CRC-Enabled Lattices for Multiuser Communication

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February 16, 2023
2023 Information Theory and Applications (ITA) Workshop

Motivation

Cyclic redundancy check (CRC) codes are widely used in communications

- If a CRC code check passes, received data is assumed valid, and is forwarded for further processing
- If check fails, the receiver requests retransmission, or other corrective action

In lattice-based communications:

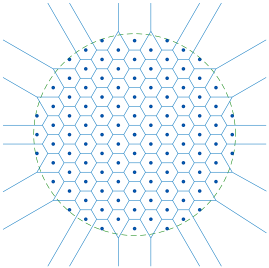
- Decoder has one or more parameters
- If lattice decoding fails, **try decoding again with a different parameter**
- Beneficial if retransmission is more expensive than re-try decoding.

We are particularly interested in the finite-length domain.

Outline

1. Background on lattices for communications
2. Genie-aided decoder for computing a bound on probability of decoder error for point-to-point AWGN channel
3. CRC-enabled lattices for the point-to-point channel
4. CRC-enabled lattices for compute-forward multiple access

Maximum Likelihood Decoding of Lattices

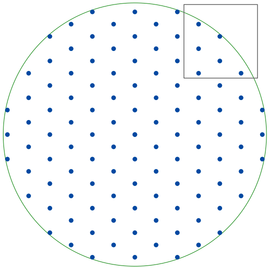


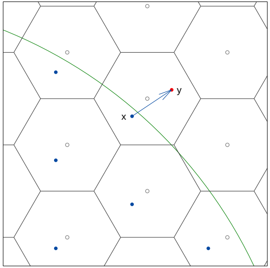
An n-dimensional lattice code is the intersection of a lattice Λ and a shaping region:

- Candidate for high-rate coded modulation. Practical to achieve shaping gain.
- Lattice codes with maximum likelihood decoding achieve the capacity of the AWGN channel [1]
- Unfortunately, Maximum likelihood decoding of lattices is not efficient

[1] Rudiger Urbanke and Bixio Rimoldi, "Lattice codes can achieve capacity on the AWGN channel", IT Trans, 1998

Motivation — Lattice Decoding Achieves Capacity





- Lattice codes with low-complexity lattice decoding can also achieve capacity
- **Lattice decoding** This is possible if scale by α_{MMSE} [2] :

$$\alpha_{MMSE} = \frac{P}{P + \sigma^2}$$

- Intuition: noise moves $\mathbf{y}$ away from the origin, so "expand" the decoding lattice

[2] Uri Erez and Ram Zamir, "Achieving $\frac{1}{2}\log(1+\mathrm{SNR})$ on the AWGN Channel With Lattice Encoding and Decoding," IT Trans, October 2004.

Scaling for Finite-Dimension Lattices

α_{MMSE} is optimal when $n \to \infty$

- How to choose scaling α when n is finite?
- Actually, α_{MMSE} is not a bad choice[3] for $n \geq 100$

What if we could try many different α? Introduce a **genie-aided decoder**

- Genie tells decoder if estimated lattice point is correct or not
- If not, decoder retries decoding using different scaling α
- Genie may be implemented using CRC codes

Goal: quantify how much the decoding can be improved by re-try decoding.

[3] N. S. Ferdinand, M. Nokleby, B. M. Kurkoski, and B. Aazhang, "MMSE scaling enhances performance in practical lattice codes," in Asilomar Conference on Signals, Systems, and Computers, November 2014

Background — Lattices

Definition (Lattices)

Lattices are additive subgroup in real number space. An n-dimension lattice Λ can be formed as:

$$\Lambda = \{\mathbf{G}\mathbf{b} : \mathbf{b} \in \mathbb{Z}^n\}$$

where $\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n] \in \mathbb{R}^{n \times n}$ and $\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n \in \mathbb{R}^n$ are linear independent. Voronoi region $\mathcal{V}$ is the set of points closer to $\mathbf{0}$ than any other lattice point, with volume:

$$V(\Lambda) = |\det(\mathbf{G})|$$

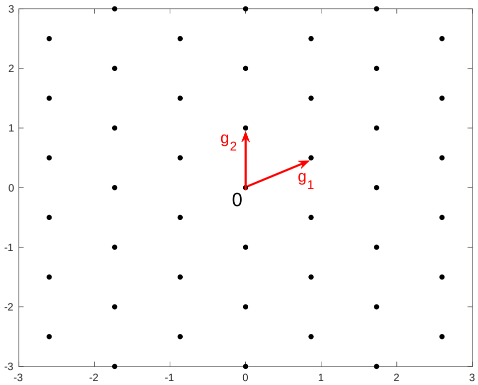


Figure 1: Example of 2-dimensional lattice spanned by $\mathbf{g}_1 = [\frac{\sqrt{3}}{2}, \frac{1}{2}]^T$ and $\mathbf{g}_2 = [0, 1]^T$.

