Nested Lattice Codes Which Are Cyclic Groups

Chengpin Luo and Brian M. Kurkoski

Japan Advanced Institute of Science and Technology

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Motivation

Lattices have potential application to various communication systems

- Shaping gain using sphere-like constellation rather than QAM
- Compute-forward relay; particularly as a type of multiple access scheme
- Physical-layer network coding such as bi-directional relay
- Integer-forcing MIMO

Finite-length lattices are needed to practically realize such systems

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This talk is about lattice codes that form a cyclic group. Potential benefit:

- Simplified encoding, since there is a single generator
- Various lattice codes may have fractional number of bits per dimension, leading to encoding loss. Using cyclic lattice code may reduce this loss.
- They are interesting

This is a cyclic group, not a cyclic code.

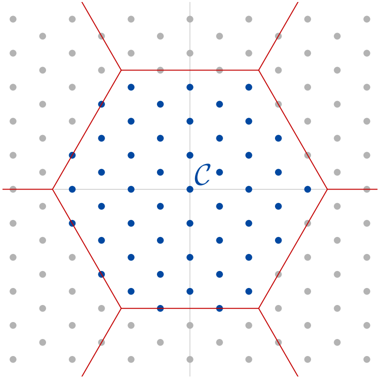
(Nested) Lattice Code

A lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$.
The generator matrix for Λ is $\mathbf{G}$:

$$\Lambda = \{\mathbf{G}\mathbf{b} \mid \mathbf{b} \in \mathbb{Z}^n\}$$

The check matrix is $\mathbf{H} = \mathbf{G}^{-1}$. A lattice code $\mathcal{C}$:

- $\mathcal{C}$ is the coset leaders of $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$
- Λ_{c} is the fine coding lattice with $\mathbf{G}_{\mathrm{c}}, \mathbf{H}_{\mathrm{c}}$
- Λ_{s} is the coarse shaping lattice with $\mathbf{G}_{\mathrm{s}}, \mathbf{H}_{\mathrm{s}}$
- Required to form lattice code:
  $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}} \Leftrightarrow \mathbf{H}_{\mathrm{c}}\mathbf{G}_{\mathrm{s}}$ is integer



A lattice is an infinite structure, a (nested) lattice code is a finite structure.

Self-Similar Lattice Codes.... Or Not?

Self-similar lattice code The shaping lattice Λ_{s} is scaled from the coding lattice Λ_{c}:

- $\mathcal{C} = \Lambda / M\Lambda$
- Sufficient for theoretical analysis (many results)

Non-self-similar lattice code[1] Practical reasons to not use self-similar lattices:

- Λ_{c} should have high coding gain and be easy to decode (e.g. lattices based on LDPC codes)
- Λ_{s} should have high shaping gain and have an efficient quantization algorithm:
    - Well-known lattices like E_8, Barnes-Wall, Leech, or
    - Convolutional code lattices with Viterbi algorithm quantization

[1] A more clever name is desired.

Rectangular Encoding

Rectangular encoding A bijective mapping from information $\mathbf{b}$ to codeword $\mathbf{x}$:

$$\begin{aligned}\mathbf{x} &= \mathbf{G}_\mathrm{c}\mathbf{b} - Q(\mathbf{G}_\mathrm{c}\mathbf{b}) \\ &= \mathbf{G}_\mathrm{c}\mathbf{b} \bmod \Lambda_\mathrm{s}\end{aligned}$$

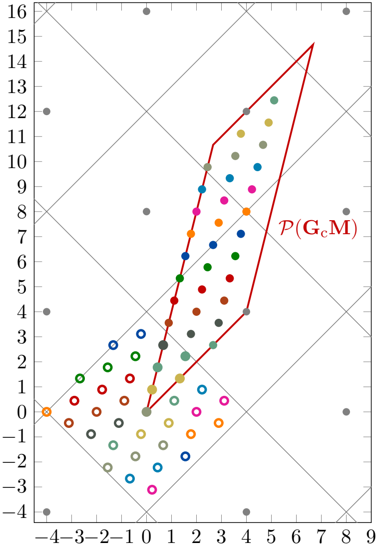
If the parallelogram $\mathcal{P}$ is a fundamental region for Λ_s:

