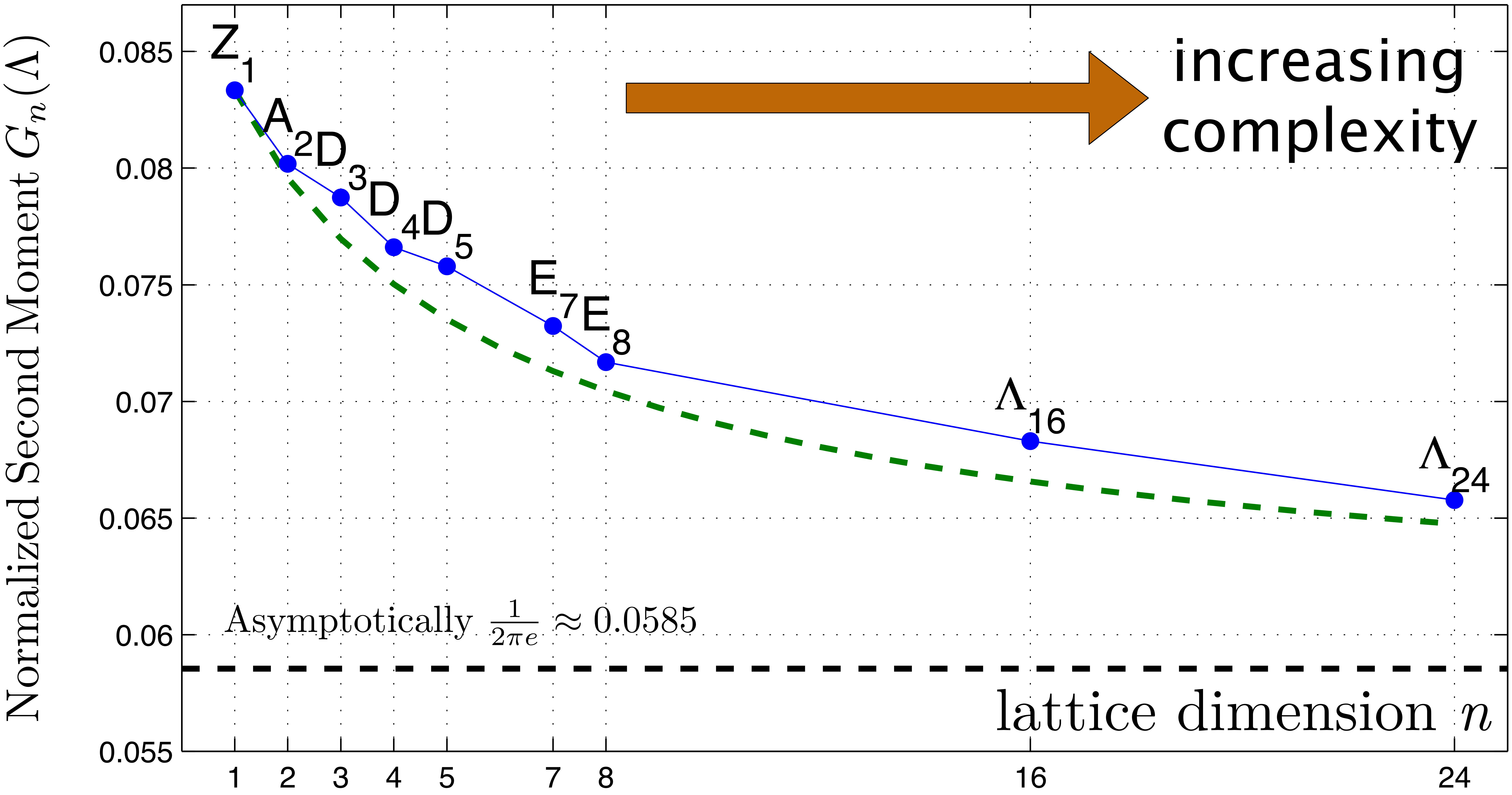
Nested Lattice Codes

Nested lattice code Λ_c/Λ_s:

- Λ_c is good for coding
- Λ_s is good for shaping, satisfies power constraint
- Quotient group Λ_c/Λ_s for network coding