Background — Nested Lattice Codes

Let two lattices Λ_c and Λ_s satisfy $\Lambda_s \subseteq \Lambda_c$ and form a quotient group Λ_c/Λ_s. A nested lattice code $\mathcal{C}$ is defined as:

$$\mathcal{C} = \{\mathbf{x} \bmod \Lambda_s : \mathbf{x} \in \Lambda_c\}$$

have code rate:

$$R = \frac{1}{n} \log \frac{V(\Lambda_s)}{V(\Lambda_c)}.$$

Nested lattice codes:

- Allow lattices to satisfy a power constraint
- Possess certain algebraic properties

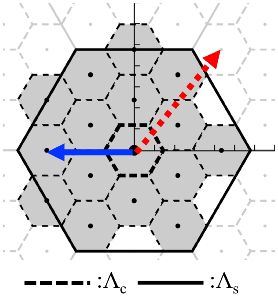
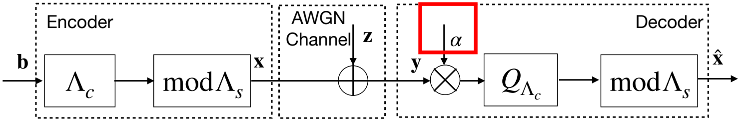


Figure 2: Nested lattice code formed using a coding lattice Λ_c and a shaping lattice Λ_s.

Background—Single-User System

Encoding and decoding scheme is given as:



Let n-dimensional lattice code $\mathcal{C} = \Lambda_c/\Lambda_s$ and $\mathbf{G}$ is generator matrix of Λ_c,

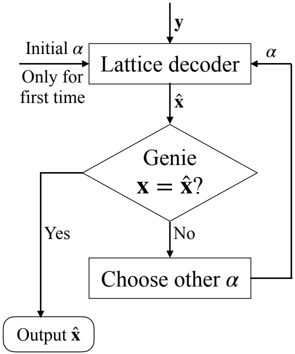
Encoder: $\mathbf{x} = \mathbf{G}\mathbf{b} \bmod \Lambda_s$, with $E[\|\mathbf{x}\|^2] = nP_x$

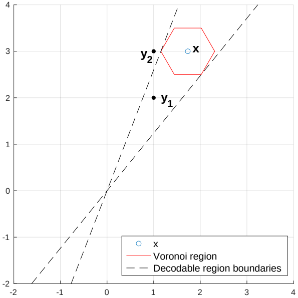
Channel: $\mathbf{y} = \mathbf{x} + \mathbf{z}$, with $\mathbf{z} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$

Decoder: $\hat{\mathbf{x}} = Q_{\Lambda_c}(\alpha\mathbf{y}) \bmod \Lambda_s$, with $\alpha \in \mathbb{R}$

Genie-Aided Decoder

The genie-aided exhaustive search decoder is allowed to use all $\alpha \in \mathbb{R}$





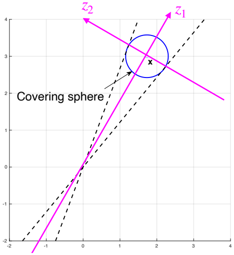
Cone region is the “decodable region”.

There exists some α such that $\mathbf{y}_1$ will be correctly decoded.

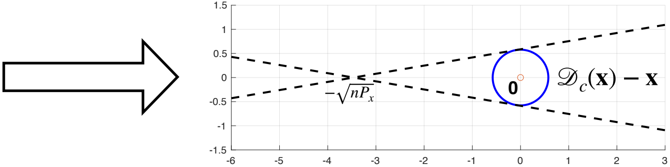
Idea: Bound the probability of decoding error by integrating noise over the cone.

Probability of Decoding Error: Lower Bound and Estimate

(1) Form a lower bound on probability of decoding error using the lattice **covering sphere**. Using this approximation because the Voronoi region is irregular.



※Draw coordinate of $\mathbf{z}$ based on rotationally invariant property.



(2) Form an estimate on the probability of decoder error using the lattice **equivalent sphere**. While not a bound, it is numerically close for some lattices.

