Algebraization of Hybrid Logic with Binders

Tadeusz Litak

School of Information Science, JAIST
Asahidai 1–1, Nomi-shi, Ishikawa-ken
923-1292 JAPAN
litak@jaist.ac.jp

Abstract. This paper introduces an algebraic semantics for hybrid logic with binders $\mathcal{H}(\downarrow, @)$. It is known that this formalism is a modal counterpart of the bounded fragment of the first-order logic, studied by Feferman in the 1960's. The algebraization process leads to an interesting class of boolean algebras with operators, called *substitution-satisfaction algebras*. We provide a representation theorem for these algebras and thus provide an algebraic proof of completeness of hybrid logic.

1 Introduction

1.1 Motivation

The aim of this paper is to provide an algebraic semantics for *hybrid logic with binders* $\mathcal{H}(\downarrow, @)$. This formalism is, as was proven in the 1990's [1], the modal counterpart of *the bounded fragment of first-order logic*. Hence, an algebraization of $\mathcal{H}(\downarrow, @)$ provides also an algebraic insight into the nature of bounded quantification, i.e., quantification of the form $\forall x(tRx \to \phi)$ and $\exists x(tRx \wedge \phi)$, where t is a term not containing x. The fragment of first-order logic obtained by allowing only such quantifiers was investigated in the 1960's by Feferman and Kreisel [2], [3]. A discovery they made is that formulas in this fragment are exactly those which are preserved by formation of *generated submodels*, as modal logicians would say, or — to use Feferman's term — *outer extensions*.

The aim of this paper is to present a class of algebras which are hybrid (or bounded) equivalent of cylindric algebras for first-order logic. These algebras are *substitution-satisfaction algebras (SSA's)*, boolean algebras equipped with three kinds of operators: $\downarrow^k$ corresponding to *binding* of variable i_k to the present state, $@_k$ saying that a formula is *satisfied* in the state named by i_k and standard modal operator $\Diamond$, corresponding to restricted quantification itself. The theory of cylindric algebras proves to be an important source of insights and methods, but not all techniques can be applied directly to SSA's. For example, cylindric algebras often happen to be simple. For locally finite dimensional ones, subdirect irreducibility is equivalent to simplicity and in the finitely dimensional case, we even have a discriminator term. SSA's are not so well-behaved. Another example: in cylindric algebras, the operation of substitution of one variable for another is always definable in terms of quantifier operators. SSA's do not allow such a feat. And yet, it turns out that their representation theory is not much more complicated than in the cylindric case.

Algebraic operators formalizing substitutions in first-order logic have been studied since Halmos started working on polyadic algebras [4]. In particular, they play a prominent role in formalisms developed by Pinter in the 1970's, cf., e.g., [5]. Nevertheless, algebras studied in the present paper do not have full substitution algebras as reducts — certain substitution operators are missing. Besides, as Halmos himself observed, the most interesting thing about satisfaction operators is their interplay with quantifiers — and bounded quantifiers do not interact with substitution operators in the same way as standard quantifiers do.

The structure of the present paper is as follows. In Section 1.2, we introduce the bounded fragment and $\mathcal{H}(\downarrow, @)$ as well as the truth preserving translation that show they are expressively equivalent. In Section 2, we introduce concrete, set-theoretical instantiation of SSA's — our counterpart of cylindric set algebras. In Section 3 we characterize SSA's axiomatically. Also, we prove some useful arithmetical facts and characterize basic algebraic notions, such as congruence filters or subdirect irreducibility. Section 4 contains main results of the paper. First, we identify Lindenbaum-Tarski algebras of hybrid theories as those which are *properly generated*. It is a more restrictive notion than the notion of *local finiteness* in the case of cylindric algebras. Then we show that every properly generated algebra of infinite (countable) dimension can be represented as a subdirect product of set algebras. In other words, we provide a representation theorem for SSA's and thus an algebraic proof of completeness of $\mathcal{H}(\downarrow, @)$. The proof was inspired by a concise proof of representation theorem for cylindric algebras by Andréka and Németi [6].