$G_n(\Lambda)$ for Well-Known Lattices



Overview

Target is a nested lattice construction $\Lambda_\mathrm{c}/\Lambda_\mathrm{s}$:

- Λ_c is an n-dimension lattice of high dimension
- Λ_s is a product of $\frac{n}{m}$ lattices, e.g. $\Lambda_\mathrm{s} = E_8 \times \cdots \times E_8$

1. Sufficient condition on Λ_c such that $\Lambda_\mathrm{s} \subseteq \Lambda_\mathrm{c}$
2. Lattice encoding technique for shaping high-dimension lattices
   - "Voronoi integers" $\mathbb{Z}^n/\Lambda_\mathrm{s}$
   - Systematic encoding of $\Lambda_\mathrm{s} \to \Lambda_\mathrm{c}$
   - Obtain 0.65 dB shaping gain of the E8 lattice (vs. 0.4 dB)

Sufficient Conditions to form a Group

Λ_s has a generator matrix G with all entries $g_{i,j}$ integers.

Definition $g_{\min}$ is the greatest common divisor: $g_{\min} = \mathrm{GCD}(|g_{i,j}|)$

Λ_c has a check matrix $H = G_\mathsf{c}^{-1}$, with entries $h_{i,j}$

Lemma

$$\text{If } \; h_{i,j} \in \frac{1}{g_{\min}}\mathbb{Z} \quad \Rightarrow \quad \Lambda_\mathrm{s} \subseteq \Lambda_\mathrm{c}$$

If $\Lambda_\mathrm{s} \subseteq \Lambda_\mathrm{c}$, then $\Lambda_\mathrm{s}/\Lambda_\mathrm{c}$ forms a quotient group.

Design of Λ_{c} : candidate values

lattice	$g_{\min}$	Design of Λ_{c} : $h_{i,j}$ candidate values
$\widetilde{E}_8$	1	$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

Design Procedure for $\Lambda_\mathrm{c}/\Lambda_\mathrm{s}$

1. Select shaping matrix Λ_s
   (a) e.g. E_8 for low shaping complexity or Leech for high shaping gain
   (b) scale to the target rate
2. Select coefficients $h_{i,j}$ for Λ_c that satisfy:
   (a) group property so $\Lambda_\mathrm{c}/\Lambda_\mathrm{s}$ exists
   (b) other design criteria for high-dimension lattice

Design Example

$$\Lambda_\mathrm{s} = 4D_2 \times 4D_2 \times 4D_2 \times 4D_2 \qquad \text{and} \qquad g_{\min} = 4$$

$$\left[\begin{array}{c|c|c|c} \begin{matrix}4 & 0\\4 & 8\end{matrix} & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}4 & 0\\4 & 8\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}4 & 0\\4 & 8\end{matrix} & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}4 & 0\\4 & 8\end{matrix} \end{array}\right] \cdot \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} & h_{17} & h_{18} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} & h_{27} & h_{28} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} & h_{37} & h_{38} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} & h_{47} & h_{48} \\ h_{51} & h_{52} & h_{53} & h_{54} & h_{55} & h_{56} & h_{57} & h_{58} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} & h_{67} & h_{68} \\ h_{71} & h_{72} & h_{73} & h_{74} & h_{75} & h_{76} & h_{77} & h_{78} \\ h_{81} & h_{82} & h_{83} & h_{84} & h_{85} & h_{86} & h_{87} & h_{88} \end{bmatrix}$$

G for Λ_s $\qquad$ H for Λ_c

Design Example

$$\left[\begin{array}{c|c|c|c} \begin{matrix}4 & 0\\ 4 & 8\end{matrix} & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}4 & 0\\ 4 & 8\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}4 & 0\\ 4 & 8\end{matrix} & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}4 & 0\\ 4 & 8\end{matrix} \end{array}\right] \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/2 & 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/4 & 0 & 1 & 0 & 0 \\ 0 & -1/4 & 0 & 0 & 1/2 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 1/2 & 0 & 1 \end{bmatrix}$$

$$\Lambda_S$$

LDLC example: • sparse matrix

• h satisfy the convergence condition

High-Dimension Lattices

- dimension $n = 1{,}000 \sim 100{,}000$
- typically decoded using belief propagation
- "integer check matrix" H

$$\begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.7 & 1.0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.7 & 1.0 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0.5 & 0 & 1.0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0.7 & 0 & 1.0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0.7 & 0 & 1.0 \end{pmatrix}$$