- coding lattice inside $\mathcal{P}$ are coset leaders
- there is a one-to-one mapping between two cosets

Integers $\mathbf{b} = [b_1, b_2, \ldots, b_n]^\mathrm{t}$ are information:

$$b_i \in \{0, 1, \ldots, M_i - 1\}$$

$\mathcal{C}$ has $M = 2^{nR} = \prod_{i=1}^{n} M_i$ codewords



Cyclic Lattice Codes

A *cyclic group* is a group which can be generated by a single element $\mathbf{g}$, called the generator.

The integers $\mathbb{Z}$ are a cyclic group generated by 1 or -1 since $1+1=2$, $1+1+1=3$, etc.

A *cyclic lattice code*[2] is a nested lattice code which forms a cyclic group.

- Any lattice code $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$ is a group (see next slide)
- But in general, a lattice code is not a cyclic group

Self-similar lattice codes do not form a cyclic group. But under certain conditions, non-self-similar lattice codes do form a cyclic group.

[2] a cyclic group, not a cyclic code

Group Structure of Lattice Codes

A lattice code $\mathcal{C}$ forms a group under addition modulo Λ_s:

$$\mathbf{x}_1 \oplus \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_2 \bmod \Lambda_\mathrm{s}$$

This group property is important for compute-forward.

A lattice code is generally not a cyclic group since $\mathbf{g}_1, \ldots, \mathbf{g}_n$ are linearly independent:

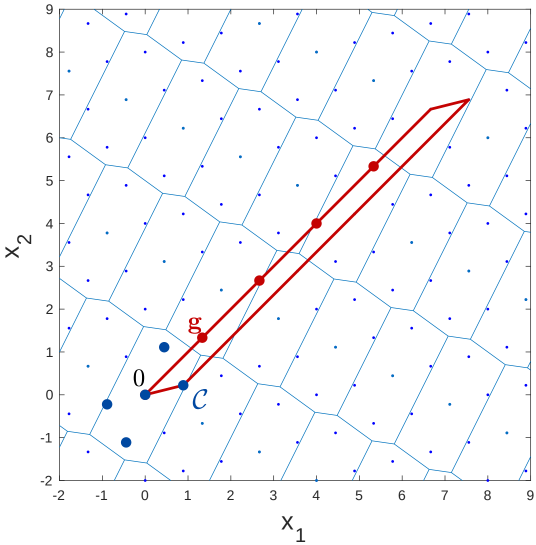
$$\mathbf{x} = \begin{bmatrix} | & | & & | \\ \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \bmod \Lambda_\mathrm{s}$$

But if we can find $[M_1, M_2, \ldots, M_n] = [1, 1, \ldots, M]$, then:

- b_1 to $b_{n-1} = 0$ and $\mathbf{g}_1$ to $\mathbf{g}_{n-1}$ are not used.
- $M_n = M$ and any codeword is given by $\mathbf{g}_n b_n \bmod \Lambda_\mathrm{s}$.
  The generator for the cyclic lattice code is $\mathbf{g}_n$.

How A Lattice Code Can Be Made Cyclic

- There is a one-to-one mapping between coset leaders in the parallelogram and $\mathcal{C}$.
- If all points in the parallelogram are generated by a single $\mathbf{g}$, then this will generate the whole group.
- Thus, $\mathcal{C}$ is cyclicly generated by $\mathbf{g}$.