The author wishes to thank heartfully Ian Hodkinson for inspiration to begin the present research and for invaluable suggestions how to tackle the issue. The author can only hope that this advice was not entirely wasted. Thanks are also due to Patrick Blackburn for his ability to seduce people into doing hybrid logic and to the anonymous referee for suggestions and comments on the first version of this paper.

1.2 $\mathcal{H}(\downarrow, @)$ and the Bounded Fragment

This subsection briefly recalls some results of Areces et al. [1]; cf. also ten Cate [7]. For any ordinal α, define $\alpha^+ = \alpha - \{0\}$. It will become clear soon that zero is going to play the role of *a distinguished variable*. Fix a countable supply of *propositional variables* $\{p_a\}_{a \in PROP}$ (the restriction on cardinality is of no importance here) and *nominal variables* $\{i_k\}_{k \in \alpha^+}$; most of the time, we assume $\alpha = \omega$. Formulas of hybrid language are given by

$$\phi ::= p_a \mid i_k \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid @_{i_k}\phi \mid \downarrow^{i_k}.\phi$$

$\Box$, $\vee$ and $\rightarrow$ are introduced as usual. Some papers introduced one more kind of syntactic objects: nominal constants, which cannot be bound by $\downarrow$. They do not increase the expressive power of the language and for our present goal the disadvantages of introducing such objects would outweigh the merits. They can be replaced by free unquantified variables.

Hybrid formulas are interpreted in *models*. A model $\mathfrak{M} := \langle W, R, V\rangle$ consists of an arbitrary non–empty set W, a binary *accessibility relation* $R \subseteq W \times W$ and *a (propositional) valuation* $V : p_a \mapsto A \in \mathbb{P}(W)$ mapping propositional variables to subsets of W. *A (nominal) assignment* in a model is any mapping $v : i_k \mapsto w \in W$ of nominal variables to elements of W. For an assignment v, $k \in \alpha^+$ and $w \in W$, define v^k_w to be the same assignment as v except for $v(i_k) = w$. The notion of satisfaction of formula at a point is defined inductively:

$w \vDash_{\mathfrak{M},v} i_k$	if $w = v(i_k)$	$w \vDash_{\mathfrak{M},v} p_a$	if $w \in V(p_a)$
$w \vDash_{\mathfrak{M},v} \psi \wedge \phi$	if $w \vDash_{\mathfrak{M},v} \psi$ and $w \vDash_{\mathfrak{M},v} \phi$	$w \vDash_{\mathfrak{M},v} \neg\psi$	if not $w \vDash_{\mathfrak{M},v} \psi$
$w \vDash_{\mathfrak{M},v} \Diamond\psi$	if $\exists y.(wRy$ and $w \vDash_{\mathfrak{M},v} \psi)$		
$w \vDash_{\mathfrak{M},v} @_{i_k}\psi$	if $v(i_k) \vDash_{\mathfrak{M},v} \psi$	$w \vDash_{\mathfrak{M},v} \downarrow^{i_k}.\psi$	if $w \vDash_{\mathfrak{M},v^k_w} \psi$.

Fix a first order-language with a fixed binary relation constant R, unary predicate constants $\{P_a\}_{a \in PROP}$ and variables in $VAR := \{x_k\}_{k \in \alpha^+} \cup \{x, y\}$. *The bounded fragment* is generated by the following grammar:

$$\phi ::= P_a(v) \mid vRv' \mid v \approx v' \mid \neg\phi \mid \phi \wedge \psi \mid \exists v.(tRv \& \psi),$$

where $v, v' \in VAR$ and t is a term which does not contain v. The last requirement is crucial.