Examples:

- Construction A lattices have a check matrix
- Low-density lattice codes

from here, assume LDLC H

- has triangular form
- dominant term in each row is on the diagonal
- assume dominant diagonal term is 1, although not strictly required

Evaluation of Nested Lattice Codes

Design H using sufficient condition

Nested lattice codes

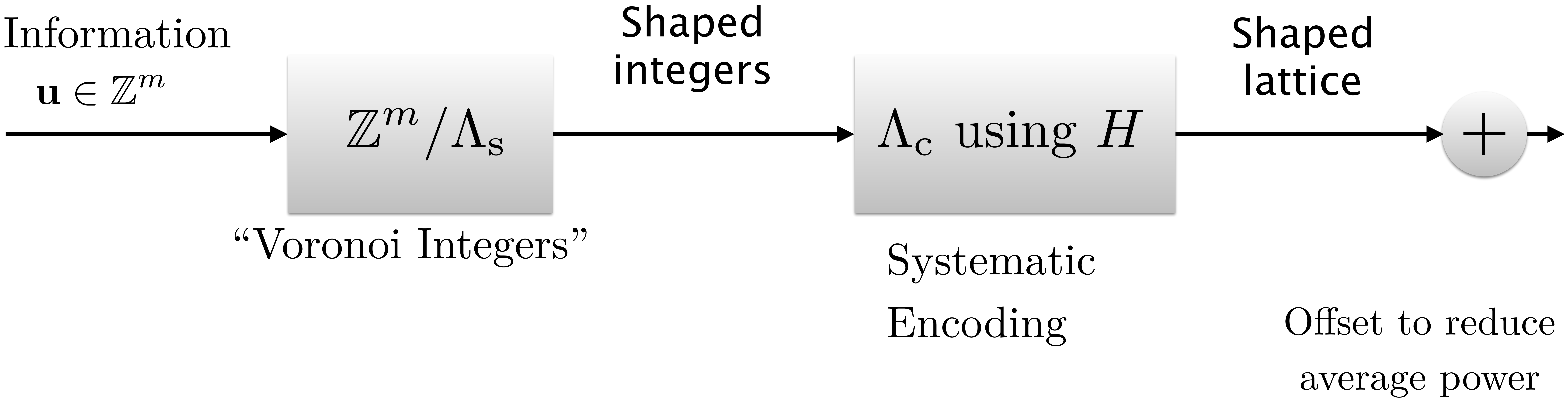
$\mathbf{x} = \mathbf{u}$ - $Q(\mathbf{u})$

(could be)