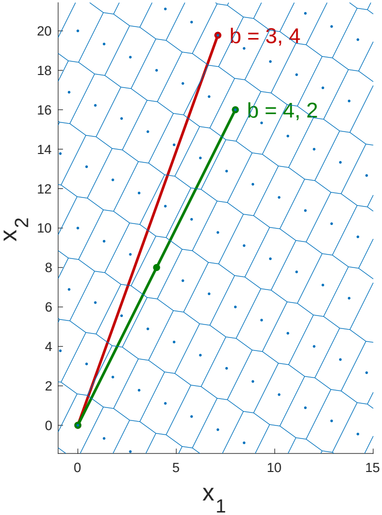
Technical Lemma

Lemma 1: Consider an n-dimension lattice Λ with generator matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \dots & \mathbf{g}_n \end{bmatrix}$$

The line segment with endpoints $\mathbf{0}$ and $\mathbf{y} = \mathbf{G} \cdot \mathbf{b}$ with $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T$ does not intersect any other point of Λ if and only if $\gcd(b_1, b_2, \dots, b_n) = 1$.

- $\mathbf{b} = [3, 4]$ are relatively prime — no other lattice point on the red segment
- $\mathbf{b} = [4, 2]$ 4 divides 2 — there is another lattice point on the green segment



Existence of Cyclic Lattice Codes

For the coding lattice Λ_{c},

$$\mathbf{G}_{\mathrm{c}} = \begin{bmatrix} | & | & & | \\ \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_n \\ | & | & & | \end{bmatrix}$$

Define $\mathbf{q}_i$ as columns of:

$$\det(\mathbf{H}_{\mathrm{c}}\mathbf{G}_{\mathrm{s}})(\mathbf{H}_{\mathrm{c}}\mathbf{G}_{\mathrm{s}})^{-1} = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$$

Lemma 2: An n dimensional nested lattice code $\mathcal{C}$ with $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$ is a cyclic lattice code with generator $\mathbf{g}_i$ if and only if $\gcd(\mathbf{q}_i) = 1$.

This $\gcd$ condition is required only for column $\mathbf{q}_i$ corresponding to the cyclic generator $\mathbf{g}_i$.

Design for $n = 2$

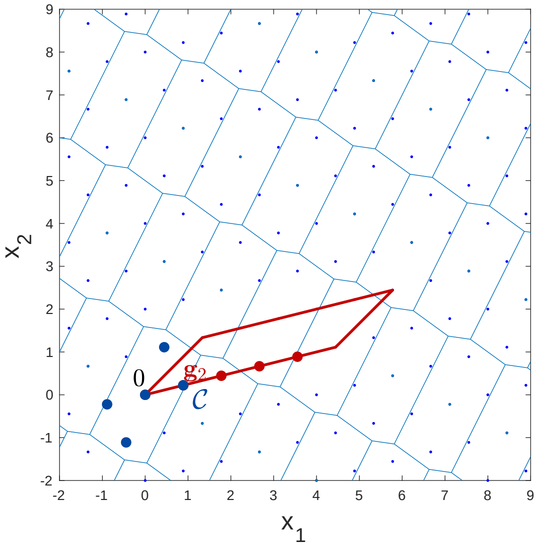
Consider coding lattice and shaping lattice with generator matrices:

$$\mathbf{G}_\mathrm{c} = \begin{bmatrix} \frac{4}{3} & \frac{8}{9} \\ \frac{4}{3} & \frac{2}{9} \end{bmatrix} \text{ and } \mathbf{G}_\mathrm{s} = \begin{bmatrix} \frac{16}{9} & \frac{4}{9} \\ \frac{22}{9} & \frac{28}{9} \end{bmatrix}$$

Then:

$$\det(\mathbf{H}_\mathrm{c}\mathbf{G}_\mathrm{s})(\mathbf{H}_\mathrm{c}\mathbf{G}_\mathrm{s})^{-1} = \begin{bmatrix} -4 & -3 \\ 1 & 2 \end{bmatrix}$$

- $4, 1$ are coprime, so $\mathbf{g}_1 = [\frac{4}{3}, \frac{4}{3}]^\mathrm{t}$ cyclicly generates $\mathcal{C}$
- $3, 2$ are coprime, so $\mathbf{g}_2 = [\frac{8}{9}, \frac{2}{9}]^\mathrm{t}$ cyclicly generates $\mathcal{C}$, also.



