Isomorphism via translation

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ABSTRACT. We observe that the known fact that the difference logic and the hybrid logic with universal modality have the same expressive power on Kripke frames can be strengthened for a far wider class of general frames. This observation, together with a general completeness result and some algebraic theory of closure operators, is used to show that lattices of difference logics and of hybrid logics are isomorphic.

Keywords: closure operators, difference operator, discrete frames, hybrid logic, universal modality, atomic algebras

Gargov and Goranko [13] proved that languages of difference logic $\mathcal{ML}(\mathsf{D}, \Diamond)$ and hybrid logic with universal modality $\mathcal{H}(\mathsf{E}, \Diamond)$ are equivalent with respect to frame definability. This observation was improved upon by Areces [1] who showed how to define a polynomial translation. It was suggested by Patrick Blackburn (personal communication) that these results should be strengthened to show something more. Namely, one would expect the existence of an isomorphism between the lattice of difference logics and the lattice of hybrid logics. We are going to show that weakly atomic frames — duals of atomic algebras — provide a natural tool to attack this problem. The apparatus behind the isomorphism proof is the standard algebraic theory of closure operators. In addition, weakly atomic frames also allow to generalize the correspondence results to the topological setting. To avoid notational complications, we work with the unimodal language, but virtually nothing hinges on it: all results transfer to the polymodal case.

The main results of this paper are Corollaries 17 and 18, Theorem 26 together with Corollaries 29 and 30. Theorem 6 is also of some independent interest. Corollary 18 allows for immediate transfers of known results on topological definability from $\mathcal{H}(\mathsf{E}, \Diamond)$ to $\mathcal{ML}(\mathsf{D}, \Diamond)$ and back. A recent example: Sustretov [19] has obtained a Goldblatt-Thomason-style characterization of topo-definability in $\mathcal{H}(\mathsf{E}, \Diamond)$, which by our result must be also a characterization of topo-definability for $\mathcal{ML}(\mathsf{D}, \Diamond)$.[1] In the converse direction, Kudinov [17] announced an axiomatization of the $\mathbf{K}_{\mathsf{D}}^{\mathbf{Name}}$-logic of Euclidean spaces of dimension at least 2. Hybrid translations of these axioms must then yield an axiomatization of the $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$-logic of these spaces.

The idea that translations in logics can be used to prove that certain lattices of logics are isomorphic occurred already in Kracht and Wolter's [16] improvement of Thomason's translation from polymodal logics to a subclass

[1] Thanks to Balder ten Cate for pointing out this example.

of unimodal logics. Actually, in their survey work [15] on modal translations and simulations, the authors mention the original result of Gargov and Goranko. However, they do not discuss the possibility of lifting it to an isomorphism between lattices of logics or otherwise put in on equal footing with other translations discussed in that paper. What they say, instead, is that *both nominals and the difference [operator] are rather nonstandard devices which work fine on Kripke structures but present special problems for generalized frames.* [15] One of the main purposes of this note is to show that those special problems are not impossible to overcome and it is possible to treat both formalisms in a general mathematical framework. More generally, the presence of *non-orthodox* (or *non-structural*) rules is not necessarily an impenetrable barrier for algebraic methods. There is more to universal algebra than varieties, quasi-varieties, structural rules and structural closure operators. Finally, our results make clear that the theory of closure operators can and should be applied to translations and embeddings between *classes* of logics.

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1 Languages and semantics

1.1 Weakly atomic frames

We are going to consider two formalisms extending the one of basic modal logic. The first is *the hybrid language with the universal modality*, obtained by extending the basic modal language with an infinite set of nominals $NOM = \{i, j, \dots\}$ and a new modal operator E. The formulas of this language are generated by the following recursive definition:

$$\phi ::= \top \mid p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid \mathsf{E}\phi,$$

where p is a proposition letter and i is a nominal. It is usually assumed that the set of nominals NOM, as well as the set of proposition letters $PROP$, is countable. $VAR := PROP \cup NOM$. The second extension arises by adding *the difference operator* D, i.e., its syntax is:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid \mathsf{D}\phi.$$

The set of hybrid formulas is denoted by $\mathcal{H}(\mathsf{E}, \Diamond)$. The set of difference formulas is denoted by $\mathcal{ML}(\mathsf{D}, \Diamond)$. To improve readability, we drop references to VAR and $PROP$. For every formula ϕ, $Sub(\phi)$ denotes the set of its subformulas. The remaining connectives — $\bot$, $\vee$, $\rightarrow$, $\Box$, A — are defined as abbreviations in a standard way. Define also $\overline{\mathsf{D}}p := \neg\mathsf{D}\neg p$, $\mathsf{O}p := p \wedge \neg\mathsf{D}p$.

DEFINITION 1. *A weakly atomic frame* is a structure of the form $\mathfrak{F} := \langle W, R, \mathbb{A}\rangle$, where $R \subseteq W \times W$ and $\mathbb{A}$ is a family of subsets of W closed under the Boolean operations, the operator $\Diamond_R X := \{w \in W \mid \exists x \in X.wRx\}$ s.t. for every non-empty $P \in \mathbb{A}$ there is $x \in P$ s.t. $\{x\} \in \mathbb{A}$. If $\{x\} \in \mathbb{A}$, x is called an *admissible element*. The set of admissible elements of W is denoted as $\mathsf{At}\mathbb{A}$. Similarly, members of $\mathbb{A}$ are called *admissible subsets*.

Thus, weakly atomic frames are those where every subset contains an admissible element. *A propositional valuation* is one assigning members of $\mathbb{A}$ to propositional variables, *a nominal valuation* is one assigning *admissible* members of W to nominal variables, *a (total) valuation* for hybrid logic is one which is both nominal and propositional. In the case of difference operator, a valuation is simply a propositional valuation. *A model* $\mathfrak{M}$ is a pair $\langle \mathfrak{F}, \mathfrak{V}\rangle$, consisting of a weakly atomic frame and a valuation in it. Depending on the kind of valuation, it is propositional or (total) hybrid model (we do not consider purely nominal models). For $x \in W$ and $c \in NOM \cup PROP$, we write $\mathfrak{M}, x \vDash c$ if $c \in PROP$ and $x \in \mathfrak{V}(c)$ or $c \in NOM$ and $x = \mathfrak{V}(c)$. Clauses for booleans and the modal operator are standard. For universal modality, the clause is $\mathfrak{M}, x \vDash \mathsf{E}\phi$ if $\exists w \in W.\mathfrak{M}, w \vDash \phi$. For difference modality, the clause is $\mathfrak{M}, x \vDash \mathsf{D}\phi$ if $\exists w \neq x.\mathfrak{M}, w \vDash \phi$. We write $\mathfrak{F}, \mathfrak{V} \vDash \phi$ if ϕ holds under $\mathfrak{V}$ at all points in W. If ϕ is a hybrid formula and $\mathfrak{V}$ is a propositional valuation, we write $\mathfrak{F}, \mathfrak{V} \vDash \phi$ if $\mathfrak{F}, \mathfrak{V}' \vDash \phi$ for every total valuation $\mathfrak{V}'$ whose propositional component coincides with $\mathfrak{V}$. We write $\mathfrak{F}, x \vDash \phi$ if ϕ holds at x under every valuation. Finally, $\mathfrak{F} \vDash \phi$ means that $\mathfrak{F}, x \vDash \phi$ for every $x \in W$.