Nested lattice codes

systematic lattice encoding

Block Diagram of Encoder



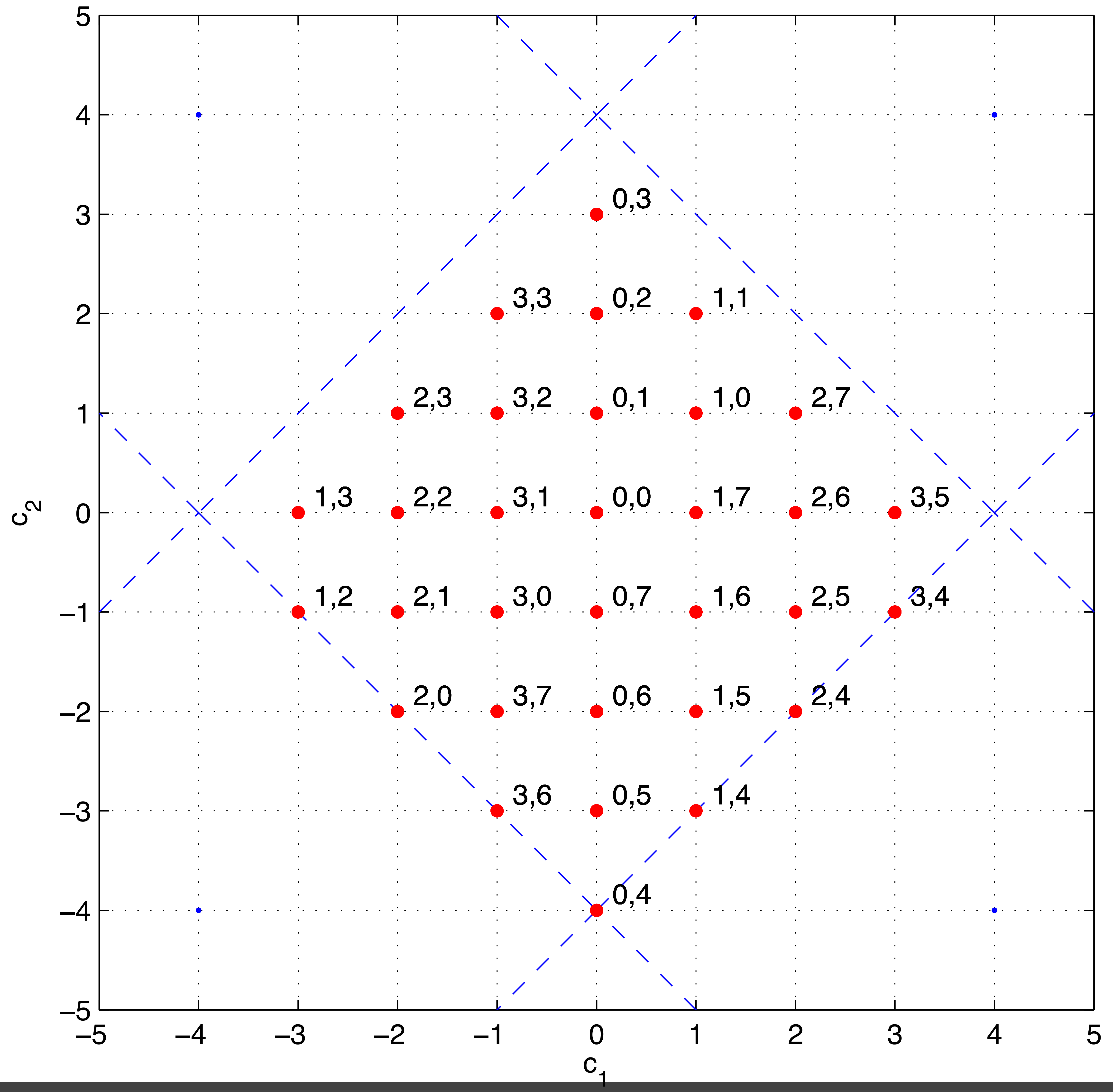
“Voronoi Integers” $\mathbb{Z}^m/\Lambda_{\text{shape}}$

Define “Voronoi Integers” Z^m/Λ_s

Integers which are shaped.

Require $\Lambda_s \subseteq \mathsf{Z}^m$, easy to satisfy:

- D_4, E_8, Barnes-Wall, Leech satisfy this condition
- Shaping (quantization) is feasible



Systematic Lattice Encoding

Encode integers $\mathbf{c}$ to lattice point $\mathbf{x}$ such that:

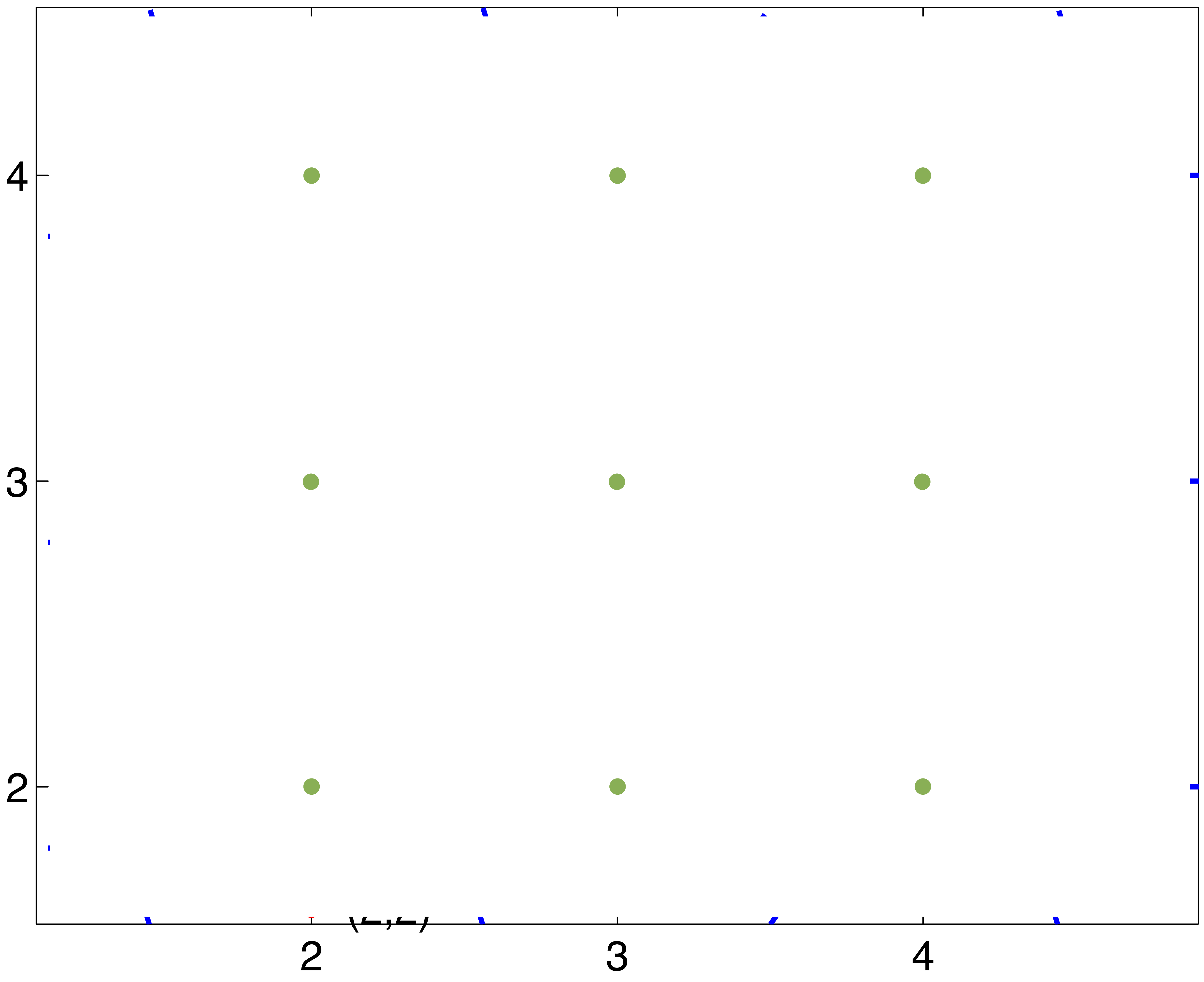
$$\mathbf{c} = \mathrm{round}(\mathbf{x})$$