Possible Design for General n

Since the design places a restriction on $\mathbf{W}^{-1} = (\mathbf{H}_\mathrm{c}\mathbf{G}_\mathrm{s})^{-1}$, define $\mathbf{W}$ in a convenient form:

$$\mathbf{W} = \begin{bmatrix} 0 & \dots & 0 & a & b & c \\ 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 \\ w_{n,1} & \dots & w_{n,n-3} & w_{n,n-2} & w_{n,n-1} & w_{n,n} \end{bmatrix}.$$

For this design, $\gcd(c-b, a) = 1$ gives a cyclic lattice code, with matrix inversion using cofactors.

Group Isomorphism

Compute-and-forward requires the lattice code satisfy a group isomorphism:

$$\mathsf{enc}(\mathbf{b}_1 \boxplus \mathbf{b}_2) = \mathsf{enc}(\mathbf{b}_1) \oplus \mathsf{enc}(\mathbf{b}_2),$$

Feng, Silva and Kschischang gave conditions on the generator matrix to possess group isomorphism:

Lemma For arbitrary nested lattice $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$, if all elements from row i of $\mathbf{H}_{\mathrm{c}}\mathbf{G}_{\mathrm{s}}$ are divisible by M_i for all $i = 1, 2, ..., n$, then an isomorphism exists between group $\mathbf{b}, \boxplus$ and $\mathcal{C}, \oplus$.

To design a cyclic lattice code with group isomorphism Write the last row of $\mathbf{W}$:

$$\begin{bmatrix} r_1 M & r_2 M & \cdots & r_n M \end{bmatrix}$$

Then $\det(\mathbf{W}) = M$ leads to a linear diophantine equation in r_i.

Design Using $n = 8$ with E_8 for Shaping

Suppose we want to design a (1) cyclic lattice code with $M = 64$ codewords (rate 0.75) which has (2) shaping gain provided by the E_8 lattice and possesses (3) group isomorphism.

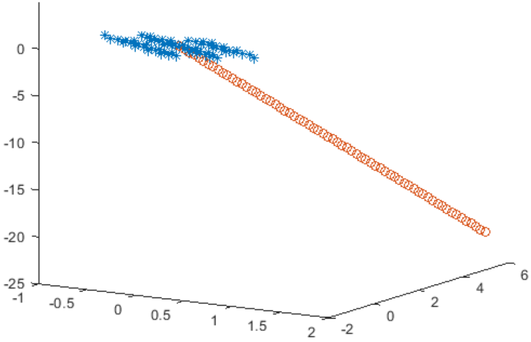
$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a & b & c \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & Mr_6 & Mr_7 & Mr_8 \end{bmatrix}.$$

Choose $(a, b, c) = (7, 17, 19)$ to make the lattice cyclic. Solve $\det(\mathbf{W}) = 64$ to obtain $(r_6, r_7, r_8) = (95, 65, 92)$.

Finally, choose $\mathbf{G}_{\mathrm{c}} = \mathbf{G}_{\mathrm{s}}\mathbf{W}^{-1}$. This gives a coding lattice with coding gain 2.67 dB

Design Using $n = 8$ with E_8 for Shaping

As a consequence of having three "design" columns in $\mathbf{W}$, the resulting lattice code is in 3 dimensions



Conclusion

- Gave conditions under which a lattice code forms a cyclic group.
- Gave a few basic constructions with dimension $n = 2$ and $n = 8$.
- Possibly simplifies encoding by replacing a generator matrix with a generator vector
- May reduce mapping overhead when the number of bits/dimension is not a power of two.