Generalized Voronoi Constellations

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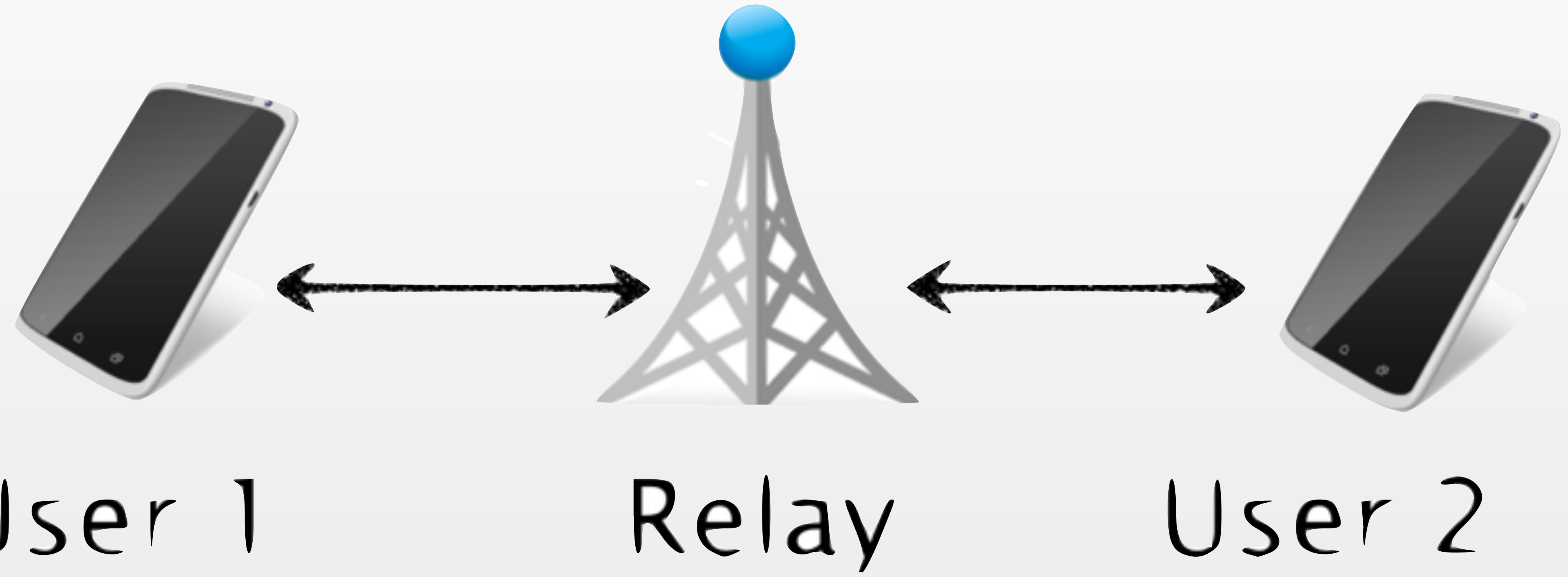
Usefulness of Lattice Codes



Lattice codes can achieve the capacity of AWGN channel [Erez and Zamir '04]

Information theoretic results and physical layer network coding using lattices:

- Lattices for relay channel e.g. [Song-Devroye '13]
- Two-way (Bidirectional) relay channel e.g. [Wilson et al.]
- Compute-forward relaying [Nazer-Gastpar '11]

"Physical layer network coding"

How to construct practical, capacity-approaching lattices?

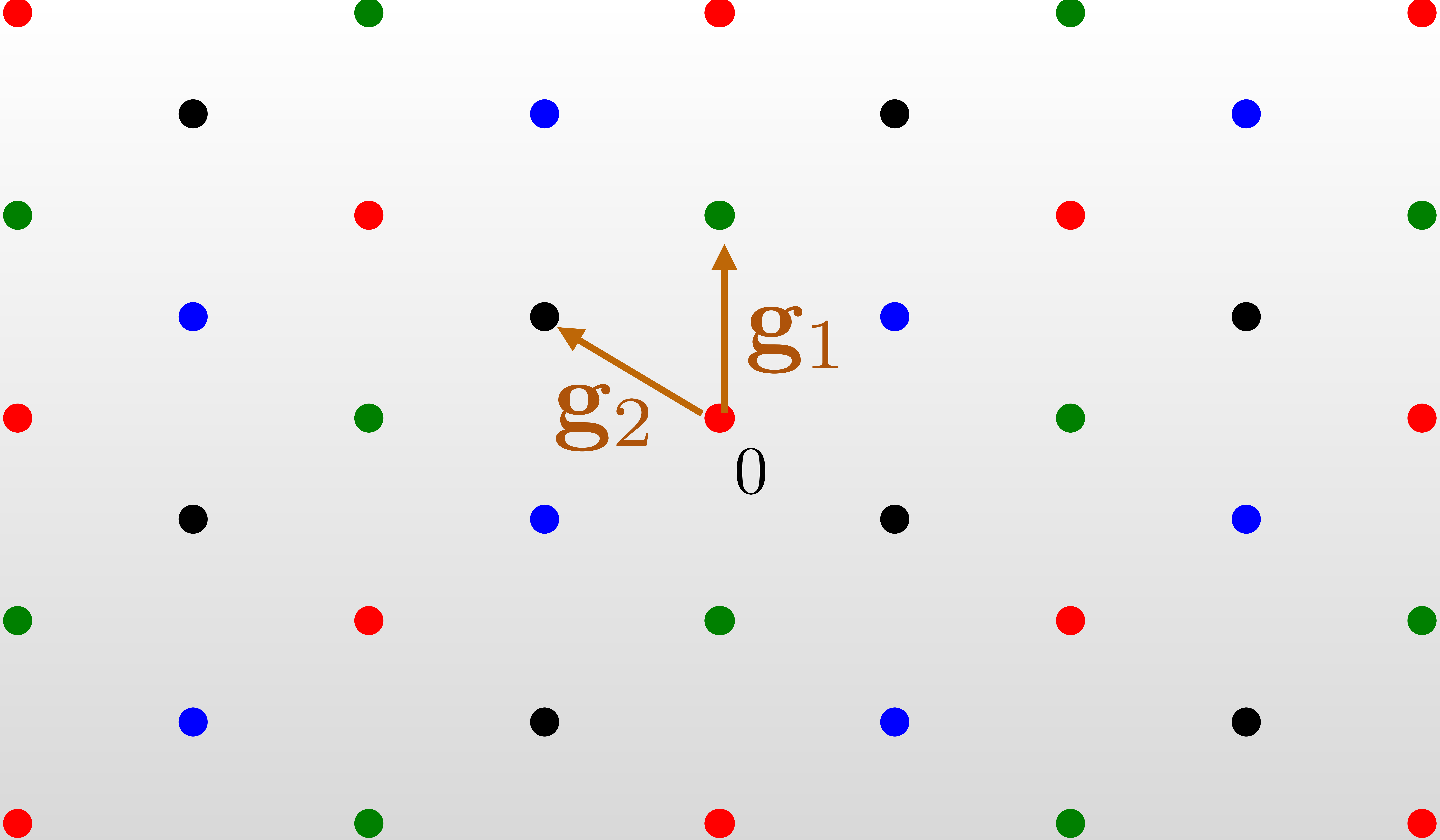
A lattice Λ is a linear additive subgroup of $\mathbb{R}^n$.

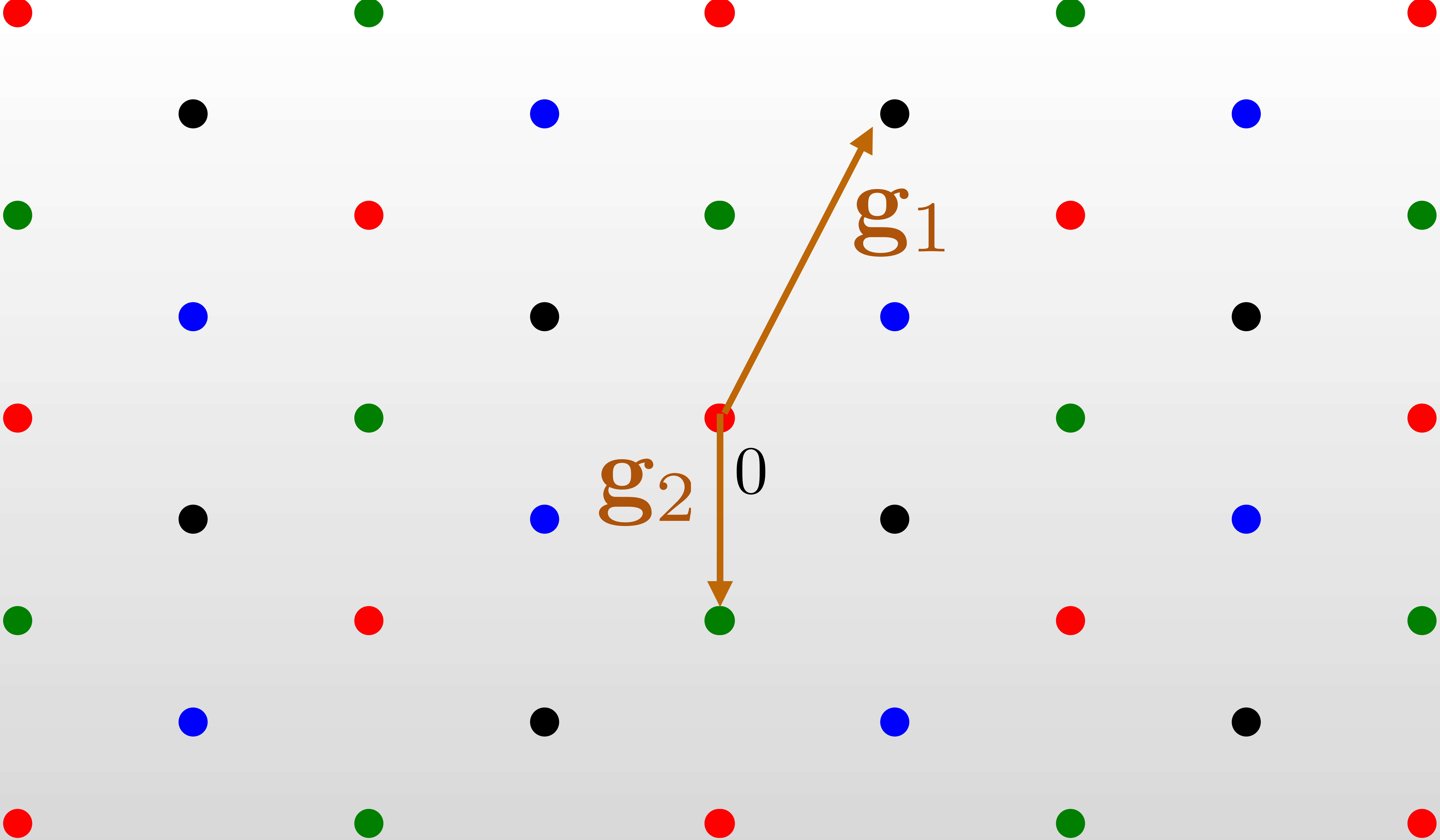
Given a basis $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$, the lattice consists of all points

$$\begin{bmatrix} | & | & & | \\ \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

for $b_i \in \mathbb{Z}$.