Say that a model $\langle \mathfrak{F}, \mathfrak{V}\rangle$ is *weakly named for* $\mathcal{H}(\mathsf{E}, \Diamond)$ if for every formula ϕ there is a nominal i s.t. $\mathfrak{V}(i) \in \mathfrak{V}(\phi)$. A model is *weakly named for* $\mathcal{ML}(\mathsf{D}, \Diamond)$ if for every formula ϕ there is a variable p s.t. $\emptyset \neq \mathfrak{V}(p \wedge \neg \mathsf{D}p) \subseteq \mathfrak{V}(\phi)$. General frames associated with weakly named models — i.e., frames where admissible subsets of W are exactly those which are values of some formula under $\mathfrak{V}$ — are weakly atomic.

Important subclasses of atomic frames are:

- *discrete frames*: frames where every singleton is admissible;
- *full frames* or *Kripke frames*: frames where every set is admissible, i.e., $\mathbb{A} = 2^W$. In such a case, we may simply drop $\mathbb{A}$ from the signature. This is how Kripke frames are usually defined. The only non-logical constant of *first-order correspondence language* for Kripke frames is a binary constant corresponding to the accessibility relation.

If for a given family of subsets $X \subseteq 2^W$ there is $A \in \mathbb{A}$ s.t. $\bigcup X \subseteq A$ and for every $B \in \mathbb{A}$, $\bigcup X \subseteq Z$ implies $A \subseteq Z$, we call A *the supremum of* X and denote it as $\bigvee X$; observe it is not necessarily equal to $\bigcup X$.

LEMMA 2. *If* $\mathfrak{F} = \langle W, R, \mathbb{A}\rangle$ *is a weakly atomic frame, then for every admissible subset* $A \in \mathbb{A}$, $A = \bigvee A \downarrow_{\mathsf{At}}$, *where* $A \downarrow_{\mathsf{At}} := \{\{x\} \mid x \in \mathsf{At}\mathbb{A} \cap A\}$.

Proof. It is clear that $\bigcup A \downarrow_{\mathsf{At}} \subseteq A$. Now assume $\bigcup A \downarrow_{\mathsf{At}} \subseteq B$ and $A \nsubseteq B$. Then $A \wedge \neg B$ is a non-empty admissible subset. Hence, by weak atomicity it has to contain an element whose singleton is in $A \downarrow_{\mathsf{At}}$ — a contradiction. ■

The reader probably recognized this lemma as a thinly disguised version of a proof that in an atomic algebra every element can be represented as the supremum of a family of atoms. The next subsection makes this connection explicit.

1.2 Connection with algebra

This subsection is addressed to readers interested in algebra, hence we do not define basic notions of duality theory appearing here: such readers are likely to know them anyways. It is not hard to recognize that weakly atomic frames are atomic modal algebras in disguise. The condition of weak atomicity readily implies that the algebra of admissible sets is atomic. Conversely, assume that the algebra is atomic and take the descriptive frame corresponding to it. Every admissible subset contains a principal ultrafilter and singletons of principal ultrafilters are admissible: hence, the frame is weakly atomic. Thus, we can obtain a full-blown duality between descriptive weakly atomic frames and atomic algebras as a restriction of standard duality between modal algebras and descriptive frames, as discussed in Chapter 5 of Blackburn et al. [2]. The only reason why we used the name *weakly atomic frames* instead of simply *atomic frames* is that the latter was sometimes used for discrete frames.

1.3 Neighborhood frames and topological spaces

The fact that we can identify so-called (normal) neighborhood frames (or Scott-Montague semantics) with a certain subclass of weakly atomic frames follows readily from the duality theory developed by Došen [8] and the above discussion. To make the paper more self-contained, let us describe it in more detail. Let us say that a weakly atomic frame $\langle W, R, \mathbb{A}\rangle$ is *set-theoretical* if for every $X \subseteq \mathbb{A}$ there exists $\bigvee X \in \mathbb{A}$. In other words, the family of admissible sets is lattice-complete. It is thus straightforward to prove that descriptive set-theoretical frames are exactly duals of atomic and complete modal algebras. Every Kripke frame is set-theoretical, but the converse does not hold for weakly atomic frames. However, a discrete frame is set-theoretical iff it is a Kripke frame.

There is another way of representing atomic and complete modal algebras: as *Scott-Montague semantics* or *normal neighborhood frames*. Such a structure consists of a family W and a function f assigning to every element of W a filter over W. Recall that a filter is a nonempty family of sets X satisfying $A, B \in X$ iff $A \cap B \in X$. $f(x)$ is called *the family of neighborhoods of* x. The dual algebra of a neighborhood frame is the powerset algebra of W together with the operator $\Box_f A = \{x \in W \mid A \in f(x)\}$.

Given a set-theoretical frame $\mathfrak{F} := \langle W, R, \mathbb{A}\rangle$, define the neighborhood frame associated with $\mathfrak{F}$ as $\mathfrak{F}_{\sqcup} := \langle \mathsf{At}\mathbb{A}, f_R \rangle$, where

$$f_R(x) := \{X \subseteq \mathsf{At}\mathbb{A} \mid x \in \Box_R \bigvee X \downarrow_{\mathsf{At}\mathbb{A}}\}.$$

Observe that for a non-admissible X this may be a larger set than $\{X \subseteq \mathsf{At}\mathbb{A} \mid x \in \Box_R X\}$: a definition of f_R using this smaller set would not work as it should. It is an instructive exercise to find a suitable counterexample. Conversely, for every neighborhood frame $\mathfrak{G}$ we can take the descriptive frame corresponding to its dual algebra to be the corresponding set-theoretical frame $\mathfrak{G}^{\sqcap}$.

FACT 3. $\mathfrak{F}_{\sqcup}$ is a neighborhood frame, $\mathfrak{G}^{\sqcap}$ is a set-theoretical frame, $\mathfrak{G} \simeq (\mathfrak{G}^{\sqcap})_{\sqcup}$ and if $\mathfrak{F}$ is a descriptive set-theoretical frame, $\mathfrak{F} \simeq (\mathfrak{F}_{\sqcup})^{\sqcap}$.

To see how this idea can be lifted to a category-theoretical equivalence, check Došen [8]. Topological spaces can be identified with neighborhood frames s.t. for every $X \in f(x)$ (1) $x \in X$ and (2) $\Box_f X \in f(x)$.

FACT 4. For any neighborhood frame $\mathfrak{G}$, t.f.a.e.

1. $\mathfrak{G}$ satisfies (1) and (2) above.
2. $\mathfrak{G}$ is a neighborhood base in the topological sense.
3. $\mathfrak{G}$ is a **S4** frame.
4. the accessibility relation of $\mathfrak{G}^{\sqcap}$ is a quasi-order.

2 Axiomatizations and completeness

2.1 The hybrid language

Axiomatization of $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$ — basic hybrid logic with non-standard $\mathbf{Name}_{\mathcal{H}}$ rule — is given in Table 1. *A* $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$*-logic* is any set of formulas containing all the axioms of $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$ and closed under all its rules. For every $\Gamma \subseteq \mathcal{H}(\mathsf{E}, \Diamond)$, $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}\Gamma$ denotes the smallest logic containing Γ.

LEMMA 5. *Every* $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$*-logic* Λ *is closed under the rule* $\mathbf{Name}_{\mathcal{H}}^{+}$*:*

from $\mathsf{A}(i \to \phi)$ *deduce* $\mathsf{A}\phi$*, for* $i \notin Sub(\phi)$

Proof.