Lower Bound on Error Rate

Theorem (1)

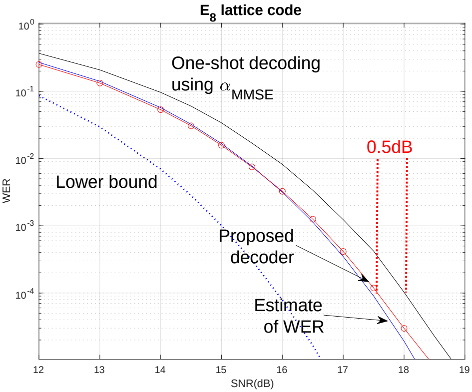
Let non-zero $\mathbf{x}$ be a lattice point of an $n \geq 2$ dimensional lattice Λ having covering radius r_c and per-dimensional power $P_{\mathbf{x}} = \|\mathbf{x}\|^2/n$. With the restriction of $r_c^2 < nP_{\mathbf{x}}$, the probability of word error for the genie-aided decoder on the AWGN channel with noise variance σ^2 is lower bounded by:

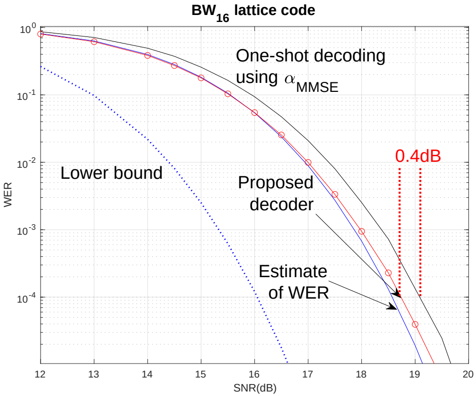
$$P_{e,Dec} > 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} (1 - h(z)) dz$$

where[1] $h(z) = e^{-t} \left(\sum_{k=0}^{(n-3)/2} \frac{t^k}{k!} \right)$ for odd n;
$h(z) = erfc(t^{1/2}) + e^{-t} \left(\sum_{k=1}^{(n-2)/2} \frac{t^{k-1/2}}{(k-1/2)!} \right)$ for even n, with $t = f^2(z)/(2\sigma^2)$
and $f(z) = \left| r_c z / \sqrt{nP_{\mathbf{x}} - r_c^2} + \sqrt{nP_{\mathbf{x}} r_c^2/(nP_{\mathbf{x}} - r_c^2)} \right|$.

[1] Tarokh, Vardy and Zeger, "Universal Bound on the Performance of Lattice Codes", IT Transactions, 1999

Numerical Evaluation of Bound for E8 and BW16 Lattices





- Genie-aided decoder gives gain over one-shot decoding
- Estimate of error-rate is quite accurate

Implementation of Genie Using CRC

- The genie may be implemented using a CRC code.
- In a typical communications system, failed CRC is used to request re-transmission.
- In this case, a failed CRC is used to adjust α. Repeating decoding is more efficient than re-transmission.

CRC-Enabled Lattice Code Λ'

Embed a codeword of $\mathcal{C}_b$ in the LSB of the lattice integer $\mathbf{b}$, where $\mathcal{C}_b$ is a block code of length n.

The original lattice Λ is:

$$\Lambda = \{\mathbf{G}\mathbf{b} \mid \mathbf{b} \in \mathbb{Z}^n\}$$

The CRC-enabled structure Λ' is:

$$\Lambda' = \{\mathbf{G}\mathbf{b} \mid \mathbf{b} \in \mathbb{Z}^n, \mathbf{b}_{\mathrm{LSB}} \in \mathcal{C}_b\}$$

$$\begin{array}{cccccc} b_1 & b_2 & b_3 & & b_n & \\ \boxed{1} & \boxed{1} & \boxed{1} & \cdots & \boxed{1} & 2^2 \\ \boxed{0} & \boxed{1} & \boxed{1} & \cdots & \boxed{1} & 2^1 \\ \boxed{1} & \boxed{1} & \boxed{0} & \cdots & \boxed{0} & 2^0 \end{array}$$

codeword of $\mathcal{C}$

Theorem

If $\mathcal{C}_b$ is an (n, k) linear block code then Λ' is a lattice.

We use a CRC as the linear block code $\mathcal{C}_b$.

CRC-Enabled Lattice Code Λ'

Λ' is a lattice with generator matrix:

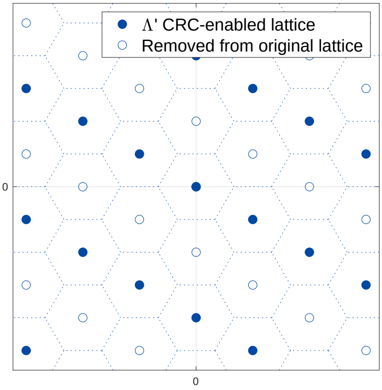
$$\mathbf{G}' = \mathbf{G} \cdot \underbrace{\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{P} & 2\mathbf{I}_{n-k} \end{bmatrix}}_{\mathrm{CA}}$$

CA is the Construction A generator matrix for $\mathcal{C}$.