Given Λ, there are arbitrarily many possible bases.


$\mathbf{g}_1$
$\mathbf{g}_2$
0



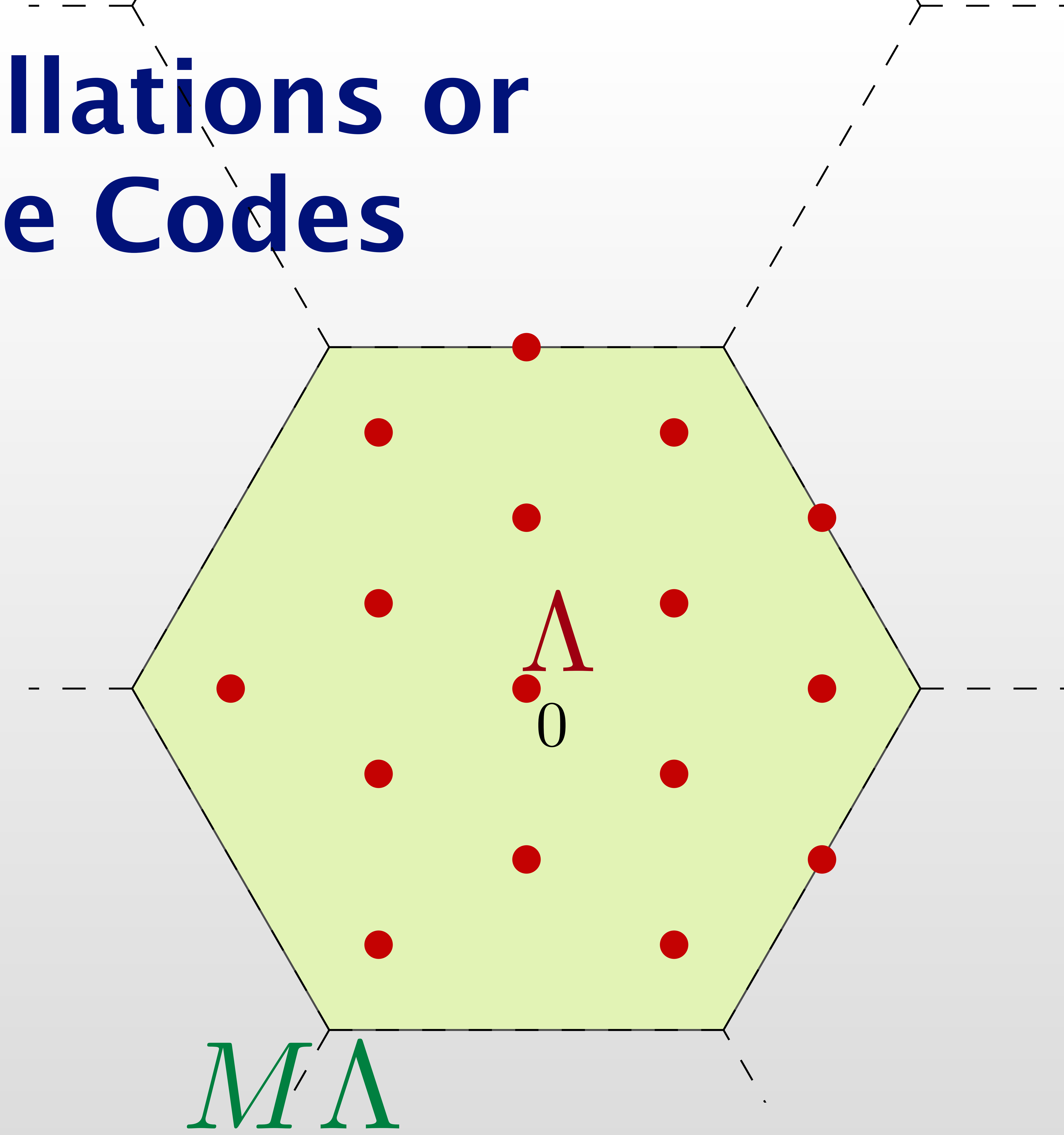
$\mathbf{g}_1$
$\mathbf{g}_2$
0

Voronoi Constellations or Nested Lattice Codes

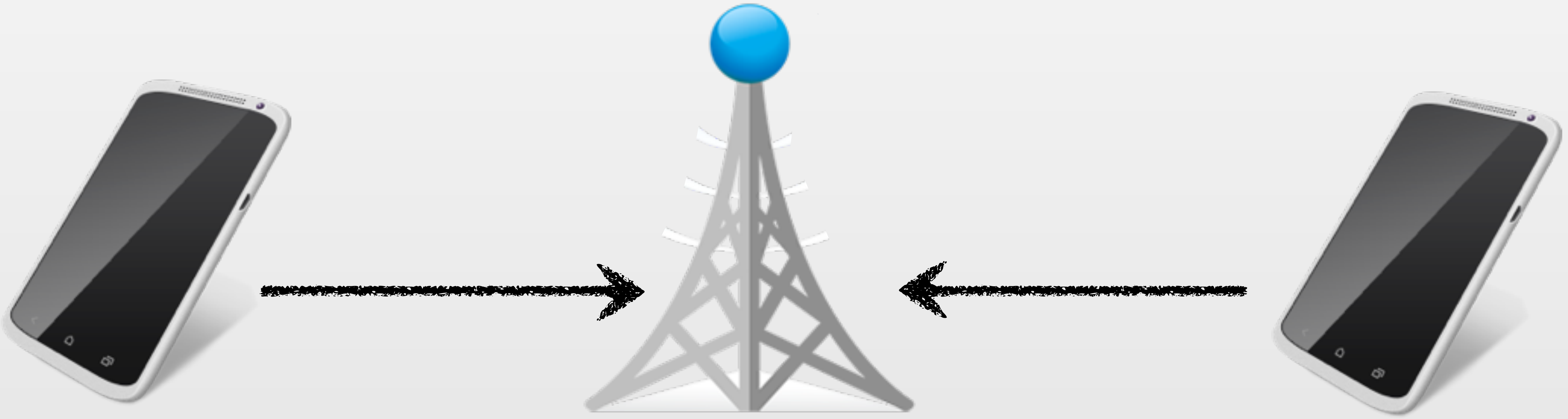
Conway and Sloane [IT 83] described *Voronoi constellations*

- Λ is a lattice
- $M\Lambda$ is scaled by M.
- $\Lambda/M\Lambda$ is a quotient group
- coset leaders Euclidean-space code

Also called *nested lattice codes*



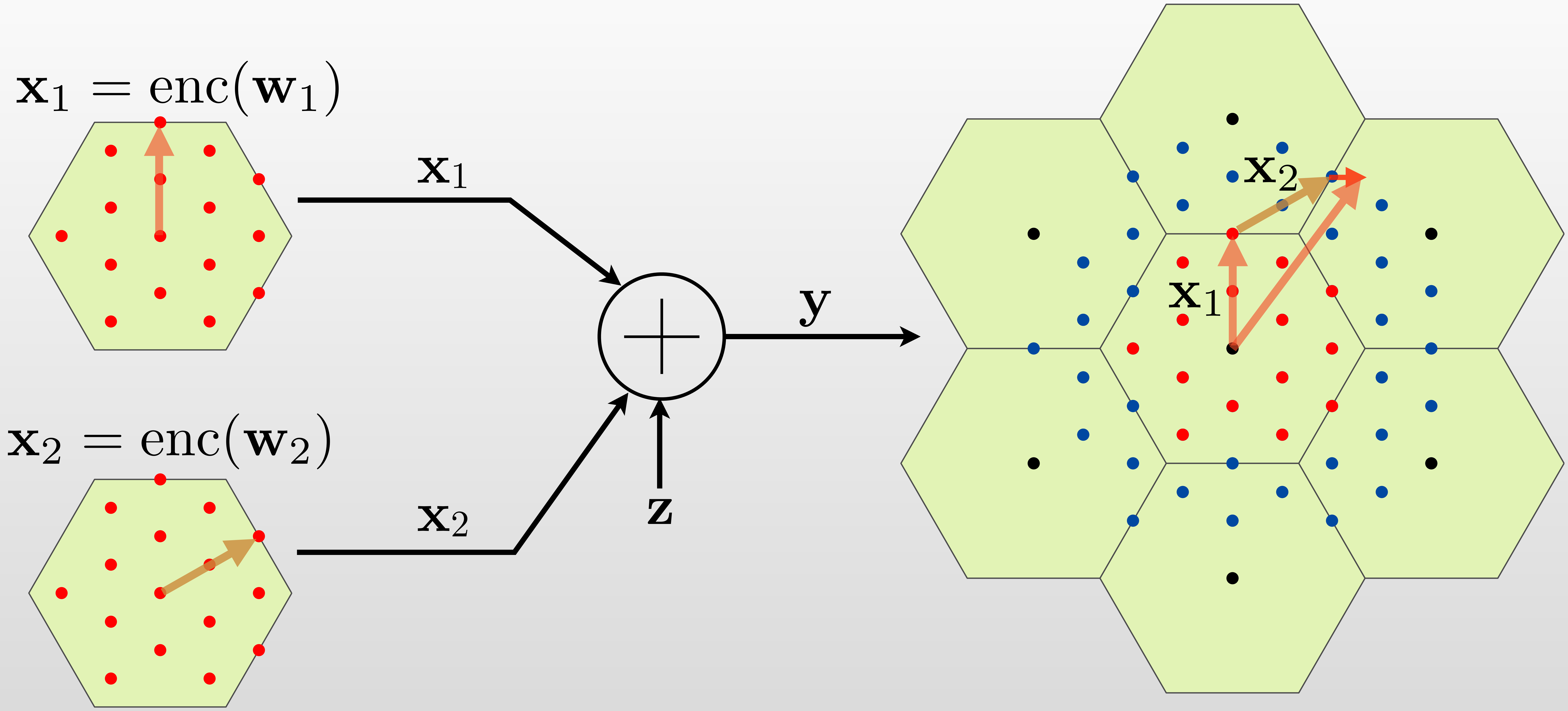
Physical Layer Network Coding: Signals Add Over the Air