1: $\mathsf{A}(i \to \phi)$ (assumption)

2: $i \to \phi$ (by the dual of $\mathbf{Ref}_{\mathsf{E}}$ and MP)

3: ϕ (by $\mathbf{Name}_{\mathcal{H}}$, as $i \notin Sub(\phi)$)

4: $\mathsf{A}\phi$ (by $\mathbf{Nec}_{\mathsf{A}}$)

■

Axioms and rules for $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$	
CT	ϕ, for all classical tautologies ϕ
K	$\Box(p \to q) \to \Box p \to \Box q$
$\mathbf{K}_{\mathsf{A}}$	$\mathsf{A}(p \to q) \to \mathsf{A}p \to \mathsf{A}q$
$\mathbf{Ref}_{\mathsf{E}}$	$p \to \mathsf{E}p$
$\mathbf{Trans}_{\mathsf{E}}$	$\mathsf{EE}p \to \mathsf{E}p$
$\mathbf{Sym}_{\mathsf{E}}$	$p \to \mathsf{AE}p$
$\mathbf{Incl}_{\Diamond}$	$\Diamond p \to \mathsf{E}p$
$\mathbf{Incl}_{i}$	$\mathsf{E}i$
Nom	$\mathsf{E}(i \wedge p) \to \mathsf{A}(i \to p)$
MP	From $\phi \to \psi$ and ϕ deduce ψ
Nec	From ϕ deduce $\Box\phi$
$\mathbf{Nec}_{\mathsf{A}}$	From ϕ deduce $\mathsf{A}\phi$
Subst	From ϕ deduce $\phi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals
$\mathbf{Name}_{\mathcal{H}}$	From $i \to \phi$ deduce ϕ, for $i \notin \phi$.
Additional rule of $\mathbf{K}_{\mathcal{H}}^{\mathbf{BG}}$	
$\mathbf{BG}_{\mathcal{H}}$	From $\mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge \phi)$ deduce $\mathsf{E}(i \wedge \Box\phi)$, for $i \neq j$ and $j \notin Sub(\phi)$

Table 1. Axiomatization for the hybrid language

Axioms and rules for $\mathbf{K}_{\mathsf{D}}^{\mathbf{Name}}$	
CT	ϕ, for all classical tautologies ϕ
K	$\Box(p \to q) \to \Box p \to \Box q$
$\mathbf{K}_{\overline{\mathsf{D}}}$	$\overline{\mathsf{D}}(p \to q) \to \overline{\mathsf{D}}p \to \overline{\mathsf{D}}q$
$\mathbf{WTrans}_{\mathsf{D}}$	$\mathsf{D}^2 p \to p \vee \mathsf{D}p$
$\mathbf{Sym}_{\mathsf{D}}$	$p \to \overline{\mathsf{D}}\mathsf{D}p$
$\mathbf{Incl}_{\mathsf{D}}$	$\Diamond p \to p \vee \mathsf{D}p$
MP	From $\phi \to \psi$ and ϕ deduce ψ
Nec	From ϕ deduce $\Box\phi$
$\mathbf{Nec}_{\overline{\mathsf{D}}}$	From ϕ deduce $\overline{\mathsf{D}}\phi$
Subst	From ϕ deduce $\vdash \phi\sigma$, where σ is an arbitrary substitution
$\mathbf{Name}_{\mathsf{D}}$	From $\mathsf{O}p \to \phi$ deduce ϕ, for any $p \notin Sub(\phi)$
Additional rule of $\mathbf{K}_{\mathsf{D}}^{\mathbf{BG}}$	
$\mathbf{BG}_{\mathcal{H}}$	From $\mathsf{E}(\mathsf{O}p \wedge \Diamond\mathsf{O}q) \to \mathsf{E}(q \wedge \phi)$ deduce $\mathsf{E}(\mathsf{O}p \wedge \Box\phi)$, for $p \neq q$ and $q \notin Sub(\phi)$

Table 2. Axiomatization for the difference language

We write $\Gamma \models \gamma$ if for every weakly atomic frame, $\mathfrak{F} \models \Gamma$ implies $\mathfrak{F} \models \gamma$. More generally, for any class K' of weakly atomic frames, we write $\Gamma \models_{K'} \gamma$ if for every $\mathfrak{F} \in K'$, $\mathfrak{F} \models \Gamma$ implies $\mathfrak{F} \models \gamma$. We say that a hybrid logic Γ is *atomically complete* if for every $\gamma \in \mathcal{H}(\mathsf{E}, \Diamond)$, $\Gamma \models \gamma$ iff $\gamma \in \mathbf{K}^{\mathbf{Name}}_{\mathcal{H}}\Gamma$. Definition of K'-completeness is analogous, with $\models$ replaced by $\models_{K'}$. We also say that a set of formulas Γ is *atomically* Λ*-consistent* if $\bot$ cannot be deduced from Γ by means of theorems of Λ, **MP**, $\mathbf{Name}_{\mathcal{H}}$ and $\mathbf{Name}^{+}_{\mathcal{H}}$. Observe that we don't allow the use of $\mathbf{Nec}_{\mathsf{A}}$ rule here, so we cannot use Lemma 5 and eliminate $\mathbf{Name}^{+}_{\mathcal{H}}$ from this definition.

THEOREM 6 (Atomic completeness for hybrid logics). *Every* $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}}$*-logic* Λ *is atomically complete.*

Proof. (sketch) It is enough to show that every $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}}$-logic Λ is complete with respect to weakly named models. Extend every atomically Λ-consistent set of formulas Γ to *a weakly distinguishing MCS* Γ^{+} — i.e., a set of formulas s.t.

- Γ is closed under all theorems of Λ and MP,
- for every ϕ, either ϕ or its negation belongs to Γ^{+}, but not both,
- there is a nominal $i \in \Gamma^{+}$ and
- for every ϕ, $\mathsf{E}\phi \in \Gamma^{+}$ only if $\mathsf{E}(i \wedge \phi) \in \Gamma^{+}$, for some $i \notin Sub(\phi)$.

The third requirement can be met because of $\mathbf{Name}_{\mathcal{H}}$-consistency of Γ, the fourth — because of $\mathbf{Name}^{+}_{\mathcal{H}}$-consistency of Γ. Compare this strategy to Lindenbaum-style lemmas in Gargov et al. [14], [13], de Rijke [18] or ten Cate, Litak [4]. Weakly distinguishing MCS's could be also called *weakly pasted MCS's* or *MCS's pasted for* E*-modality*, to show both similarities and differences with *distinguishing* or *pasted* MCS's used in these papers.

Our model is then built out of all MCS's Δ s.t. for every $\phi \in \Delta$, $\mathsf{E}\phi \in \Gamma^{+}$. Observe that — as opposed to proofs in papers mentioned above — we *don't* assume that every MCS in the model is a weakly distinguishing one and hence named (i.e., contains a nominal). The accessibility relation $R_{\Diamond}$ is then defined as usual in canonical models: $\Delta_1 R_{\Diamond} \Delta_2$ iff for every $\Box\phi \in \Delta_1$, $\phi \in \Delta_2$. The model thus obtained is weakly named and the general frame associated with it is a weakly atomic frame for the logic in question. ■

This strategy then is a mixture of the standard canonical model technique and the hybrid technique of surjectively named models [13], which gives rise to discrete frames. The relationship with "mainstream" hybrid techniques and rules is discussed further in Section 2.3.