Define the following mapping from the hybrid language to the first-order language by mutual recursion between two functions ST_x and ST_y:

	ST_x	ST_y
i_k	$x \approx x_k$	$y \approx x_k$
p_a	$P_a(x)$	$P_a(y)$
$\psi \wedge \phi$	$ST_x(\psi) \wedge ST_x(\phi)$	$ST_y(\psi) \wedge ST_y(\phi)$
$\neg\psi$	$\neg ST_x(\psi)$	$\neg ST_y(\psi)$
$\Diamond\phi$	$\exists y.(xRy \wedge ST_y(\phi))$	$\exists x.(yRx \wedge ST_x(\phi))$
$@_{i_k}\phi$	$\exists x.(x \approx x_k \wedge ST_x(\phi))$	$\exists y.(y \approx x_k \wedge ST_y(\phi))$
$\downarrow^{i_k}.\phi$	$\exists x_k.(x \approx x_k \wedge ST_x(\phi))$	$\exists x_k.(y \approx x_k \wedge ST_y(\phi))$

This mapping is known as *the standard translation.*

Theorem 1. *Let $\mathfrak{M} := \langle W, R, V\rangle$ be a hybrid model, v a nominal assignment. Let also ν be a valuation of first-order individual variables satisfying $\nu(x_k) = v(i_k)$, $\nu(x)$ and $\nu(y)$ being arbitrary; recall that unary predicate constants correspond to propositional variables. For every $w \in W$ and every hybrid formula ϕ, $w \vDash_{\mathfrak{M},v} \phi$ iff $\nu^x_w, w \vDash ST_x(\phi)$.*

Proof: See, e.g., Section 3.1 of Areces et al. [1] or Section 9.1 of ten Cate [7]. ⊣

The special role, then, is played by x: we sometimes call it *the distinguished variable* and identify it with x_0. The role of y is purely auxiliary. It is never used as a non-bound variable.

The apparatus of binders and satisfaction operators makes also the reverse translation possible. Let the supply of individual variables be $\{x_k\}_{k\in\alpha^+}$; no distinguished variables this time. Define

$$HT(P_a(x_k)) := @_{i_k} p_a,\ HT(x_k \approx x_l) := @_{i_k} i_l,\ HT(x_k R x_l) := @_{i_k} \Diamond i_l,$$
$$HT(\neg\phi) := \neg H(\phi), \quad HT(\phi \wedge \psi) := HT(\phi) \wedge HT(\psi),$$
$$HT(\exists x_k . x_l R x_k \wedge \psi) := @_{i_l} \Diamond \downarrow^{i_k} . HT(\psi),$$

Theorem 2. *Let $\mathfrak{M}$ be a first-order model in the signature $\{P_a\} \cup \{R\}$ and ν be a valuation of first-order individual variables. Define a nominal assignment $v(i_k) := \nu(x_k)$. For every formula ψ in the bounded fragment and every point $x \in \mathfrak{M}$, $\nu, x \vDash \phi$ iff $x \vDash_{\mathfrak{M},v} HT(\psi)$.*

Proof: See Section 3.1 of Areces et al. [1] or Section 9.1 of ten Cate [7]. $\dashv$

In short, $\mathcal{H}(\downarrow, @)$ and the bounded fragment of first-order logic have the same expressive power. There is a beautiful semantic characterization of first-order formulas equivalent to those in the bounded fragment: these are exactly formulas *invariant for generated submodels*. Unfortunately, we cannot enter into details here: cf. Feferman [3] or Areces et al. [1].

2 Concrete Algebras

A set substitution-satisfaction algebra or *a concrete substitution-satisfaction algebra of dimension α (CSSA$_\alpha$) with base $\langle W, R\rangle$*, where $R \subseteq W^2$, is defined as a structure $\mathfrak{A} := \langle A, \vee, \neg, \emptyset, A, @_i, \mathbf{s}^i_0, \mathbf{d}_i, \Diamond\rangle_{i\in\alpha^+}$, where A is a field of subsets of W^α closed under all operations defined below and for every $i \in \alpha^+$ and $X \in \mathbb{P}(W^\alpha)$:

- $\mathbf{d}_i := \{x \mid x_0 = x_i\}$,
- $@_i X := \{y \mid \exists x \in X . x_0 = x_i \& \forall j \neq 0 . x_j = y_j\}$,
- $\downarrow^i X := \{y \mid \exists x \in X . x_0 = x_i \& \forall j \neq i . x_j = y_j\}$,
- $\Diamond X := \{y \mid \exists x \in X . y_0 R x_0 \& \forall j \neq 0 . x_j = y_j\}$.