Volume is $V(\Lambda') = 2^{n-k} V(\Lambda)$.

With N bits/codeword on AWGN channel, we must pay:

$$\mathrm{Rate\ penalty} = 10 \log \frac{N}{N-(n-k)} \ \mathrm{dB}$$



Original Λ is A_2 hexagonal lattice.

Using single-parity check code $\mathcal{C}_b$, points are removed to form Λ'.

Polar Code Lattices Using CRC-Enabled Genie

- Dimension $n = 128$ polar code lattice[1].
- $R = 1.74$ with 223 bits per codeword; 4 CRC bits.
- 3 decoding attempts.
- (SC decoding in standard Construction D multilevel decoder)
- 0.1 dB improvement in WER.
  - Includes SNR penalty 0.078 dB due to CRC bits

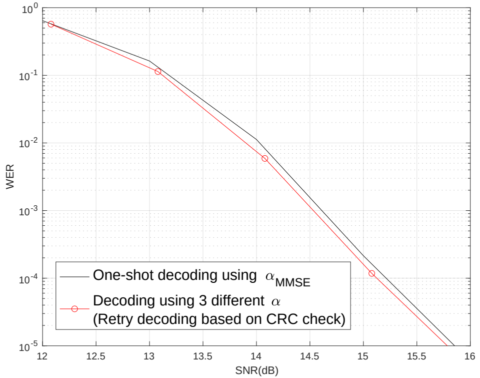


Figure 3: SNR vs WER of 128-dim polar code lattice with one-shot decoding and 3-times decoding.

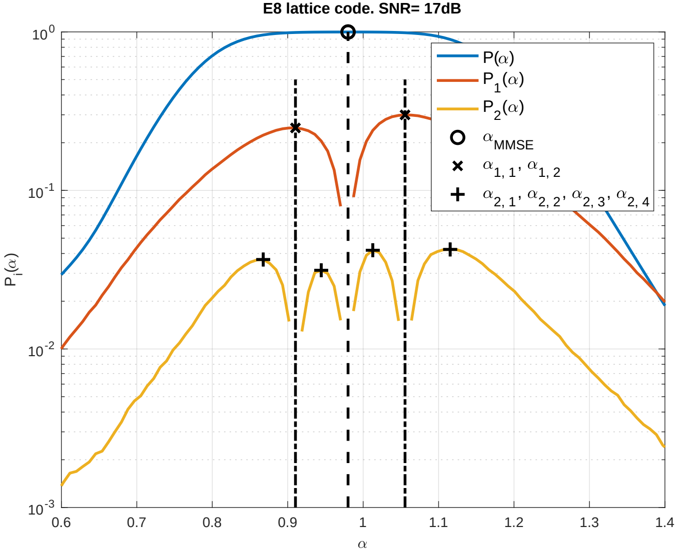
[1] Ludwiniananda, Liu, Anwar, and Kurkoski, "Design of polar code lattices of small dimension," ISIT 2021.

α for Successive Attempts

On the first decoding attempt, choose $\alpha = \alpha_{\text{MMSE}}$.

If we need to re-attempt, which value of α should be used?

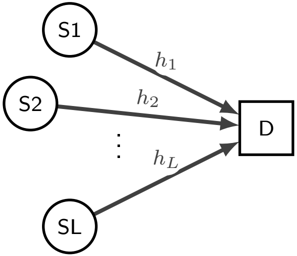
$P_1(\alpha)$ is the probability of correct on second try, given first try failed.
$\Rightarrow$ The local optimums $\alpha_{1,1}, \alpha_{1,2}$ are next candidates.



Multiple-Access Using Compute-Forward Relaying[1]

L transmitters transmit $\mathbf{x}_i \in \mathcal{C}$ over channel with fading coefficient h_i.

One receiver with one linear combination shown; need L independent linear combinations.



$$h_1\mathbf{x}_1 + \cdots h_L\mathbf{x}_L + \mathbf{z} \qquad h \in \mathbb{R}$$

decoder should produce

$$a_1\mathbf{x}_1 + \cdots a_L\mathbf{x}_L \in \Lambda \qquad a_i \in \mathbb{Z}$$

Receiver must determine $a_i \in \mathbb{Z}$. Usual strategy is to maximize computation rate:

$$\mathbf{a} = \arg\max \log^+ \left(||\mathbf{a}||^2 - \frac{P|\mathbf{h}^t\mathbf{a}|^2}{\sigma^2 + P||\mathbf{h}||^2}\right)$$

a_i is considered to be an integer approximation of real h_i.