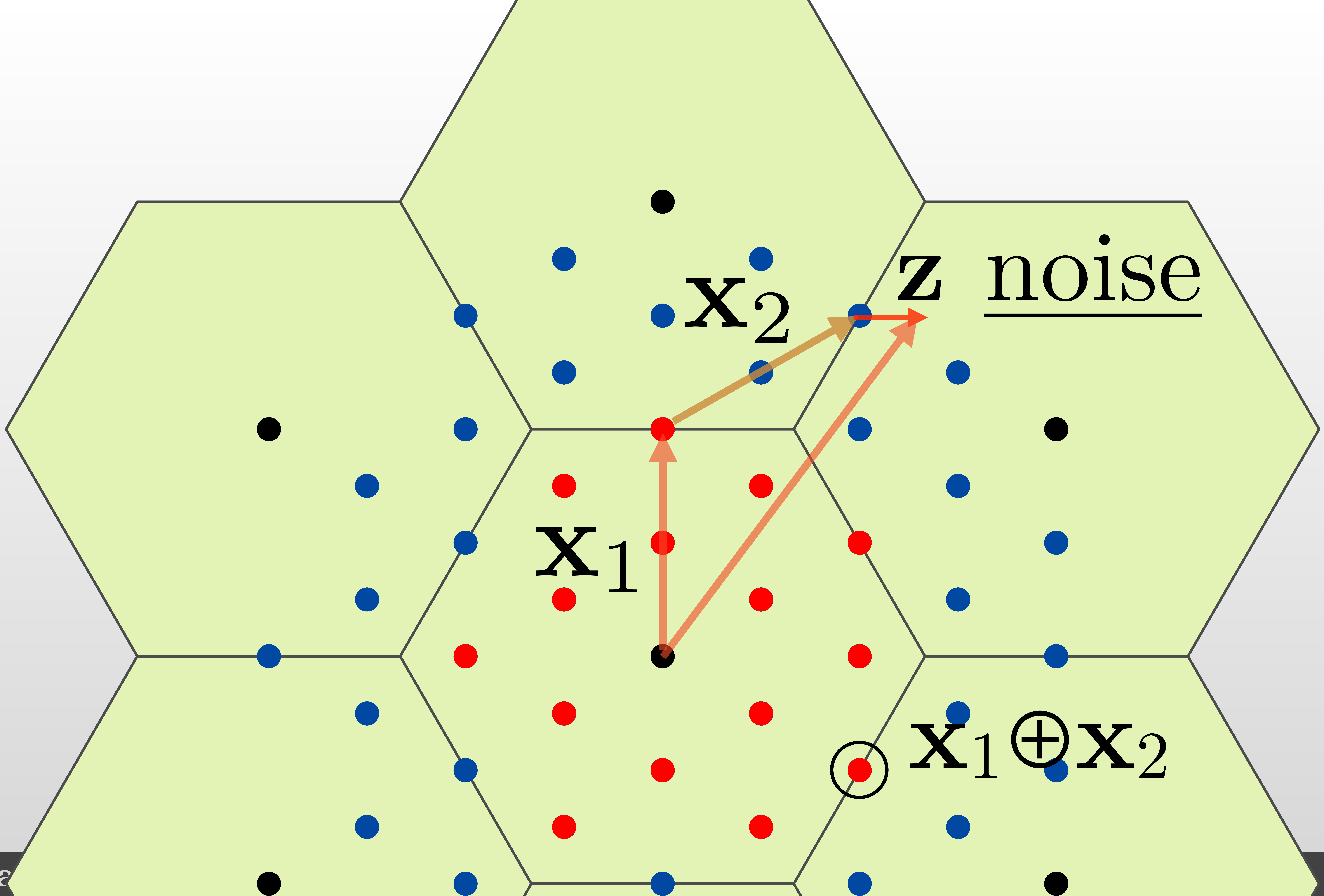


User 1 | Relay | User 2

has w_1 | wants $w_1 + w_2$ | has w_2

Two Users Transmit to Relay




$\mathbf{z}$
noise
$\mathbf{x}_2$
$\mathbf{x}_1$
$\mathbf{x}_1 \oplus \mathbf{x}_2$

Background Summary

Lattices are codes over the real numbers.

Network coding for wireless networks: signals add over the air: **physical layer network coding**

Voronoi constellations (nested lattice codes) have three properties:

1. Coding lattice Λ — good for error correction
2. Shaping lattice $M\Lambda$ —
   - As $n \to \infty$ Voronoi region is sphere like
   - A sphere achieves optimal AWGN input distribution (and Shannon capacity). Optimal 1.53 dB shaping gain
3. Forms a quotient group required for physical layer network coding

Good for theoretical results, difficult to construct capacity-achieving codes

Contributions

Generalized Voronoi Constellations — Practical lattice codes

- Shaping lattice is not a scaled coding lattice:

$$\underset{\substack{\text{is high dimension,}\\ \text{capacity-approaching}}}{\textit{coding lattice}} \longrightarrow \quad \Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}} \quad \longleftarrow \underset{\substack{\text{high shaping gain}\\ \text{efficient shaping algorithm}}}{\textit{shaping lattice}}$$

- Give necessary and sufficient condition so $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$ is a group
- Encoding for triangular coding matrices: easy
- Encoding for general coding matrices: not so easy

How to Design a Coding Lattice

Approach unconstrained lattice capacity, lattice dimension n should be large

Construction A and Construction D

- Construction D using LDPC codes [Sadeghi et al IT 2006]
- Construction A using non-binary LDPC codes [Huang et al ISIT 2014]
- Construction D using polar codes [Yan et al ITW 2012]

Derive generator matrix G, and check matrix $H = G^{-1}$ from the design

Low-Density Lattice Codes (LDLC lattices)

- [Sommer et al, 2008]
- Spatially-coupled LDLCs [Uchikawa et al, ISIT 2012]

Design the H matrix to be sparse and other easy conditions.

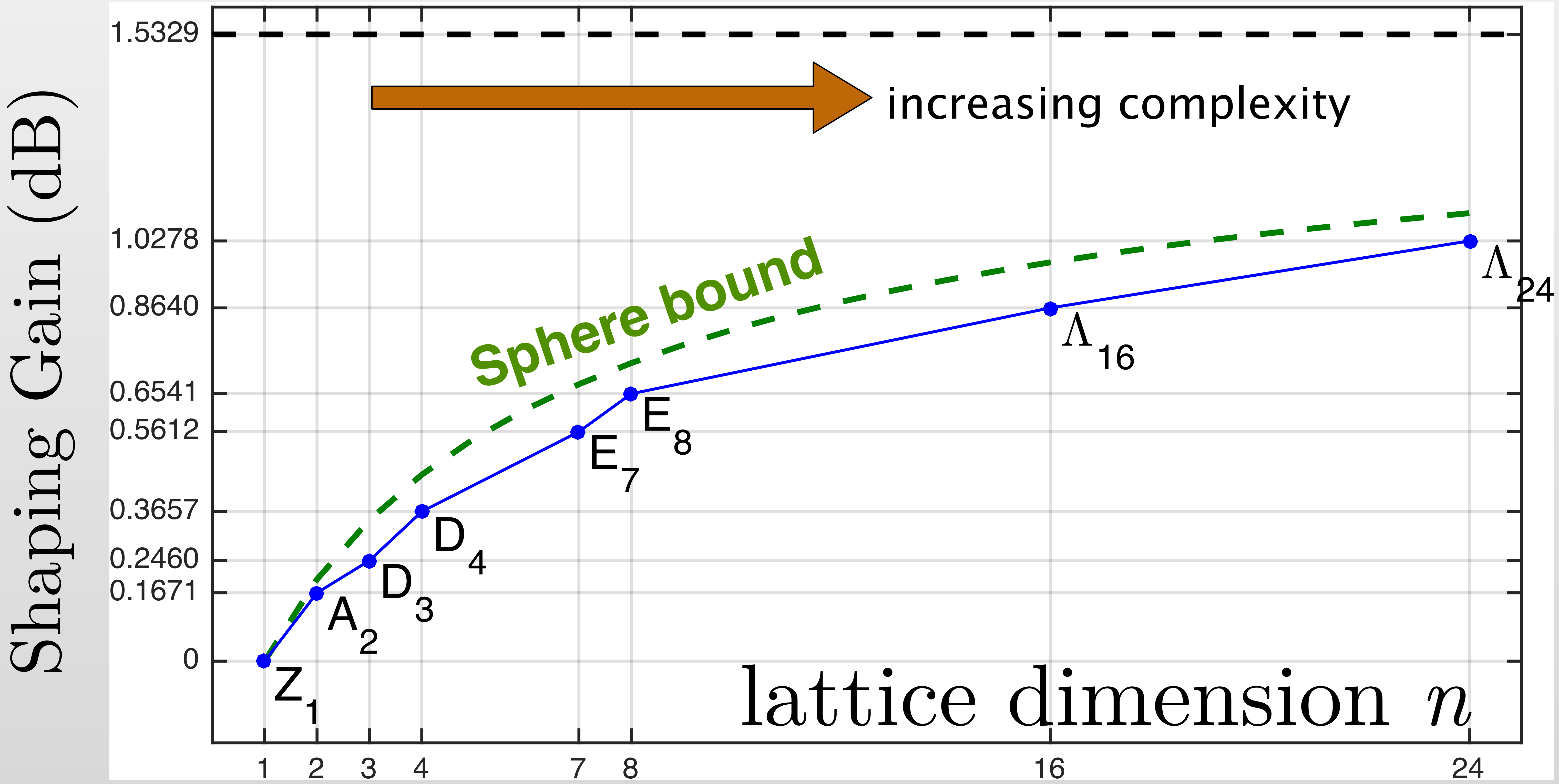
How to Design a Shaping Lattice

Ideally, want a shaping lattice with efficient maximum-likelihood decoding:

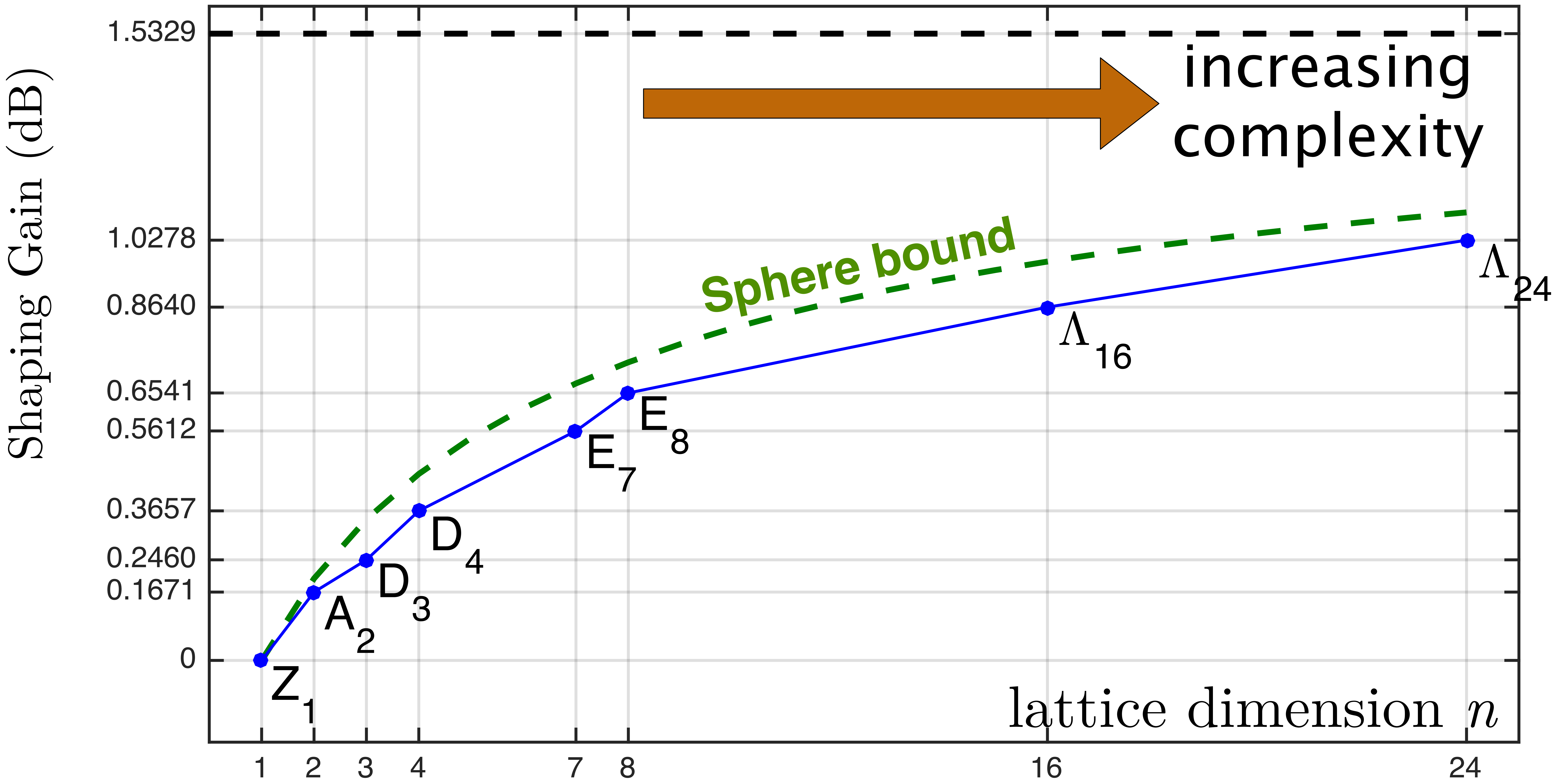
1. Lattices based on convolutional codes (Viterbi-based decoding)
2. Low-dimension lattices, E8, BW16, etc.

Shaping lattice is concatenation of low-dimension lattices:

$$\underbrace{\Lambda_s \times \Lambda_s \times \cdots \times \Lambda_s}_{\text{dimension } n}$$



Shaping Gain for Well-Known Lattices

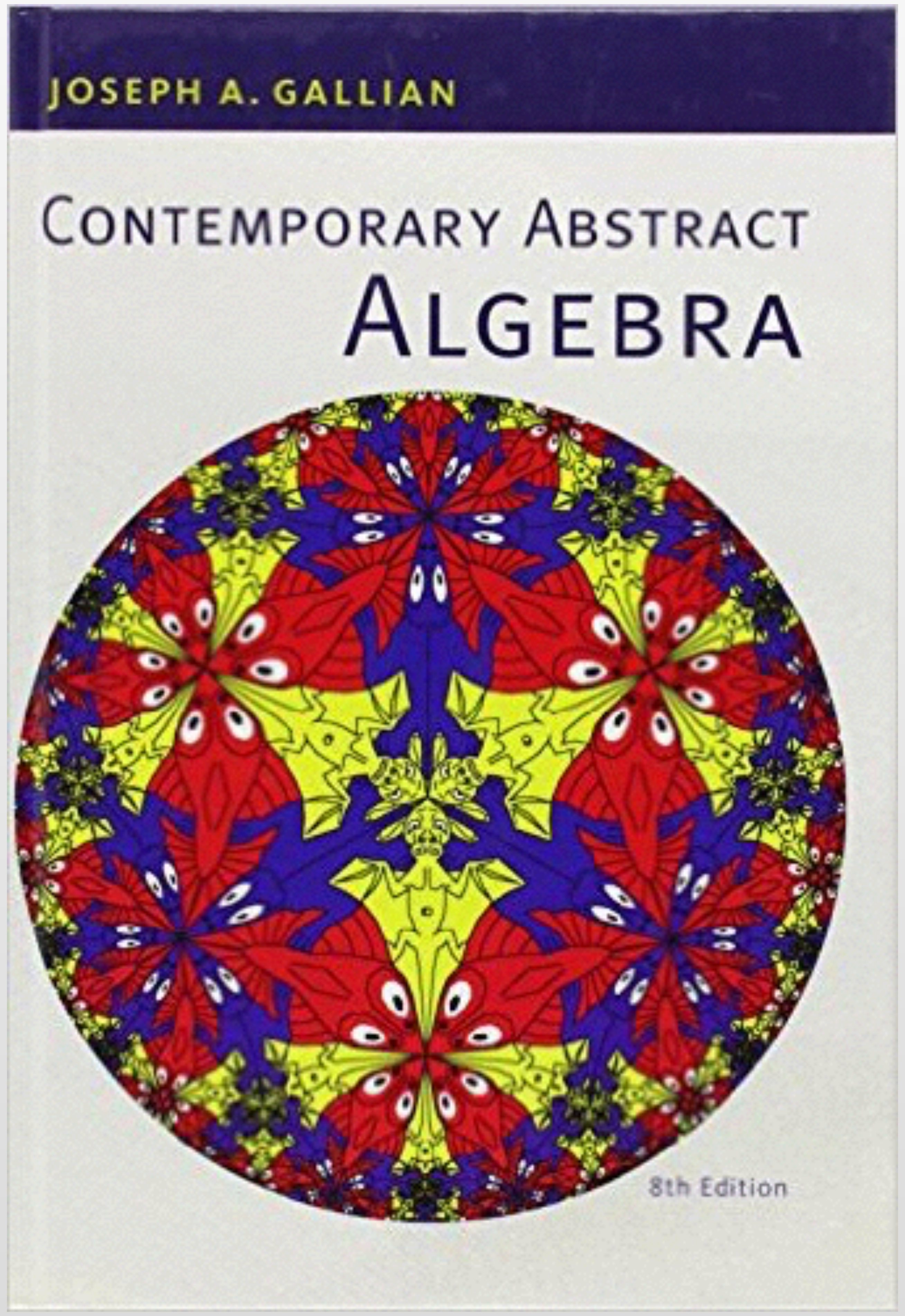


Basic Group Theory

If G is a group, and $H \subseteq G$ is a subgroup then G/H is a quotient group.

$$\boxed{\text{If } \Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}} \Rightarrow \Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}} \text{ is a quotient group.}}$$

Conway and Sloane: $\Lambda/M\Lambda$ is a quotient group.



Joseph A. Gallian, *Contemporary Abstract Algebra*, 2012


Sublattice (subgroup)
$\Lambda_{\mathrm{s}} = 2A_2$


subgroup
group but
not subgroup
Lattice (group) $\Lambda_c = A_2$

$+$	$\mathbf{c}_0$	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
$\mathbf{c}_0$	$\mathbf{c}_0$	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
$\mathbf{c}_1$	$\mathbf{c}_1$	$\mathbf{c}_0$	$\mathbf{c}_3$	$\mathbf{c}_2$
$\mathbf{c}_2$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_0$	$\mathbf{c}_1$
$\mathbf{c}_3$	$\mathbf{c}_3$	$\mathbf{c}_2$	$\mathbf{c}_1$	$\mathbf{c}_0$

4 cosets. Coset containing $\mathbf{c}_0$, $\mathbf{c}_1$,$\mathbf{c}_2$,$\mathbf{c}_3$.

The Subgroup Condition

- Shaping lattice Λ_{s} has generator matrix G_{s}.
- Coding lattice Λ_{c} has check matrix H_{c}.

Lemma Let Λ_{s} have an all-integer generator matrix G_{s}. $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$ if and only if $H_{\mathrm{c}} G_{\mathrm{s}}$ is a matrix of integers.

- Simple test for $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}}$.
- If $\Lambda_{\mathrm{s}} \subseteq \Lambda_{\mathrm{c}} \Rightarrow$ quotient group $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$ exists, and is a candidate for physical layer network coding.

The Subgroup Condition: Example

- dimension $n = 8$ coding lattice Λ_{c} is LDLC-style
- shaping lattice Λ_{s} on D_4

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 1 \end{bmatrix}}_{\text{coding: LDLC } H_c} \cdot \underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 8 \end{bmatrix}}_{\text{shaping } 4D_4 \times 4D_4} = \begin{array}{c}\text{matrix} \\ \text{of} \\ \text{integers}\end{array}$$

- Condition is satisfied. Thus $\Lambda_{\mathrm{c}}/\Lambda_{\mathrm{s}}$ is a quotient group.

Encoding Λ_c/Λ_s

Encoding is mapping information to lattice points Λ_c/Λ_s.