$\mathbf{d}_i$ corresponds to $\mathbf{d}_{0i}$ in cylindric algebras, hence our present notation. $\emptyset$ is often denoted as $\bot$ and $\mathbb{P}(W^\alpha)$ as $\top$. The zero coordinate of an element from W^α is called *the distinguished axis*. Geometrically, $@_k X$ corresponds to the effect of intersecting X with the hyperplane $\mathbf{d}_k$ and moving the set thus obtained parallel to the distinguished axis. Analogously, $\downarrow^k X$ corresponds to the effect of intersecting X with the hyperplane $\mathbf{d}_k$ and moving the set thus obtained parallel to the k-axis. Logical counterparts of these operations will be made explicit in the next section, but the notation should suggest the proper interpretation. Those CSSA$_\alpha$'s whose universe consists of the whole W^α form the class of *full-set substitution-satisfaction algebras of dimension α*, denoted by FSSA$_\alpha$. Thus, CSSA$_\alpha = S($FSSA$_\alpha)$. The class of *representable substitution-satisfaction algebras of dimension α* is defined as RSSA$_\alpha = ISP($FSSA$_\alpha)$.

2.1 Connection with Logic

With every model $\mathfrak{M} = \langle W, R, V\rangle$ we can associate the structure $\mathbf{Ss}\mathfrak{M}$. Namely, with every formula ϕ whose nominal variables are in $\{i_k\}_{k\in\alpha^+}$ we can associate the set $\phi^{\mathfrak{M}} = \{v \in W^\alpha \mid v_0 \vDash_{\mathfrak{M},v|_{\alpha^+}} \phi\}$; $v|_{\alpha^+}$ is identified with the corresponding assignment of nominal variables. Such sets form a field of sets closed under $\downarrow^k$, $@_k$, $\Diamond_R$ and all diagonals. This is exactly the algebra $\mathbf{Ss}\mathfrak{M} \in \textsc{cssa}_\alpha$. Let us record the following basic

Fact 1 *For all hybrid formulas ϕ, ψ, every $k \in \alpha^+$, every hybrid model $\mathfrak{M}$, $i_k^{\mathfrak{M}} = \mathbf{d}_k$, $(\psi \wedge \phi)^{\mathfrak{M}} = \psi^{\mathfrak{M}} \wedge \phi^{\mathfrak{M}}$, $(\neg\psi)^{\mathfrak{M}} = \neg\psi^{\mathfrak{M}}$, $(\Diamond\psi)^{\mathfrak{M}} = \Diamond_R\psi^{\mathfrak{M}}$, $(@_{i_k}\psi)^{\mathfrak{M}} = @_k\psi^{\mathfrak{M}}$, $(\downarrow^{i_k}.\psi)^{\mathfrak{M}} = \downarrow^k\psi^{\mathfrak{M}}$.*

In order to characterize those $\textsc{cssa}_\alpha$'s which are of the form $\mathbf{Ss}\mathfrak{M}$ for some $\mathfrak{M}$, let us introduce the notion of *a dimension set* $\Delta a := \{i \in \alpha^+ \mid \downarrow^k a \neq a\}$. An element a is *zero-dimensional* if $\Delta a = 0$. The family of all zero-dimensional elements of $\mathfrak{A}$ is denoted by $A^{[0]}$. The algebra generated (as $\textsc{cssa}_\alpha$; of course, all constant elements are also treated as generators) from $A^{[0]}$ is denoted as $[A^0]$. $\mathfrak{A}$ is called *properly generated*[1] if $\mathfrak{A} = [A^0]$. $\mathfrak{A}$ is called *locally finitely dimensional* if $\#\Delta a < \omega$ for every a. Finally, $\mathfrak{A} \in \textsc{cssa}_\alpha$ with base $\langle W, R\rangle$ is called *0-regular* if for every $a \in A$, every $v, w \in W^\alpha$, $v \in a$ and $v_0 = w_0$ implies $w \in a$.