2.2 The difference language

Axiomatization of $\mathbf{K}^{\mathbf{Name}}_{\mathsf{D}}$-logic — difference logic with the non-standard $\mathbf{Name}_{\mathsf{D}}$ rule — is given in Table 2. Note that the universal modality is definable in this system: in the case of difference operator, $\mathsf{E}\phi$ is defined as an abbreviation of $\phi \vee \mathsf{D}\phi$. The definition of atomic completeness is the same as in the hybrid case and we can prove

THEOREM 7 (Atomic completeness for difference logics). *Every difference logic is atomically complete.*

Proof. (sketch) Essentially the same as Theorem 6. The role of names for the points in the weakly named model construction is performed by formulas $\mathsf{O}p$. The only point one has to take care of is that R_{D} is really irreflexive, as the canonical model construction for difference logic — as opposed to the technique of *distinguishing sets* [21] — does not preclude that for some Δ, $\Delta R_{\mathsf{D}} \Delta$. Nevertheless, weak namedness implies that for every ϕ and every Δ, $\mathsf{E}\phi \in \Delta$ iff $\mathsf{E}(\mathsf{O}p \wedge \phi) \in \Delta$ for some p. If $\mathsf{O}p \in \Delta$, then Δ must be R_{D}-irreflexive. Hence, the variant of the canonical model obtained by deleting all pairs $\langle x, x \rangle$ from the interpretation of R_{D} validates exactly the same formulas. ■

2.3 The role of non-standard rules

The axiomatizations used above are weaker than those used in Gargov and Goranko [13], Gargov et al. [14], Venema [21], ten Cate and the present author [4] and other papers on hybrid and difference logic. The difference lies not in the choice of axioms, but in non-standard rules. We are using only $\mathbf{Name}_{\mathcal{H}}$ for $\mathcal{H}(\mathsf{E}, \diamond)$ and $\mathbf{Name}_{\mathsf{D}}$ for $\mathcal{ML}(\mathsf{D}, \diamond)$. The Bulgarian logicians used a rule scheme called COV for $\mathcal{H}(\mathsf{E}, \diamond)$. $\mathbf{Name}_{\mathcal{H}}$ only is not enough to derive all instances of COV. Venema [21] made an analogous observation concerning $\mathbf{Name}_{\mathsf{D}}$, which he denoted as IR, and replaced it with a (set of) rule(s) IR^{*}_{D}. Blackburn et al. [2] used both $\mathbf{Name}_{\mathcal{H}}$ and a rule called $PASTE$. Our counterparts of these stronger rules are $\mathbf{BG}_{\mathcal{H}}$ (Table 1) for $\mathcal{H}(\mathsf{E}, \diamond)$ and $\mathbf{BG}_{\mathsf{D}}$ (Table 2) for $\mathcal{ML}(\mathsf{D}, \diamond)$. It can be proven that — as we have the universal modality in the language — both in the hybrid case and in the difference case all these strengthenings are equivalent. $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}}$-logics closed under $\mathbf{BG}_{\mathcal{H}}$ are called $\mathbf{K}^{\mathbf{BG}}_{\mathcal{H}}$-logics, $\mathbf{K}^{\mathbf{BG}}_{\mathsf{D}}$-logics are defined analogously.

Every $\mathbf{K}^{\mathbf{BG}}_{\mathcal{H}}$-logic is complete with respect to *surjectively named models* (Theorem 5.4 in [13]), i.e., models where every point is named by a nominal. Analogously, every $\mathbf{K}^{\mathbf{BG}}_{\mathsf{D}}$-logic is complete with respect to models where every point is named by $\mathsf{O}p$. This gives us the following

THEOREM 8. *Every $\mathbf{K}^{\mathbf{BG}}_{\mathcal{H}}$-logic and every $\mathbf{K}^{\mathbf{BG}}_{\mathsf{D}}$-logic is complete with respect to discrete frames.*

How and why are these stronger rules non-conservative? It was observed first by Gabbay [10] and later restated in Venema [21] or Gargov et al. [13] that $\mathbf{Name}_{\mathcal{H}}$ ($\mathbf{Name}_{\mathsf{D}}$) is enough if $\diamond$ is conjugated.

FACT 9. If Λ is $\mathbf{K}^{\mathbf{Name}}_{\mathcal{H}}$-logic ($\mathbf{K}^{\mathbf{Name}}_{\mathsf{D}}$-logic)containing $p \to \Box\diamond p$, then Λ is also closed under $\mathbf{BG}_{\mathcal{H}}$ ($\mathbf{BG}_{\mathsf{D}}$). Consequently, Λ is complete with respect to discrete frames.

As every other result in the paper, this observation can be easily generalized to the polymodal context, e.g., for tense logics. Nevertheless, in general atomic completeness does not imply di-completeness. The logic of the so-called van Benthem frame [20] is atomically complete but not di-complete. Here, let us consider a more natural example taken from ten Cate and the present author [5]. That paper contains a more thorough discussion of non-standard rules in the topological context.

FACT 10. The $\mathbf{BG}_{\mathcal{H}}$ ($\mathbf{BG}_{\mathsf{D}}$) rule does not preserve validity on the real line, and, indeed, on any non-discrete T_1 space.

2.4 Properties of logics

Let us sum up by compiling a list of some standard properties one would like to be preserved and/or reflected by mappings between lattices of logics.

Decidability, finite axiomatizability. Definitions are standard.

Di-completeness, neighborhood completeness, Kripke completeness, finite model property. Substitute a suitable K' in definition of K'-completeness.

Elementary generation. Completeness with respect to a first-order definable class of Kripke frames.

Elementarity. The class of Kripke frames validating theorems of the logic is first-order definable.

At-persistence, di-persistence. If a weakly atomic (discrete) frame validates Λ, its underlying Kripke frame validates Λ as well.

Sahlqvist property. We present the notion of a Sahlqvist formula along the lines of Venema [21]. Let $c_1, c_2, c_3 \dots$ be syntactic metavariables ranging over VAR, let $\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \dots$ be syntactic metavariables ranging over arbitrary combinations of $\{\Diamond, \mathsf{E}\}$ in the hybrid case ($\{\Diamond, \mathsf{D}\}$ in the in the difference case) and define $\blacksquare_1, \blacksquare_2, \blacksquare_3 \dots$ dually, i.e., a syntactic metavariable ranging over words in $\{\Box, \overline{\mathsf{D}}\}$ in the difference case ($\{\Box, \mathsf{A}\}$ in the hybrid case). *A strongly positive* formula is a conjunction of formulas of the form $\blacksquare c_i$. A formula is *positive* (*negative*) if every c_i occurs under an even (odd) number of negation symbols. A formula is *untied* if it obtained from strongly positive and negative formulas by applying only $\wedge$ and $\blacklozenge_1, \dots \blacklozenge_n$. Formulas of the form $UNTIED \to POS$ (i.e., where antecedent is untied and consequent is positive) are called *Sahlqvist formulas.* Logics axiomatizable with Sahlqvist formulas are called Sahlqvist logics.

3 From $\mathcal{ML}(\mathsf{D}, \Diamond)$ to $\mathcal{H}(\mathsf{E}, \Diamond)$ and back

This section is based on the ideas of Gargov and Goranko [13] and Areces [1, Chapter 7].

DEFINITION 11 (Translation from $\mathcal{ML}(\mathsf{D}, \Diamond)$ to $\mathcal{H}(\mathsf{E}, \Diamond)$).