For Conway and Sloane, indexing $\Lambda/M\Lambda$ is easy:

$$\{0, 1, \cdots, M-1\}^n \to \Lambda/M\Lambda$$

For Λ_c/Λ_s satisfying the subgroup condition

1. If coding check matrix H_c is triangular, then indexing is also easy
2. If H_c is full, then indexing is harder.

1. Encoding When Hc is Triangular

g_{ii} are diagonal elements of G_s, h_{ii} are diagonal elements of H_c, then:

$$\text{information is } \{0, 1, \cdots, g_{ii}h_{ii}\}$$

and encoding info to $\Lambda_\mathrm{c}/\Lambda_\mathrm{s}$ is straightforward.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 1 \end{bmatrix}}_{\text{coding } H_c} \cdot \underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 8 \end{bmatrix}}_{\text{shaping } G_\mathrm{s}}$$

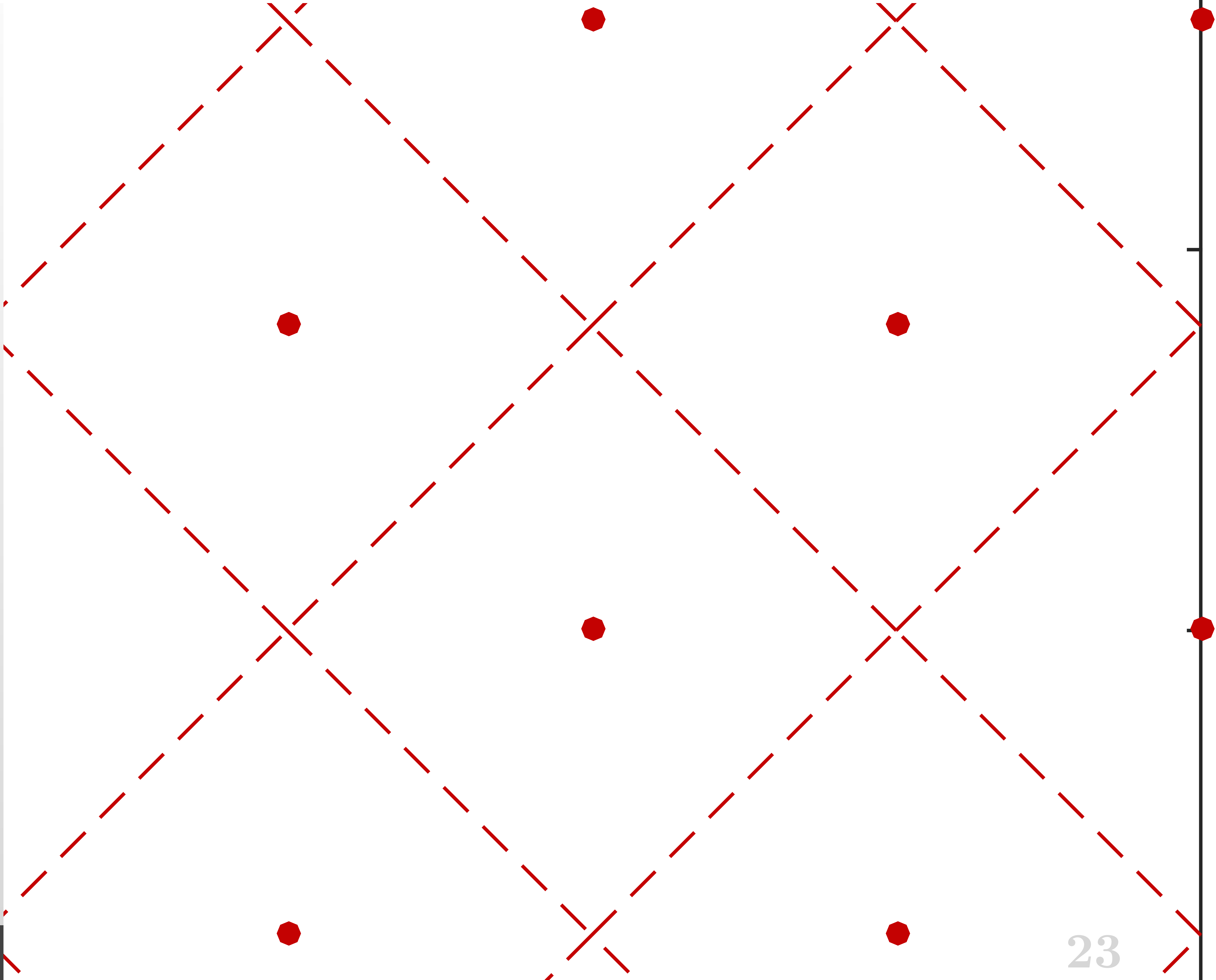
Encoding Lattice Codes, Conway and Sloane Style

Easy when $\Lambda_\mathrm{s} = M\Lambda_\mathrm{c}$ (Conway and Sloane 1983). Example:

$$G_\mathrm{s} = \begin{bmatrix} 4 & 0 \\ 4 & 8 \end{bmatrix} (\Lambda_\mathrm{s})$$

$$G_\mathrm{c} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} (\Lambda_\mathrm{c})$$

$\Lambda_\mathrm{s} = 4\Lambda_\mathrm{c}$ nested lattice code



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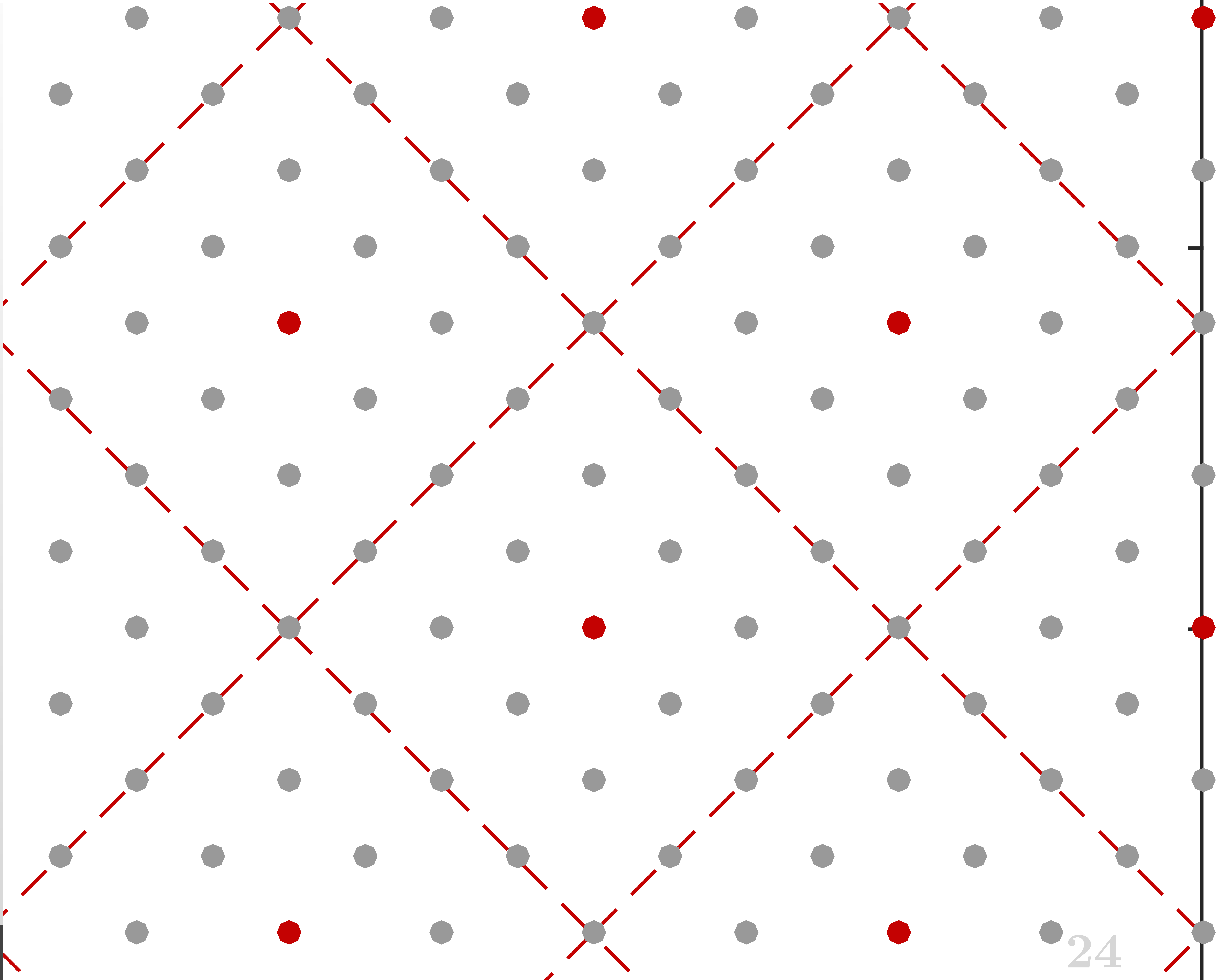
$\Lambda_s = 4\Lambda_c$ nested lattice code

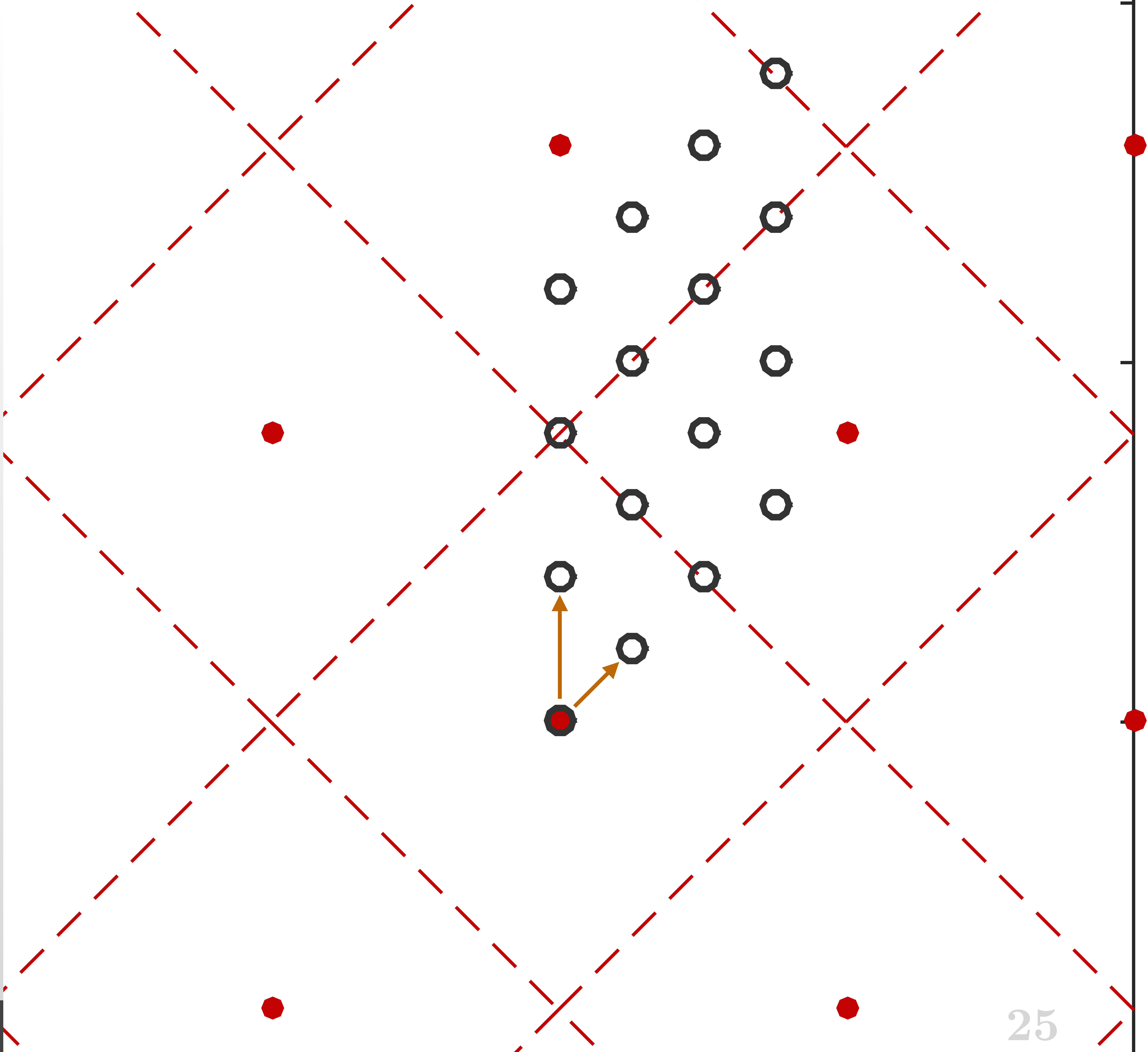


Information is $b_i \in \{0, 1, 2, 3\}$,

Indexing Step 1:

$$G\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

(clearly these points form coset representatives)