Lemma 2. *Every algebra of the form $\mathbf{Ss}\mathfrak{M}$ for some hybrid model $\mathfrak{M} = \langle W, R, V\rangle$ is properly generated and 0-regular.*

Proof: Proof of proper generation consists of three straightforward claims. First, every hybrid formula is by definition built from $\{p_a\}_{a\in PROP}$ and $\{i_k\}_{k\in\alpha^+}$ by finitely many applications of $\neg$, $\wedge$, $\Diamond$, $@_{i_k}$ and $\downarrow^{i_k}$. Second, by Fact 1, the connectives are interpreted by corresponding operations in algebra. Third, for every propositional variable p, $p^{\mathfrak{M}}$ is zero-dimensional, as $v \in p^{\mathfrak{M}}$ iff $v_0 \in V(p)$ iff $(v^i_{v_0})_0 \in V(p)$ iff $v \in \downarrow^k p^{\mathfrak{M}}$.

For 0-regularity, for any 0-dimensional $\phi^{\mathfrak{M}}$ let $var(\phi) = \{k \in \alpha^+ \mid i_k \text{ occurs in } \phi\}$. If $f \in \phi^{\mathfrak{M}}$, $g|_{var(\phi)} = f|_{var(\phi)}$, then $g \in \phi^{\mathfrak{M}}$, as it is irrelevant what values g assigns to variables which do not occur in ϕ. Assume now $g_0 = f_0$. In case $var(\phi)$ is non-empty, let $i_{v(0)}, \ldots, i_{v(n)}$ be an enumeration of it. Let f' (g') be a valuation obtained from f (g) by substituting $f_0 (= g_0)$ for every $i_{v(k)}$, where $k \in \{0, \ldots, n\}$. $f \in \phi^{\mathfrak{M}} = \downarrow^{v(0)} \ldots \downarrow^{v(k)} \phi^{\mathfrak{M}}$, hence $f' \in \phi^{\mathfrak{M}}$, by the above observation $g' \in \phi^{\mathfrak{M}}$, thus $g \in \downarrow^{v(0)} \ldots \downarrow^{v(k)} \phi^{\mathfrak{M}} = \phi^{\mathfrak{M}}$. ⊣

The above observation can be strengthened to an equivalence.

Theorem 3. *$\mathfrak{A} \in \textsc{cssa}_\alpha$ based on $\langle W, R\rangle$ is of the form $\mathbf{Ss}\mathfrak{M}$ for some hybrid model $\mathfrak{M}$ with the same base iff it is properly generated and 0-regular.*

---

[1] We avoid the notion *zero-generated* as it could be misleading: algebraists usually call this way the smallest subalgebra, i.e., the algebra generated from constants

Proof: The left-to-right direction has already been proven. For the converse, let $\mathfrak{A} \in \text{FSSA}_\alpha$ be a properly generated and regular algebra based on $\mathfrak{F} = \langle W, R \rangle$. For any $a \in A^{[0]}$, let $V(p_a) = \{w \in W \mid \exists v \in a.w = v_0\}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$. We want to show $\mathfrak{A} = \mathbf{Ss}\mathfrak{M}$. The bases of both algebras and hence the fundamental operations on the intersection of both universes coincide. Thus, in order to show the $\subseteq$-direction, it is enough to show that for every $a \in A^{[0]}$, $a = p_a^{\mathfrak{M}}$. For every $v \in W^\alpha$, $v \in p_a^{\mathfrak{M}}$ iff $v_0 \in V(p_a)$ iff $v_0 = w_0$ for some $w \in a$ iff (by 0-regularity) $v \in a$. For the reverse inclusion, observe that the atomic formulas in the language of $\mathfrak{M}$ are always of the form p_a or i_k. The proof proceeds then by standard induction on the complexity of formulas. ⊣

3 Abstract Approach

3.1 Axioms and Basic Arithmetics

Let $i, j, k \ldots$ be arbitrary ordinals in α^+. The *class of substitution-satisfaction algebras of dimension* α SSA_α is defined as the class of algebras satisfying the following axioms:[2]

Ax1. Axioms for boolean algebras

Ax2. Axioms for the modal operator
- (a) $\Diamond\bot = \bot$
- (b) $\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