That is, $|x_i - c_i| \leq \frac{1}{2}$

Example using

$$H = \begin{bmatrix} 1 & 0 \\ -0.3 & 1 \end{bmatrix}$$

Voronoi volume $\det(H) = \det(\mathbb{Z}^n) = 1$

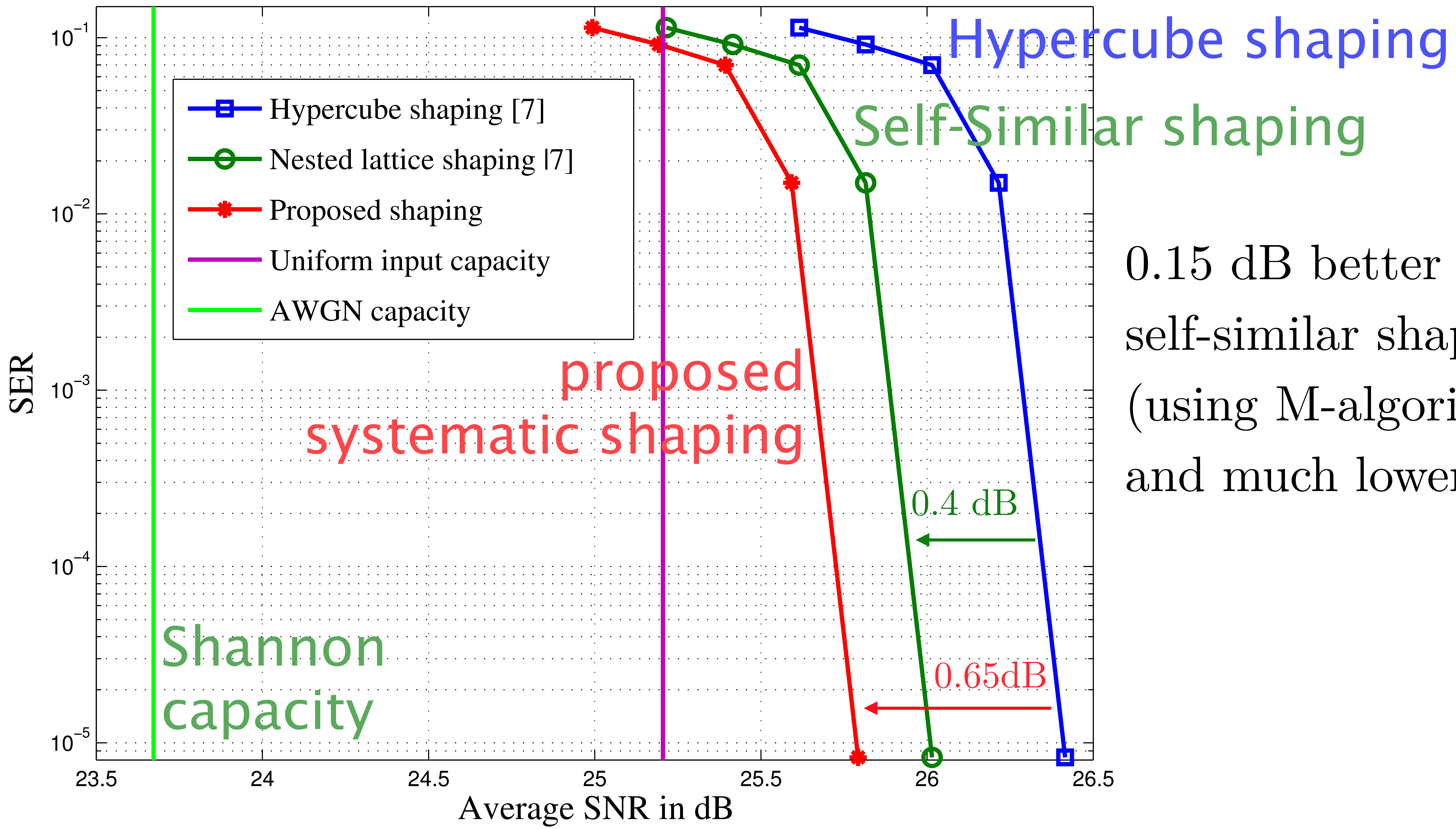


Power-Constrained AWGN Channel

AWGN channel with average power constraint

- 5 bits/dimension
- coding: LDLC lattice dimension $n = 10{,}000$
- shaping: E8 lattice with $m = 8$
- Compare with M-Algorithm LDLC shaping of Sommer et al

LDLCs: 0.65 dB Gain Over Hypercube



0.15 dB better than self-similar shaping (using M-algorithm) and much lower complexity

Conclusion

- Gave conditions on check matrix H for Λ_c such that $\Lambda_s \subset \Lambda_c$:
  - Structured lattices Λ_s for modest shaping gain
  - High-dimension lattice Λ_c for high coding gain
  - Nested lattice code Λ_c/Λ_s for physical layer network coding
- Open problems
  - Numerical evaluation of LDLCs under the sufficient condition
  - Suitability for Construction A, etc. lattices