Fix any recursive $1-1$ mapping $f : \mathcal{ML}(\mathsf{D}, \Diamond) \mapsto \mathbb{N}$ s.t. $\mathbb{N} - f[\mathbb{N}]$ is an infinite recursive set and any recursive $1-1$ mapping $g : \mathbb{N} \mapsto \mathbb{N} - f[\mathbb{N}]$. Let $\tau' : \mathcal{ML}(\mathsf{D}, \Diamond) \mapsto \mathcal{ML}(\mathsf{D}, \Diamond)$ and $\tau : \mathcal{ML}(\mathsf{D}, \Diamond) \mapsto \mathcal{ML}(\mathsf{D}, \Diamond)$ be preprocessing functions s.t. τ' replaces every propositional variable p_n in ψ by $p_{g(n)}$ and $\tau(\psi)$ replaces all occurrences of subformulas of the form $\mathsf{D}\phi$

in $\tau'(\psi)$ by $p_{f(\mathsf{D}\tau(\phi))}$ by induction on the number of nested D operators. Denote $h(\phi) := f(\mathsf{D}\tau(\phi))$. Define

$$\begin{aligned}\sigma(\psi) \;:=\; & \tau(\psi) \wedge \bigwedge_{\mathsf{D}\phi \in Sub(\tau'(\psi))} (\mathsf{A}p_{h(\phi)} \vee \mathsf{A}\neg p_{h(\phi)} \vee (\mathsf{A}(p_{h(\phi)} \leftrightarrow \neg i_{h(\phi)}) \wedge \mathsf{E}p_{h(\phi)})) \wedge \\ & (\mathsf{A}p_{h(\phi)} \to \mathsf{E}(\tau(\phi) \wedge i_{h(\phi)}) \wedge \mathsf{E}(\tau(\phi) \wedge \neg i_{h(\phi)})) \wedge \\ & (\mathsf{A}\neg p_{h(\phi)} \to \mathsf{A}\neg\tau(\phi)) \wedge \\ & ((\mathsf{A}(p_{h(\phi)} \leftrightarrow \neg i_{h(\phi)}) \wedge \mathsf{E}p_{h(\phi)}) \to \mathsf{A}(\tau(\phi) \leftrightarrow i_{h(\phi)})).\end{aligned}$$

As opposed to Areces [1, Chapter 7], we tried to avoid adding new variables to the language: we want to keep the language fixed, hence slightly more involved formulation. Nevertheless, the translation is in fact the same.

THEOREM 12. *For every weakly atomic frame* $\mathfrak{F}$ *and every* $\psi \in \mathcal{ML}(\mathsf{D}, \Diamond)$, $\mathfrak{F} \vDash \psi$ *iff* $\mathfrak{F} \vDash \sigma(\psi)$.

Proof. (sketch) First, observe that $\mathfrak{F} \vDash \psi$ iff $\mathfrak{F} \vDash \tau'(\psi)$, as the logic of an arbitrary frame is closed under substitution. Let $\mathfrak{V}$ be a propositional valuation in $\mathfrak{F}$. Take $\mathfrak{V}'$ to be any total valuation s.t. $\mathfrak{V}'$ agrees with $\mathfrak{V}$ on variables with indices from $g[\mathbb{N}]$, $\mathfrak{V}'(p_{h(\phi)}) = \mathfrak{V}(\mathsf{D}\tau(\phi))$, $\mathfrak{V}'(i_{h(\phi)})$ is some admissible singleton in $\mathfrak{V}(\tau(\phi))$ if this set is non-empty (here we use weak atomicity) and arbitrary otherwise. Clearly, $\mathfrak{V}(\tau'(\psi)) = \mathfrak{V}'(\sigma(\psi))$. Conversely, for every total hybrid model $\mathfrak{M} := \langle \mathfrak{F}, \mathfrak{V}\rangle$ and $\psi \in \mathcal{ML}(\mathsf{D}, \Diamond)$, if $\mathfrak{V}(\sigma(\psi)) \neq \emptyset$, then for every $\mathsf{D}\phi \in Sub(\tau'(\psi))$, $\mathfrak{V}(p_{h(\phi)}) = \mathfrak{V}(\mathsf{D}\tau(\phi))$ and hence $\mathfrak{V}(\sigma(\psi)) = \mathfrak{V}(\tau'(\psi))$.

■

The converse direction is quite simple. As we already saw, by means of the difference operator, we can explicitly force a variable to serve as a name.

DEFINITION 13 (Translation from $\mathcal{H}(\mathsf{E}, \Diamond)$ to $\mathcal{ML}(\mathsf{D}, \Diamond)$).

Choose any $1-1$ and onto recursive mapping $\theta : PROP \cup NOM \mapsto PROP$ and extend it inductively to all formulas $\phi \in \mathcal{H}(\mathsf{E}, \Diamond)$. Let $\pi(\phi) := \bigwedge_{i \in NOM \cap Sub(\phi)} \mathsf{EO}\theta(i) \to \theta(\phi)$. In addition, for every hybrid model $\mathfrak{M} = \langle W, R, \mathfrak{V}\rangle$ define $\mathfrak{M}_{\neq} := \langle W, R, \mathfrak{V}_{\neq}\rangle$, where $\mathfrak{V}_{\neq}(p) := \mathfrak{V}(\theta^{-1}(p))$ for each $p \in PROP$.

THEOREM 14. *For every weakly atomic frame* $\mathfrak{F}$ *and every* $\phi \in \mathcal{H}(\mathsf{E}, \Diamond)$, $\mathfrak{F} \vDash \phi$ *iff* $\mathfrak{F} \vDash \pi(\phi)$.

Proof. (sketch) Let $\mathfrak{F} := \langle W, R, \mathbb{A}\rangle$ be a weakly atomic frame, $x \in W$ and $\phi \in \mathcal{H}(\mathsf{E}, \Diamond)$. Then

$$\mathfrak{F}, x \vDash \phi \text{ iff } \mathfrak{F}, x \vDash \pi(\phi)$$

and thus $\mathfrak{F} \vDash \phi$ iff $\mathfrak{F} \vDash \pi(\phi)$. The proof of the above fact is based on two claims whose proofs can be adopted from Gargov and Goranko [13]:

CLAIM 15. For every hybrid valuation $\mathfrak{V}$, $\mathfrak{V}(\phi) = \mathfrak{V}_{\neq}(\pi(\phi))$.

CLAIM 16. For every propositional valuation $\mathfrak{V}$, $x \notin \mathfrak{V}(\pi(\phi))$ only if there is a hybrid valuation $\mathfrak{V}_\phi$ s.t. $\mathfrak{V}_\phi(\phi) = \mathfrak{V}(\theta(\phi)) = \mathfrak{V}(\pi(\phi))$.

Proof. (of claim, sketch) Fix an admissible $w \in W$. Define

$$\mathfrak{V}_\phi(q) = \begin{cases} \mathfrak{V}(\theta(q)) & : \ \theta(q) \in Sub(\pi(\phi)), \\ \{w\} & : \ \theta(q) \notin Sub(\pi(\phi)). \end{cases}$$

■

■

COROLLARY 17. *$\mathcal{ML}(\mathsf{D}, \Diamond)$ and $\mathcal{H}(\mathsf{E}, \Diamond)$ are equally expressive with respect to weakly atomic frames.*

This in turn using the observations of Section 1 gives us the following

COROLLARY 18. *$\mathcal{ML}(\mathsf{D}, \Diamond)$ and $\mathcal{H}(\mathsf{E}, \Diamond)$ are equally expressive with respect to discrete frames, normal neighborhood frames and topological spaces.*

4 Isomorphism between lattices of logics

4.1 Equivalence of closure operators

Ever since the early work of Tarski and the Polish school in the 1930's, it became clear that the study of logics should be intimately connected with the study of *closure operators.* Let us recall (cf. [7]) that given an arbitrary set X and a function $C : 2^X \mapsto 2^X$, we say that C is *a closure operator* on X if the following three conditions are satisfied for every $A, B \in 2^X$:

C1. $A \subseteq C(A)$,

C2. $A \subseteq B$ implies $C(A) \subseteq C(B)$,

C3. $C(C(A)) = C(A)$.