Ax3. Axioms governing $@_k$
- (a) $\neg @_k p = @_k \neg p$
- (b) $@_k(p \vee q) = @_k p \vee @_k q$
- (c) $@_k \mathbf{d}_k = \top$
- (d) $@_k @_j p = @_j p$
- (e) $\mathbf{d}_k \leq p \leftrightarrow @_k p$

Ax4. Interaction of $\Diamond$ and $@_k$: $\Diamond @_k p \leq @_k p$

Ax5. Axioms governing $\downarrow^k$
- (a) $\neg \downarrow^k p = \downarrow^k \neg p$
- (b) $\downarrow^k (p \vee q) = \downarrow^k p \vee \downarrow^k q$
- (c) $\downarrow^k \downarrow^j p = \downarrow^j \downarrow^k p$
- (d) $\downarrow^k \downarrow^k p = \downarrow^k p$
- (e) $\downarrow^j \mathbf{d}_k = \mathbf{d}_k$ for $j \neq k$ and $\downarrow^k \mathbf{d}_k = \top$

Ax6. Interaction of $\downarrow^k$ and $@_j$
- (a) $\downarrow^k @_j \downarrow^k p = @_j \downarrow^k p$ for $j \neq k$
- (b) $\downarrow^k @_k p = \downarrow^k p$
- (c) $@_k \downarrow^k p = @_k p$

Ax7. Interaction of $\downarrow^k$ and $\Diamond$: $\downarrow^k \Diamond \downarrow^k p = \Diamond \downarrow^k p$

Ax8. The Blackburn-ten Cate axiom BG: $@_k \Box \downarrow^j @_k \Diamond \mathbf{d}_j = \top$

Fact 3 $\text{FSSA}_\alpha \subseteq \text{SSA}_\alpha$ *and thus* $\text{RSSA}_\alpha \subseteq \text{SSA}_\alpha$.

[2] Added in the on-line version: Ax6b and Ax5d can be derived from Ax3e, Ax5e and Ax6c. It remains to be checked if there are other redundancies.

Lemma 4. *The following are deriviable:*

Ar1. $\Box(p \to q) \leq \Diamond p \to \Diamond q$
Ar2. $@_k(p \to q) = @_k p \to @_k q$
Ar3. $\downarrow^k (p \to q) = \downarrow^k p \to \downarrow^k q$
Ar4. $@_k \mathbf{d}_j \leq @_k p \leftrightarrow @_j x$
Ar5. $@_k \mathbf{d}_j \leq @_j \mathbf{d}_k$
Ar6. $\mathbf{d}_j \wedge p \leq @_j p$
Ar7. $@_k \Diamond \mathbf{d}_j \wedge @_j p \leq @_k \Diamond p$.

Proof: The only one which requires some calculation is Ar7 and we will need this inequality later. From Ax3e we get that $@_j p \leq \mathbf{d}_j \to p$, this by Ax4 gives us $@_j p \leq \Box(\mathbf{d}_j \to p)$. Using Ax3d, we get $@_j p \leq @_k \Box(\mathbf{d}_j \to p)$. By Ar1, this gives us $@_j p \leq @_k(\Diamond \mathbf{d}_j \to \Diamond p)$. By Ar2, we get the desired conclusion. $\dashv$

3.2 Proper Generation and Finite Dimensionality

The notions of *dimension* of an element, *locally finitely dimensional* and *properly generated* algebra are introduced in exactly the same way as in the concrete case. The class of locally finite algebras of dimension α is denoted as $\mathbf{Lf}_\alpha$, the properly generated ones — as $\mathbf{Prop}_\alpha$. [3]

Fact 5 $\Delta \mathbf{d}_k = \{k\}$, $\Delta \Diamond a \subseteq \Delta a$, $\Delta \neg a = \Delta a$, $\Delta(a \wedge b) \subseteq \Delta a \cup \Delta b$, $\Delta @_k a \subseteq \Delta a \cup \{k\}$, $\Delta \downarrow^k a \subseteq \Delta a - \{k\}$.

Corollary 1. $\mathbf{Prop}_\alpha \subseteq \mathbf{Lf}_\alpha$.