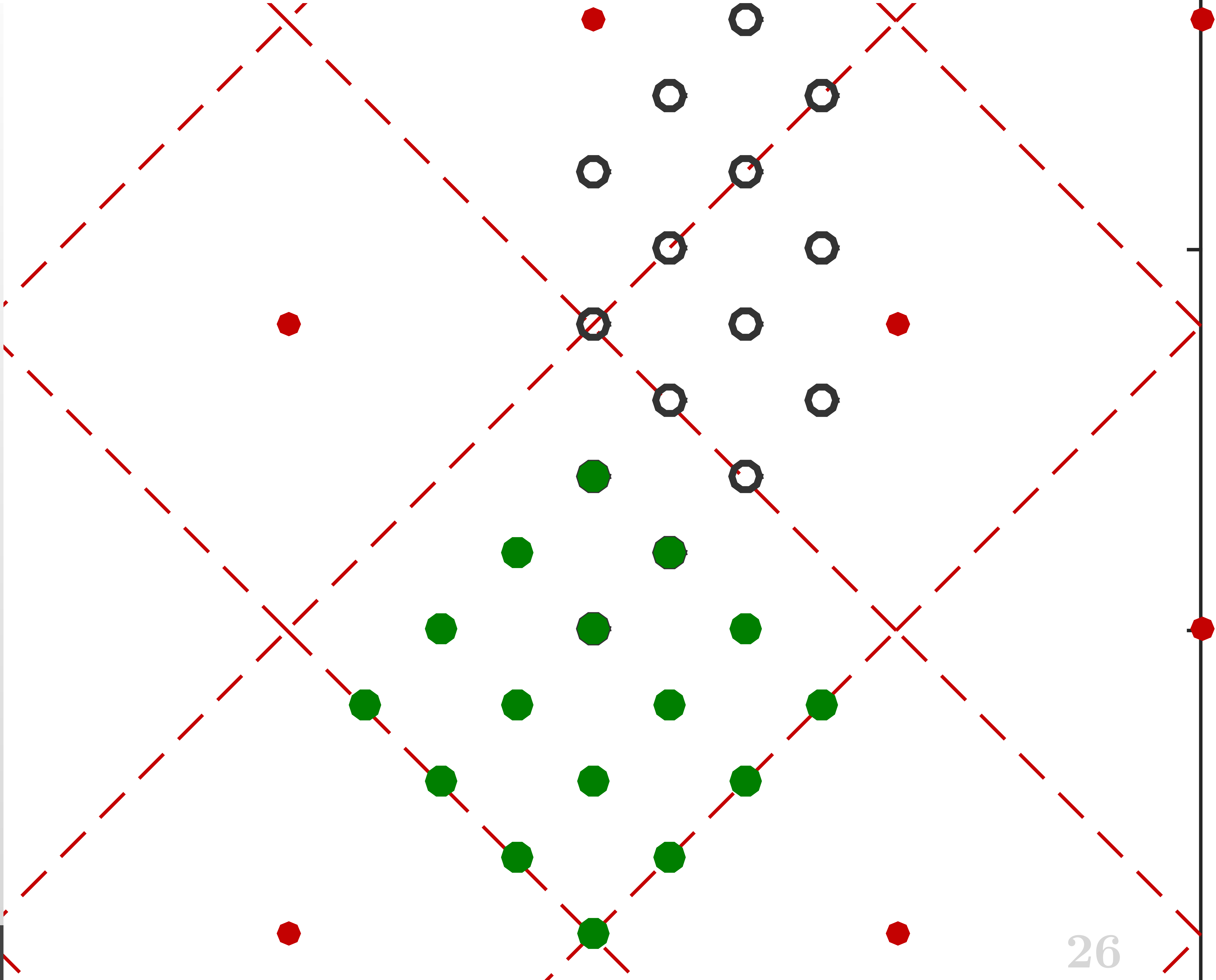


Encoding Lattice Codes, Conway and Sloane Style

Codebook is coset representatives inside Voronoi region for Λ_s around 0:

Indexing Step 2:

$$x = G\mathbf{b} - Q_{\Lambda_s}(G\mathbf{b})$$



2. Encoding When Hc is Full Matrix

Weakness of Hc triangular:

- Hc may not be available in triangular form
- triangular form reducing coding gain or rate.

If Hc full:

- Give conditions under which encoding is possible
- requires solving a diophantine equation

Encoding Lattice Codes, Generalized Voronoi Constellations

What if the lattices are not nested? Recall we want to use distinct lattices for coding and shaping.

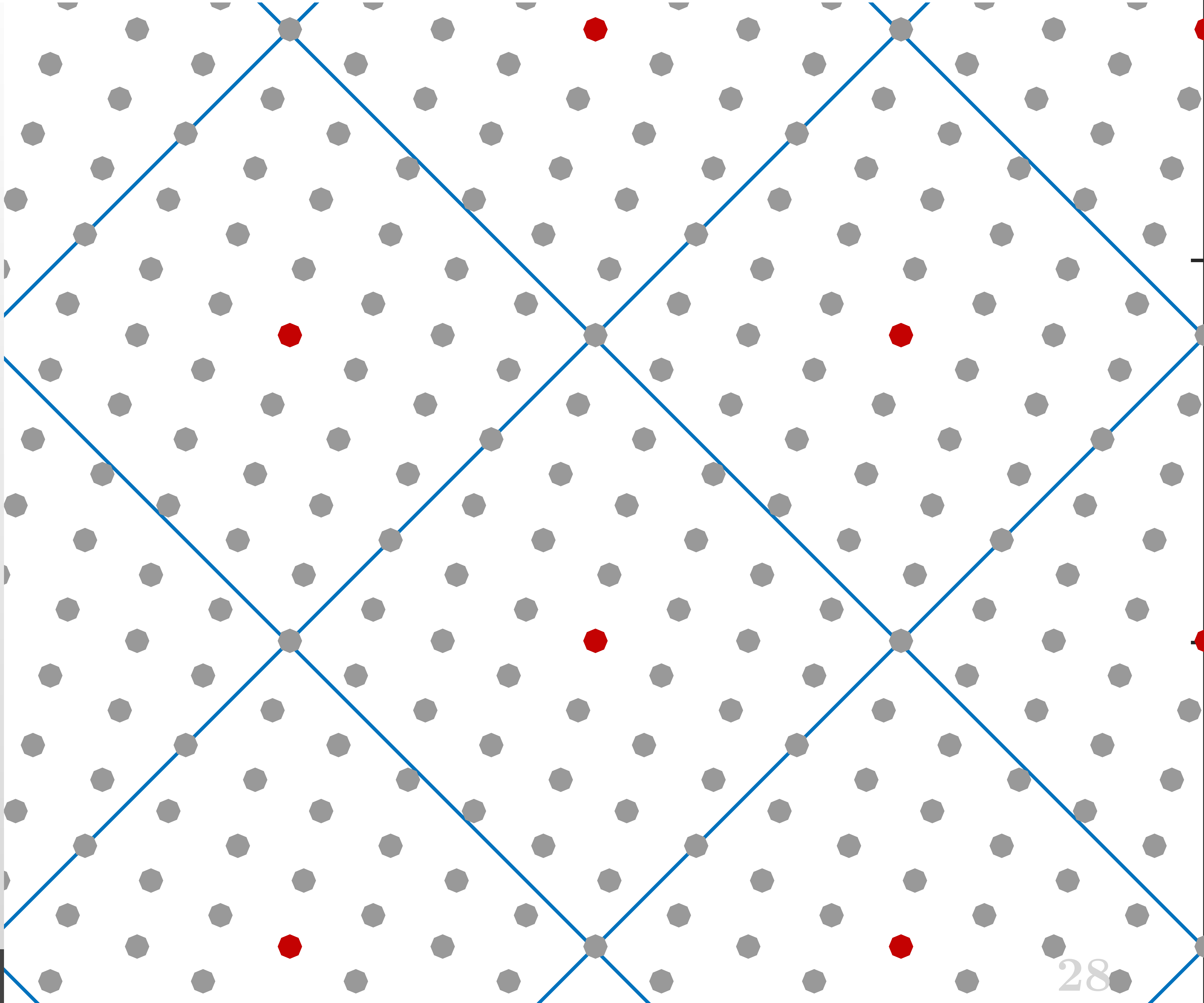
Example:

$$G_{\mathrm{s}} = \begin{bmatrix} 4 & 0 \\ 4 & 8 \end{bmatrix} (\Lambda_{\mathrm{s}})$$

$$G_{\mathrm{c}} = \begin{bmatrix} 8/9 & 2/9 \\ -4/9 & 8/9 \end{bmatrix} (\Lambda_{\mathrm{c}})$$

$$\left(G_{\mathrm{c}}^{-1} = \begin{bmatrix} 1 & -1/4 \\ 1/2 & 1 \end{bmatrix} \right)$$

Not a nested lattice code!