For convenience, a pair $\langle X, C\rangle$ consisting of a set and a closure operator on it will be called *a closure space.* A logic is often identified with a deductive consequence operator, which is indeed a closure operator on the set of formulas. The problem with this approach is that distinct consequence operators can often generate the same set of theorems. That is, the notion of *a theory* (deductively closed set of sentences) may differ even if the set of *tautologies* (deductive closure of the empty set) is the same.

Here, we take a more Hilbert-style approach: we identify logics with sets of formulas. But it doesn't mean that the theory of consequence operators is of no use for us. Because of its generality, it works well in contexts, where other algebraic techniques may pose certain problems, such as those mentioned in the introduction. There is nothing which prevents us from studying in this manner deductive consequence *with the substitution rule* and also *with non-orthodox rules.* Set of formulas which are closed under the substitution rule are *logics* rather than theories. It is a standard fact that for any closure operator C, C-closed sets form a complete lattice, where arbitrary meets coincide with set-theoretical intersections (cf. [7, Chapter 2]). Recall also that an element a of a complete lattice L is called *compact* if for every $A \subseteq L$, $a \leq \bigvee A$ implies the existence of $B \subseteq_{fin} A$ s.t. $a \leq \bigvee B$. If $\langle A, C\rangle$ is a closure space, C is *algebraic* if for every X, $C(X) = \bigcup\{C(Y) \mid Y \subseteq_{fin} X\}$. For algebraic closure operators, compact closed sets are those of the form $C(Y)$ for a finite Y.

LEMMA 19 (Isomorphism of lattices of closed elements). *Let* $\mathfrak{X} := \langle X, C_X\rangle$ *and* $\mathfrak{Y} := \langle Y, C_Y\rangle$ *be two closure spaces and assume there are mappings* $\Sigma : X \mapsto Y$, $\Pi : Y \mapsto X$ *s.t. for every* $\{a\}, A \subseteq X$, $\{b\}, B \subseteq Y$*:*

I1. $a \in C_X(A)$ *iff* $\Sigma(a) \in C_Y(\Sigma[A])$,

I2. $b \in C_Y(B)$ *iff* $b \in C_Y(\Sigma\Pi[B])$.

Then the lattices of C_X *and* C_Y*-closed sets are isomorphic by* $\bar{\Sigma}(A) := C_Y(\Sigma[A])$. *Moreover,* $\bar{\Pi}(B) := C_X(\Pi[B])$ *is the converse isomorphism, i.e.,* $B = \bar{\Sigma}(\bar{\Pi}(B))$ *for any* C_Y*-closed* B. *For algebraic closure operators, this mapping preserves and reflects compactness; that is,* A *is compact iff* $\bar{\Sigma}$ *is.*

Proof. For the sake of readability, we omit almost all parentheses, both round and square. It is straightforward to see that $\bar{\Sigma}$ is well-defined and preserves order. To see that it also reflects the order and hence is $1-1$, assume A_1, A_2 are C_X-closed, $a \in C_X A_1$ and $C_Y\Sigma C_X A_1 \subseteq C_Y\Sigma C_X A_2$. Then by I1, $\Sigma a \in C_Y\Sigma C_X A_1$ and by assumption and another application of I1, it gives us $a \in C_X A_2$.

To prove the mapping is onto and the 'moreover' part, take any C_Y-closed B and let $A := C_X\Pi B$. Then $b \in C_Y\Sigma C_X\Pi B$ iff (C1–C3) $C_Y b \subseteq C_Y\Sigma C_X\Pi B$ iff (I2) $C_Y\Sigma\Pi b \subseteq C_Y\Sigma C_X\Pi B$ iff (C1–C3) $\Sigma\Pi b \in C_Y\Sigma C_X\Pi B$ iff (I1 and C3) $\Pi b \in C_X\Pi B$ iff (I1) $\Sigma\Pi b \in C_Y\Sigma\Pi B$ iff (I2) $\Sigma\Pi b \in C_Y B$ iff (C1–C3) $C_Y\Sigma\Pi b \subseteq C_Y B$ iff (I2) $C_Y b \subseteq C_Y B$ iff (C1–C3) $b \in C_Y B$. Thus, $\bar{\Sigma}A = C_Y B = B$ and surjectivity follows. Preservation and anti-preservation of compactness is straightforward. ■

Cf. Blok, Jónsson [3, Theorem 3.7] or Galatos, Tsinakis [12] for more general results of this kind.

4.2 Isomorphism via translation

Lemma 19 allows us to obtain a lattice isomorphism result as soon as we have the following ingredients:

- a class of frames or algebras K;
- two languages L_1 and L_2 and two closure operators C_1 and C_2 on them — the sets of formulas closed under C_1 (C_2) are called L_1-logics (L_2-logics);

- a proof that every L_1-logic and every L_2-logic is complete with respect to a subclass of K;
- two translations $F_1 : L_1 \mapsto L_2$ and $F_2 : L_2 \mapsto L_1$ s.t. every $\mathfrak{A} \in K$ satisfies $\phi \in L_1$ iff it satisfies $F_1(\phi)$ and $\mathfrak{A}$ satisfies $\psi \in L_2$ iff it satisfies $F_2(\psi)$.

We will be also able to prove that this isomorphism preserves and reflects many desirable properties, such as finite axiomatizability or completeness with respect to some well-behaved subclass of K. The idea is clear, but in order to prove it formally we need to introduce some definitions in the spirit of Abstract Algebraic Logic.

DEFINITION 20 (Logical family). $\langle F, C_\vdash, K, \vDash_K\rangle$ is called *a logical family* if

- F is an arbitrary *set of formulas.* We assume here that this set is recursive.
- $C_\vdash$ is an algebraic closure operator on F. $C_\vdash$-closed sets are called *logics.* Compact $C_\vdash$-closed sets are called *finitely axiomatizable logics.*
- K is an arbitrary class of structures (frames, algebras, topological spaces ...) called *a semantics.*
- $\vDash_K \subseteq K \times F$ is called *a validity relation.* If $\mathfrak{A} \vDash \phi$, we say ϕ *holds* in $\mathfrak{A}$.

This definition is so general that it has to be unsatisfying. For a start, it does not say anything about the relationship between $\vdash$ and $\vDash$. For $\Gamma \subseteq F$ and $K' \subseteq K$, denote *closure operators induced on F by K'* as

$$C_{K'}(\Gamma) := \{\phi \in F \mid \forall \mathfrak{A} \in K'(\forall \gamma \in \Gamma.\mathfrak{A} \vDash \gamma \Rightarrow \mathfrak{A} \vDash \phi)\}.$$

It is straightforward to see $C_{K'}$ is a closure operator; we say that $C_{K'}(\Gamma)$ is the *K'-closure of* Γ. If $\Gamma = C_{K'}(\Gamma)$, we say Γ is *a K'-complete logic.* Conversely, for any $\Gamma \subseteq F$, we can define $Mod(\Gamma) := \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma\}$. Thus, Γ is a K'-complete logic iff $\Gamma = C_{K' \cap Mod(\Gamma)}(\Gamma)$.

DEFINITION 21 (Soundness, completeness, complete family).