From now on, we use the fact that $\downarrow^k$ and $@_k$ distribute over all boolean connectives without explicit reference to Ax3a, Ax3b, Ax5a, Ax5b and Ar2. A straightforward consequence of Ax6b is

Fact 6 *For every* $k \notin \Delta p$, $p \neq \bot$ *implies* $@_k p \neq \bot$. *Consequently for any* $a \in \mathfrak{A} \in \mathbf{Lf}_\alpha$, $a \neq \bot$ *iff there is* k *s.t.* $@_k a \neq \bot$.

The following result is an algebraic counterpart of an observation of ten Cate and Blackburn. [8], [7]

Lemma 7. *Assume* $\alpha \geq \omega$, $\mathfrak{A} \in \mathbf{Lf}_\alpha$, $p \in \mathfrak{A}$. *Then*

$$@_j \Diamond p = \bigvee_{l \notin \Delta p} (@_j \Diamond \mathbf{d}_l \wedge @_l p) \tag{1}$$

[3] Added in the on-line version: in fact, the only relevant property of $\mathbf{Lf}_\alpha$ for infinite α is that for every element p, the complement of Δp with respect to α^+ is infinite. This is the defining property of *dimension-complemented* cylindric algebras. All results concerning locally finite algebras in the present paper— in particular, Lemma 7 — would hold also for dimension-complemented SSA's.

Proof: $@_j\Diamond\mathbf{d}_l \wedge @_l p \leq @_j\Diamond p$ by Ar7. Thus, in order to show 1, it is enough to prove that for any z, if $@_j\Diamond\mathbf{d}_l \wedge @_l p \leq z$ for every $l \in \Delta p$, then $@_j\Diamond p \leq z$. Choose some $l, k \notin \Delta p \cup \Delta z$ (here is where we use assumptions on $\mathfrak{A}$). By assumption and by Ax3d, $@_j\Diamond\mathbf{d}_l \wedge @_l p \leq @_k z$. This in turn, by $l \notin \Delta z \cup \Delta p$ and Fact 5 implies $\downarrow^l @_j\Diamond\mathbf{d}_l \wedge p \to @_k z = \top$, from which we get $@_j\Box(\downarrow^l @_j\Diamond\mathbf{d}_l \to (p \to @_k z)) = \top$. Here is where we use Ax8 to obtain $@_j\Box(p \to @_k z) = \top$. This implies $@_j\Diamond p = @_j\Diamond p \wedge @_j\Box(p \to @_k z)$. By Ar1, we get thus $@_j\Diamond p \leq @_j\Diamond @_k z$ and by Ax4 and Ax3d we obtain $@_j\Diamond p \leq @_k z$. By $k \notin \Delta z \cup \Delta p$ and Fact 5, the conclusion follows. ⊣

3.3 Ideals, Homomorphisms, The Rasiowa-Sikorski Lemma

Let us introduce several standard algebraic notions concerning the structure of SSA_α's. *An open ideal* is a lattice-theoretical ideal closed under $\Diamond$, all $@_i$ and $\downarrow^i$. It is a standard observation that congruence ideals correspond to homomorphisms. *An ideal generated by* p is the smallest open ideal containing p; it is denoted by $\mathbf{Gen}(p)$. Let $\mathbf{Mod}_\alpha$ be the set of words in the alphabet $\{\Diamond, \downarrow^i, @_i \mid i \in \alpha^+\}$.

Fact 8 $\mathbf{Gen}(p) = \{q \mid q \leq \blacklozenge_1 p \vee \cdots \vee \blacklozenge_n p, \blacklozenge_1, \ldots, \blacklozenge_n \in \mathbf{Mod}_\alpha\}$

A subdirectly irreducible algebra is one which contains smallest nontrivial open ideal. By the above observation, we can reformulate it as follows.

Corollary 2. $\mathfrak{A} \in \text{SSA}_\alpha$ *is subdirectly irreducible iff there exists* $\mathsf{o} \neq \bot$ *s.t. for every* $p \neq \bot$ *there are* $\blacklozenge_1, \ldots, \blacklozenge_n \in \mathbf{Mod}_\alpha$ *s.t.* $\mathsf{o} \leq \blacklozenge_1 p \vee \cdots \vee \blacklozenge_n p$. o *is called* a (dual) opremum element.