Encoding Lattice Codes, Generalized Voronoi Constellations

Number of codewords:

$$\frac{\det(G_s)}{\det(G_c)} = 36$$

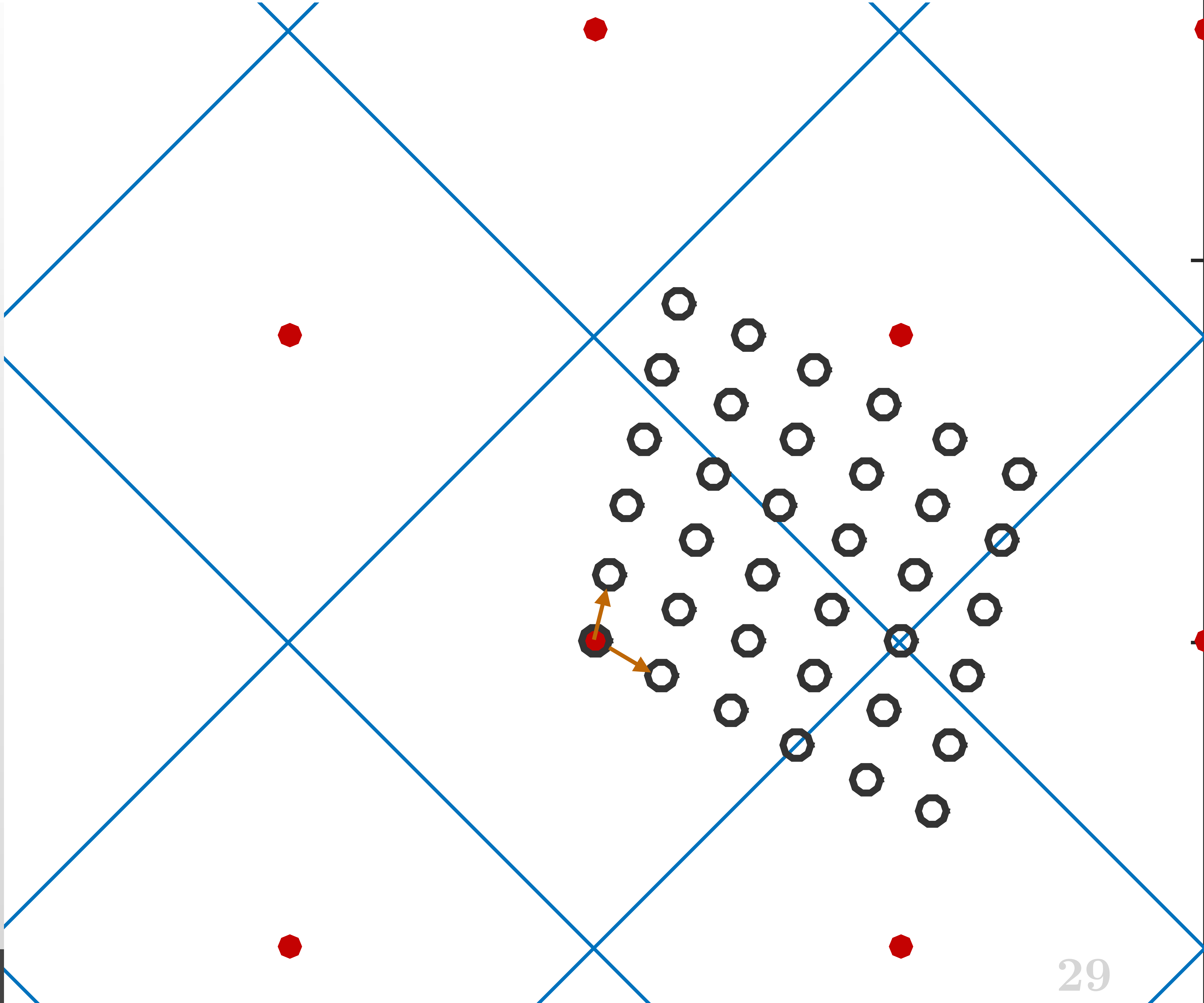
Natural candidate:

$$b_1 \in \{0,1,2,3,4,5\}$$
$$b_2 \in \{0,1,2,3,4,5\}$$

Indexing Step 1:

$$G\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Do these points form coset representatives?

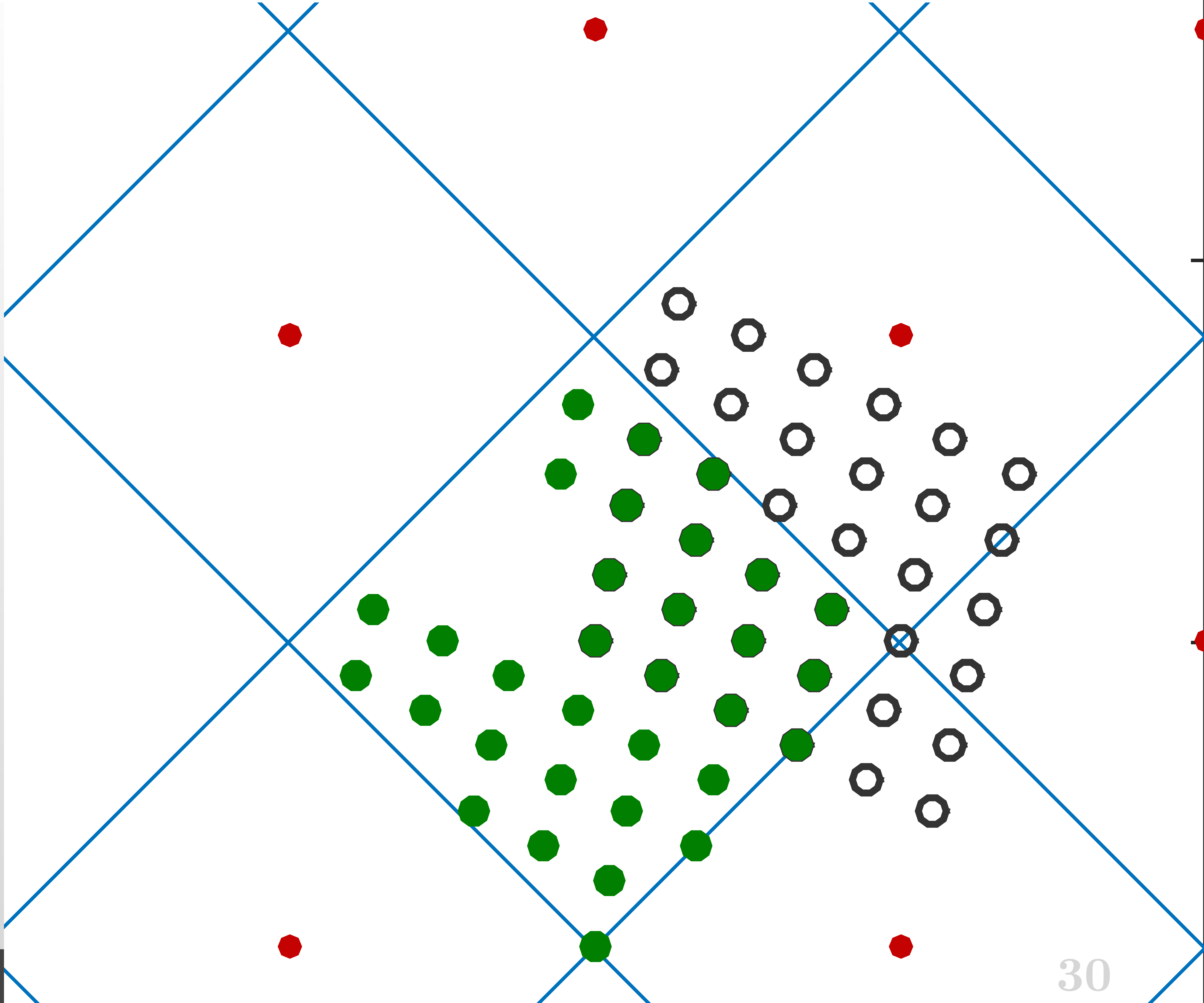


Encoding Lattice Codes, Generalized Voronoi Constellations

Indexing Step 2:

$$x = G\mathbf{b} - Q_{\Lambda_\mathrm{s}}(G\mathbf{b})$$

No! Coset representatives not formed.



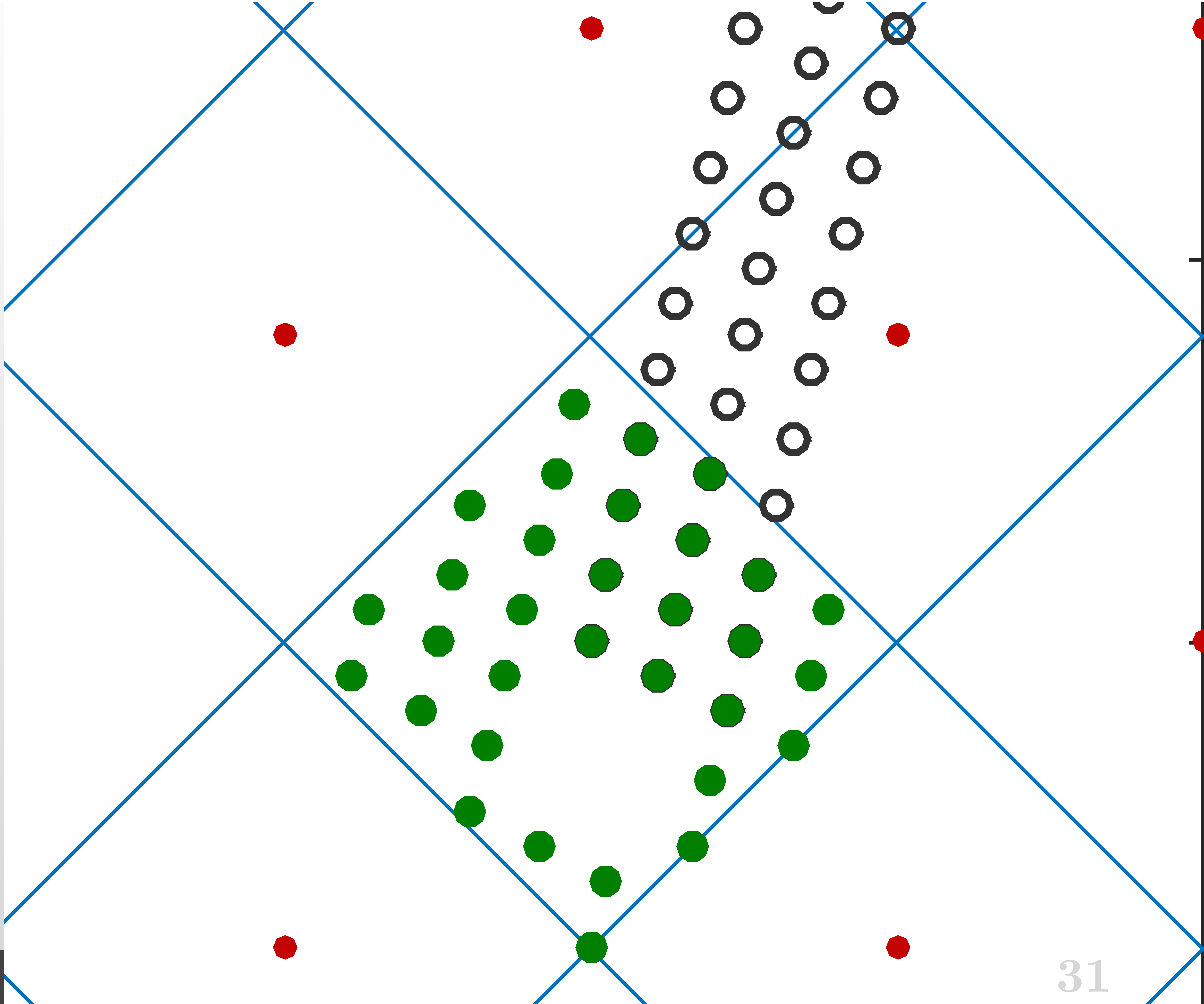
Encoding Lattice Codes, Generalized Voronoi Constellation

Another candidate:

$$b_1 \in \{0, 1, 2\}$$
$$b_2 \in \{0, 1, 2, 3, \ldots, 11\}$$

Still, no coset representatives found

What about a change of basis for G_{c}?