Let $\langle F, C_\vdash, K, \vDash_K\rangle$ be a logical family. If $C_\vdash(\Gamma) \subseteq C_K(\Gamma)$ for every $\Gamma \subseteq F$, K is *a sound semantics.* If the converse inclusion holds, K is called *a complete semantics.* A logical family with sound and complete semantics is called *a complete family.*

FACT 22. In every complete family, $C_\vdash$-closed elements are exactly C_K-complete logics.

DEFINITION 23 (Persistence and relative soundness).

Assume $\mathcal{F}_1 = \langle F_1, C_1, K, \vDash_1\rangle$, $K' \subseteq K$ and $\nu : K' \mapsto K'$ is a mapping s.t.

- restriction of ν to $\nu[K']$ is the identity mapping;

- for every $\mathfrak{A} \in K'$ and every $\phi \in F$, $\nu\mathfrak{A} \models \phi$ implies $\mathfrak{A} \models \phi$.

$\nu\mathfrak{A}$ is called then *the underlying ν-frame of* $\mathfrak{A}$. ν itself is called *a carrier mapping.* We say that $\Gamma \subseteq F$ is *K'-persistent (relative to ν)* if $K' \ni \mathfrak{A} \models \Gamma$ implies $\nu\mathfrak{A} \models \Gamma$. Also, assume $\mathcal{F}_2 := \langle F_2, C_2, \nu[K'], \models_2 \rangle$, i.e., the semantics of $\mathfrak{F}_2$ is the range of ν. We say that $\mathcal{F}_1$-logic Γ is *sound relative to* $\mathcal{F}_2$ if there is $\Gamma' \subseteq F_2$ s.t. $Mod_{\mathcal{F}_1}(\Gamma) \cap \nu[K'] = Mod_{\mathcal{F}_2}(\Gamma')$.

A paradigm example of such a ν is the mapping assigning to a weakly atomic frame its underlying Kripke frame. We allowed the case when the domain of ν is smaller than K itself to cover the case of di-persistence too. The notion of relative soundness generalizes the notion of elementarity. To see how, take $\mathfrak{F}_2$ to be the family of first-order theories in the frame correspondence language.

To sum up: we saw how to generalize notions such as topo-completeness, di-completeness, Kripke completeness, finite model property, elementary generation (K'-completeness, with K' replaced by respective class of frames), at-persistence, di-persistence (K'-persistence), and elementarity (relative soundness). Now let us see how to generalize the idea of translations preserving and reflecting validity — and how to use this notion to prove the existence of isomorphism preserving and reflecting the above-defined properties.

DEFINITION 24 (Equivalent families).

Let $\mathcal{F}_1 := \langle F_1, C_1, K, \models_1 \rangle$, $\mathcal{F}_2 := \langle F_2, C_2, K, \models_2 \rangle$ be two complete logical families sharing the same class of structures as semantics and let f_1, f_2 be a pair of functions s.t.

T1. $f_1 : F_1 \mapsto F_2$ and $f_2 : F_2 \mapsto F_1$;

T2. for every $\mathfrak{A} \in K$ and every $\phi_1 \in F_1$, $\mathfrak{A} \models_1 \phi_1$ iff $\mathfrak{A} \models_2 f_1(\phi_1)$;

T3. for every $\mathfrak{A} \in K$ and every $\phi_2 \in F_2$, $\mathfrak{A} \models_2 \phi_2$ iff $\mathfrak{A} \models_1 f_2(\phi_2)$.

We say then $\mathcal{F}_1$ and $\mathcal{F}_2$ are *K-equivalent by* $\langle f_1, f_2 \rangle$.

LEMMA 25. *Assume $\mathcal{F}_1 := \langle F_1, C_1, K, \models_1 \rangle$ and $\mathcal{F}_2 := \langle F_2, C_2, K, \models_2 \rangle$ are K-equivalent by $\langle f_1, f_2 \rangle$ and let $K' \subseteq K$. Then $\langle F_1, C_{1K'}, K', \models_1 \rangle$ and $\mathcal{F}_2 := \langle F_2, C_{2K'}, K', \models_2 \rangle$ are K'-equivalent by $\langle f_1, f_2 \rangle$, where $C_{1K'}$ and $C_{2K'}$ are closure operators induced by K' on F_1 and F_2, respectively.*

THEOREM 26. *Assume $\mathcal{F}_1 := \langle F_1, C_1, K, \models_1 \rangle$ and $\mathcal{F}_2 := \langle F_2, C_2, K, \models_2 \rangle$ are K-equivalent by $\langle f_1, f_2 \rangle$. For $\Gamma \subseteq F_1$, define $F(\Gamma) = C_2(f_1[\Gamma])$. This mapping restricted to C_1-closed sets is an isomorphism onto the lattice of C_2-closed sets, which preserves and reflects*

- *finite axiomatizability;*
- *for every $K' \subseteq K$, K'-completeness. Moreover, lattices of K'-complete $\mathcal{F}_1$-logics and $\mathcal{F}_2$-logics are isomorphic too;*

- *for any carrier mapping $\nu : K' \mapsto K'$, K'-persistence and for any logical family $\mathcal{G}$ whose semantics is $\nu[K']$, relative $\mathcal{G}$-soundness;*
- *if both f_1 and f_2 are effectively defined, the isomorphism preserves and reflects decidability. If, moreover, both are computable in polynomial time, the isomorphism preserves complexity up to a polynomial.*

Proof. The existence of such an isomorphism follows directly from Lemma 19. All we have to do is to check that the following two conditions hold for arbitrary $\{\delta\} \cup \Delta \subseteq F_1$, $\{\gamma\} \cup \{\Gamma\} \subseteq F_2$:

CLAIM 27. $\delta \in C_1(\Delta)$ iff $f_1(\delta) \in C_2(f_1[\Delta])$.

Proof. (of claim) $\delta \in C_1(\Delta)$
iff $\Delta \models_{\mathcal{F}_1} \delta$ ($\mathcal{F}_1$ is a complete family)
iff for every $\mathfrak{F} \in K$, $\mathfrak{F} \models_1 \Delta$ implies $\mathfrak{F} \models_1 \delta$
iff for every $\mathfrak{F}$, $\mathfrak{F} \models_2 f_1[\Delta]$ implies $\mathfrak{F} \models_2 f_1(\delta)$ (T2)
iff $f_1(\delta) \in C_2(f_1[\Delta])$ ($\mathcal{F}_2$ is a complete family). ■

CLAIM 28. $\gamma \in C_2(f_2[f_1[\Gamma]])$ iff $\gamma \in C_2(\Gamma)$.

Proof. (of claim) $\gamma \in C_2(f_1[f_2[\Gamma]])$
iff for every $\mathfrak{F} \in K$, $\mathfrak{F} \models_2 f_1[f_2[\Gamma]]$ implies $\mathfrak{F} \models_2 \gamma$ ($\mathcal{F}_2$ is a complete family)
iff $\mathfrak{F} \models_2 \Gamma$ implies $\mathfrak{F} \models_2 \gamma$ (by T2 and T3)
iff $\gamma \in C_2(\Gamma)$ ($\mathcal{F}_2$ is a complete family). ■

Preservation and antipreservation of

- finite axiomatizability: follows from preservation and antipreservation of compactness;
- decidability: assume $\Delta = C_1(\Delta)$ is a decidable $\mathcal{F}_1$-logic. The problem whether $\gamma \in F_2$ is in $F(\Delta)$ reduces then to checking if $f_2(\gamma)$ is in Δ. So, if f_2 is computable, computability is preserved and if f_1 is computable, computability is reflected. Reasoning for complexity is analogous;
- K'-completeness: is a consequence of Lemma 25;
- K'-persistence:

  $K' \ni \mathfrak{A} \models \Gamma$ implies $\nu\mathfrak{A} \models \Gamma$

  is equivalent to

  $K' \ni \mathfrak{A} \models f_1(\Gamma)$ implies $\nu\mathfrak{A} \models f_1(\Gamma)$;
- relative soundness: because $Mod_{\mathcal{F}_1}(\Gamma) \cap \nu[K'] = Mod_{\mathcal{F}_2}(f_1[\Gamma]) \cap \nu[K']$.