[1] Nazer and Gastpar, "Compute-and-forward: Harnessing interference through structured codes," IT Trans 2011

Linear Combinations of CRC-Enabled Lattice

For the unconstrained lattice, the linear combinations preserve the CRC:

Theorem

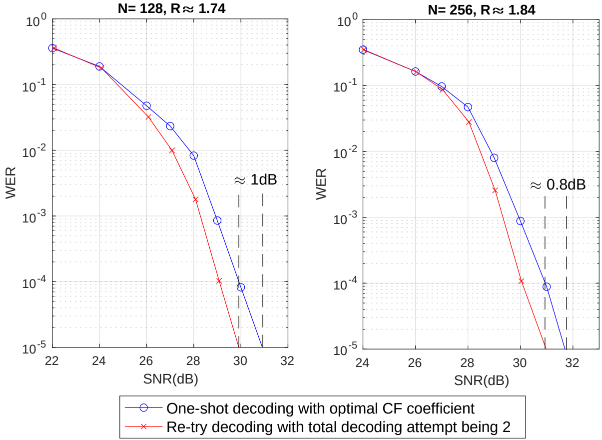
Let $\mathbf{x}_i \in \Lambda'$. Then for any $a_i \in \mathbb{Z}$, the linear combination $\sum_{i=1}^{L} a_i \mathbf{x}_i \in \Lambda'$.

However, for a nested lattice code, there is a minor restriction on the lattice code design for linearity to hold:

Theorem

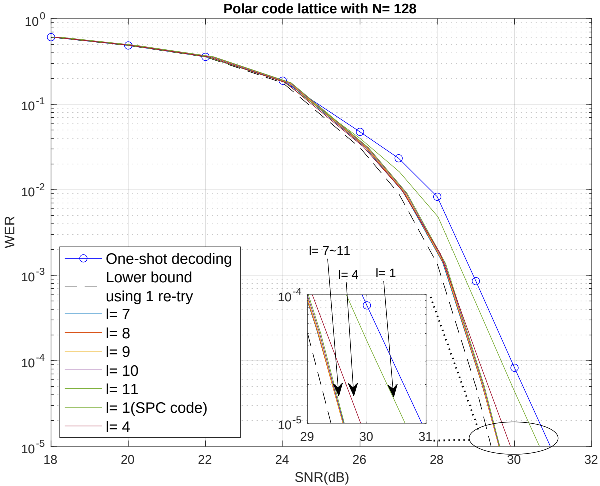
For a nested lattice code $\mathcal{C} = \Lambda'/\Lambda_s$, let Λ' and Λ_s have generator matrices $\mathbf{G}_c$ and $\mathbf{G}_s$, respectively. If $\mathbf{G}_c^{-1}\mathbf{G}_s$ has only even integers, then $\sum_{i=1}^{L} a_i \mathbf{x}_i \in \mathcal{C}$.

Compute-Forward with CRC-Enabled Lattice Codes



- $\mathbf{a}_0$ and $\mathbf{a}_1$ give the maximum and second maximum computation rate.
- 2 decoding attempts using $\mathbf{a}_0$, then $\mathbf{a}_1$.
- 1.0 dB gain with $n = 128, R = 1.74$ polar code lattice
- 0.8 dB gain with $n = 256, R = 1.84$ polar code lattice
- 4 bits CRC.

Optimized CRC Length



- The WER after retry decoding can be estimated analytically as a function of CRC length l.
- CRC length can be optimized by combining the estimated WER after retry decoding and SNR penalty due to CRC bits.
- Gain increased to 1.3 dB for CRC length of $l = 7$ to 11 ($n = 128$ polar code lattice).

Conclusion

Lattice-based decoders have asymptotically-optimal parameters,

but in the finite-length domain, retrying decoding with other parameters can improve error-rate performance.

For the point-to-point channel:

- For genie-aided decoding and CRC-enabled lattices, the benefit of retry decoding is greatest for small dimension n.
- But at small n, the rate penalty due to the CRC is significant.
- For example, an $n = 128$ polar code lattice has a gain of 0.1 dB

For the two-user multiple-access channel using compute-forward:

- Retry decoding allows using second-best $\mathbf{a}$ when first-best fails.
- Shows a benefit of 1.3 and 0.8 dB for $n = 128$ and 256 polar code lattices.