Finding a Basis Suitable for Encoding

Transform the basis of Λ_c from G_c to G'_c should "align" with Λ_s.

Basis transformation:

$\mathbf{g}_i$ from shaping lattice

$$G'_\mathrm{c} = G_\mathrm{c} W$$

where W is has integer entires and $\det W = 1$.

New basis is:

$$G'_\mathrm{c} = \begin{bmatrix} \frac{\mathbf{g}_1}{M_1} & \frac{\mathbf{g}_2}{M_2} & \cdots & \frac{\mathbf{g}_{n-1}}{M_{n-1}} & \mathbf{q} \end{bmatrix}$$

where $\mathbf{q}$ is some vector to be found.

Finding a Basis Suitable for Encoding

Find basis transformation W:

$$G_c' = G_c \cdot W$$
$$(G_c)^{-1} \cdot G_c' = W$$

linearly dependent

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1,n-1} & z_1 \\ w_{21} & w_{22} & \cdots & w_{2,n-1} & z_2 \\ & \vdots & & & \\ w_{n,1} & w_{n,2} & \cdots & w_{n,n-1} & z_n \end{bmatrix}$$

Then $\det W = 1$ is a **linear diophantine equation** in $z_1, z_2, \ldots, z_n$.

Example

$$G_c^{-1} \cdot G_c' = W$$

$$W = \begin{bmatrix} 1 & -1/4 \\ 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 & q_1 \\ 4/3 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 2 & z_2 \end{bmatrix}$$

$$\det W = 1$$

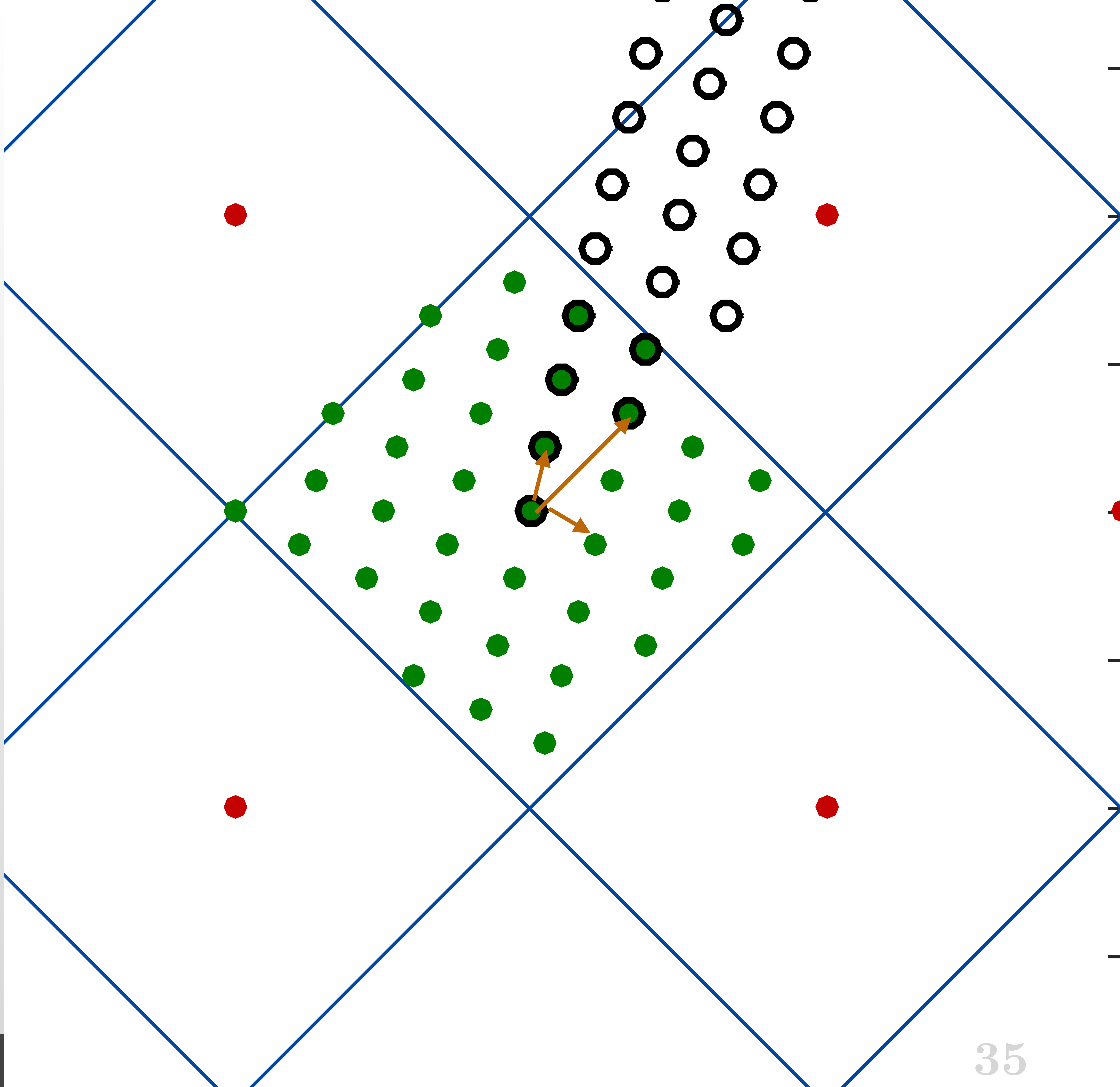
$$1z_2 - 2z_1 = 1 \qquad \leftarrow \text{diophantine equation}$$

$$\{z_1, z_2\} = \{0, 1\} \qquad \leftarrow \text{one of many solutions}$$

Encoding Non-Nested Lattice Codes Using a Suitable Basis

$$\begin{bmatrix} 1 & -1/4 \\ 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 & q_1 \\ 4/3 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 2 & z_2 \end{bmatrix}$$

$\det W = 1 \Rightarrow 1z_2 - 2z_1 = 1$ has numerous solutions.



Discussion

Shaping gain means designing the codebook to have a sphere-like shape, to approximate the Gaussian input distribution of the AWGN channel

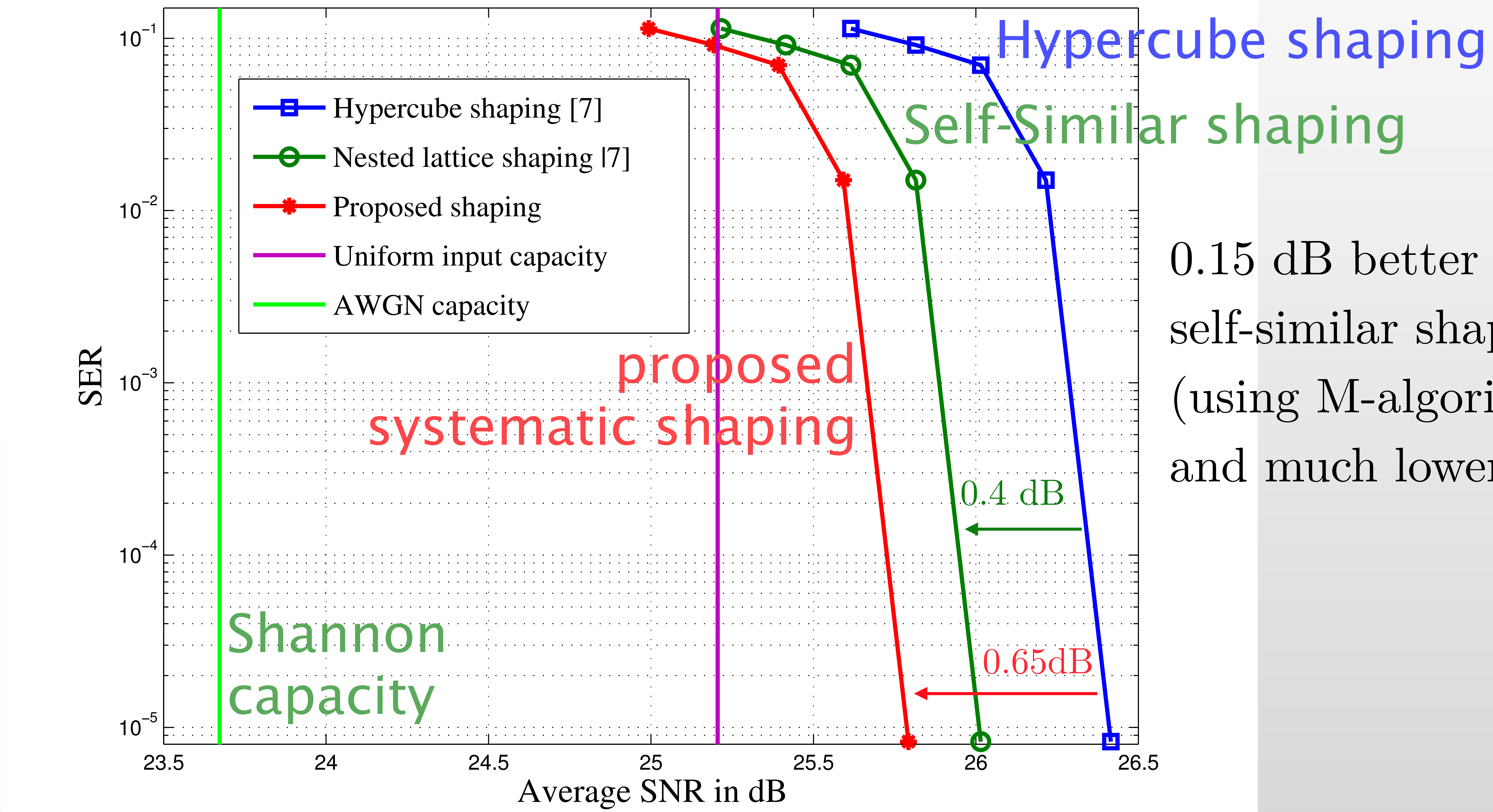
For coding lattice (LDLC, Constuction A LDPC, etc) triangular matrix:

- No shaping/cubic is easy but no shaping gain.
- Shaping using E8, BW16, etc. is also easy. Gain 0.65 to 0.86 dB
- A little bit of effort gives a big gain!
- But, triangular matrix lattices may not perform as well

For a general (non-triangular) matrices

- Shaping if we can solve a linear diophantine equation,
- Currently those coefficients get large quickly in n, practical solutions still needed

LDLCs: 0.65 dB Gain Over Hypercube



0.15 dB better than self-similar shaping (using M-algorithm) and much lower complexity