■

COROLLARY 29. *The lattices of* $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$*-logics and* $\mathbf{K}_{\mathsf{D}}^{\mathbf{Name}}$*-logics are isomorphic. This isomorphism preserves and reflects di-completeness, topo-completeness, Kripke completeness, finite model property, elementary generation, elementarity, at-persistence, di-persistence, finite axiomatizability and decidability.*

COROLLARY 30. *The lattices of* $\mathbf{K}_{\mathcal{H}}^{\mathbf{BG}}$*-logics and* $\mathbf{K}_{\mathsf{D}}^{\mathbf{BG}}$*-logics are isomorphic. This isomorphism preserves and reflects Kripke completeness, finite model property, elementary generation, elementarity, di-persistence, finite axiomatizability and decidability.*

4.3 The Sahlqvist problem

One more property whose preservation under modal translations is desirable is the property of being Sahlqvist. Its preservation and/or reflection are usually discussed while introducing translations and interpretations in modal logic; cf. [15]. Nevertheless, as observed by Conradie et al. [6], the problem with the syntactic definition of the Sahlqvist property is that it *is extremely fragile as it does not withstand even simple boolean transformations, or even substitutions changing the polarity of propositional variables.* And so, even if $\phi \in \mathcal{H}(\mathsf{E}, \Diamond)$ is Sahlqvist, $\pi(\phi)$ can, strictly speaking, fail to be one. The antecedent $\bigwedge\limits_{i \in NOM \cap Sub(\phi)} \mathsf{EO}\theta(i)$ is indeed an untied formula, but the consequent, which is $\theta(\phi)$ itself, is not necessarily positive. However, it is of course a matter of trivial boolean pre-processing to show that if ϕ is a Sahlqvist formula of the form $\alpha \to \beta$, then $\pi(\phi)$ can be taken to be $\bigwedge\limits_{i \in NOM \cap Sub(\phi)} \mathsf{EO}\theta(i) \wedge \theta(\alpha) \to \theta(\beta)$, and this is again a Sahlqvist formula. Therefore, it is safe to say that π preserves the property of being Sahlqvist as well. Moreover, it is clear that this slightly modified version of π (i.e. with the clause that formulas whose main connective is implication are translated as described above) yields a Sahlqvist formula *iff* the input was a Sahlqvist formula. Thus, it is justified to say that this translation *preserves and reflects* the property of being Sahlqvist. Things, however, look different if one takes σ as a starting point. It is enough to glance at the definition of σ to see there is no straightforward way of ensuring that this translation preserves the Sahlqvist property.

5 Concluding remarks

5.1 Further applications

There is nothing in the isomorphism proof from Section 4 which crucially depends on the nature of weakly atomic frames, hybrid logic or the difference operator. Therefore, our note is meant as a methodological suggestion: a proof that two languages are equivalent with respect to expressivity over certain class K of frames, algebras, spaces etc. yields automatically that lattices of K-complete logics in both languages are isomorphic. If K is large enough to provide a general completeness proof, it proves the isomorphism of lattices of *all* logics in both languages. Moreover, this isomorphism preserves many desirable properties. We feel this method can find other applications. First idea: using results of Gabbay et al. [11], one can try to apply ideas presented here to the correspondence between lattice of modal logics with Stavi connectives and the lattice of weak second-order theories (in the sense of van Benthem [20]) of linear orders. It would be also interesting to find other examples of this kind in the modal realm or elsewhere.

Besides, it should be possible to relate the isomorphism-via-translation techniques to the theory of residuation and Galois connections. It was not necessary to study this connection in depth for our present purposes, but anyone aiming to develop a more general mathematical theory of translations between classes of logics along the lines of Section 4 should investigate this option seriously. A good starting point is Erne et al. [9]

5.2 Life without rules

Coming back to the correspondence between $\mathbf{K}_{\mathcal{H}}^{\mathbf{Name}}$-logics and $\mathbf{K}_{\mathsf{D}}^{\mathbf{Name}}$-logics, the reader may wonder now what was the role of non-standard rules in the isomorphism proof. As we have shown, general completeness and isomorphism results can be proven both with and without the paste-like rules ($\mathbf{BG}_{\mathcal{H}}$ and $\mathbf{BG}_{\mathsf{D}}$, respectively). They are geared towards discrete frames — algebraically, they force complete additivity of corresponding algebras. It means that behaviour of $\Diamond$ on all sets is completely determined by its behaviour on admissible individuals. There is, however, nothing in the translation which prevents semantics of more topological character. In these semantics, constraints on $\Diamond$ imposed by paste-like rules are not natural.

But the situation with $\mathbf{Name}_{\mathcal{H}}$ and $\mathbf{Name}_{\mathsf{D}}$ seems to be different. These rules apparently capture something very fundamental about the nature of the difference operator and nominals. Let us consider briefly what could happen if we delete these rules. Of course, we could not take weak namedness and atomicity for granted anymore. So, in the hybrid case the set of admissible singletons could become arbitrarily small. And in the $\mathcal{ML}(\mathsf{D}, \Diamond)$ case, as was already noted, these rules are also necessary to ensure that we can restrict our attention to frames where R_{D} is irreflexive. So, the idea of translation based on non-standard semantics for such weak deductive systems would be to treat exactly the set of those points for which R_{D} is irreflexive as the set of admissible singletons. However, the axiom $\mathbf{Incl}_i$ poses an immediate problem. In the $\mathcal{H}(\mathsf{E}, \Diamond)$-case, it forces non-emptiness of the collection of admissible singletons. In the $\mathcal{ML}(\mathsf{D}, \Diamond)$-case, this would correspond to the requirement that every frame contains at least one point on which R_{D} is irreflexive. But this condition cannot be forced on non-standard semantics by any modal formula.

Balder ten Cate suggested two ways out of this predicament. One was to retain a very weak form of $\mathbf{Name}_{\mathsf{D}}$ for $\mathcal{ML}(\mathsf{D}, \Diamond)$, with $\mathsf{O}p$ replaced by $\mathsf{EO}p$. Another was to remove the problematic axiom from the hybrid axiomatization. The present author is not happy with either choice. The first one, while sacrificing a nice completeness result, would fail to achieve the main goal of eliminating non-standard rules from both languages. The second option feels, if anything, even worse. It would remove not only the axiom whose roots in hybrid logics can be traced back to Prior, but also the underlying fundamental idea: that nominals should behave like genuine individual names and hence be true not just at *at most* one point, but at *exactly* one point. Still more unacceptably, $\mathbf{Incl}_i$ would not make these pseudo-semantics specialize to standard semantics in hybrid logic: $\mathsf{E}i$ would define an *empty* class of frames

And it is doubtful anyways that either of bad solutions would restore the isomorphism result. An interested reader may investigate this question. The present author feels contended with the conclusion that every general (meta-)theory of logics with individual names has to take the non-orthodox rules seriously.

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