Of course, we don't really have to consider all members of $\mathbf{Mod}_\alpha$; it is possible to restrict the set significantly. In particular, for zero-dimensional p we can restrict attention to $\blacklozenge_1, \ldots, \blacklozenge_n \in \{\Diamond^n, @_i\Diamond^n \mid i \in \alpha^+, n \in \omega\}$.

Combining Corollary 2 and Fact 6 we arrive at the following:

Corollary 3. $\mathfrak{A} \in \mathbf{Lf}_\alpha$ *is subdirectly irreducible iff there is* $\mathsf{o} \in \mathfrak{A}$ *and* $k \in \alpha^+$ *s.t.* $@_k\mathsf{o} \neq \bot$ *and for every* $p \neq \bot$ *there are* $\blacklozenge_1, \ldots, \blacklozenge_n \in \mathbf{Mod}_\alpha$ *s.t.* $@_k\mathsf{o} \leq @_k\blacklozenge_1 p \vee \cdots \vee @_k\blacklozenge_n p$. *Of course, we can simply replace* o *with* $@_k\mathsf{o}$*, as this element is an opremum as well.*

Definition 1. *Let* $\mathfrak{A} \in \text{SSA}_\alpha$. *An ultrafilter* H *of* $\mathfrak{A}$ *is* elegant *if for every* $i \in \alpha^+$ *and every* $p \in \mathfrak{A}$, $@_i\Diamond p \in H$ *iff there is* $j \in \alpha^+$ *s.t.* $@_i\Diamond\mathbf{d}_j \wedge @_j p \in H$.

Lemma 9. *Assume* $\alpha \geq \omega$, $\mathfrak{A} \in \mathbf{Lf}_\alpha$, $\#\mathfrak{A} \leq \omega$. *For every* $a \neq \bot$*, there exists an elegant ultrafilter containing* a.

Proof: Follows from Lemma 7 and The Rasiowa-Sikorski Lemma: cf. Koppelberg [9, Theorem 2.21] . We briefly sketch the proof here to make the paper more self-contained. Let $b_0, b_1, b_2 \ldots$ be an enumeration of all elements of the form $@_j\Diamond p$ for some $j \in \alpha^+$ and $p \in \mathfrak{A}$: here is where we use the fact that universe of $\mathfrak{A}$ is countable. Define $a_0 := a$. If a_n is defined, let $a_{n+1} := a_n$ if $a_n \wedge b_n = \bot$. Otherwise, assume $b_n = @_j\Diamond p$. Lemma 7 implies there is $k \in \alpha^+$ s.t. $a_{n+1} := a_n \wedge @_j\Diamond\mathbf{d}_k \wedge @_k p \neq \bot$. In this way we obtain an infinite descending chain of nonzero points. Any ultrafilter containing $\{a_n\}_{n\in\omega}$ is elegant. ⊣

4 The Representation Theorem

This section proves the main result of the paper. We identify those SSA's which correspond to Lindenbaum-Tarski algebras of $\mathcal{H}(\downarrow, @)$-theories and prove a representation theorem for them.

4.1 Transformations, Retractions, Replacements

Halmos [4] developed general theory of transformations and used it as a foundation for theory of polyadic algebras. Let us recall some basic results. A *transformation* of α^+ is any mapping of α^+ into itself. We call a transformation τ *finite* if $\tau(i) = i$ for almost all i (i.e., cofinitely many). We will be interested only in finite transformations. The intuitive reason is that transformations will correspond to substitutions of variables — and, for a given formula, only finitely many variables are relevant. From now on, finiteness of transformations is assumed tacitly.

A transformation τ is called *a transposition* if for some k and l, $\tau(k) = l$, $\tau(l) = k$ and for all other arguments, τ is equal to identity. Such mappings are denoted as (k, l). A product of transpositions is called *a permutation.* τ is called *a replacement* if τ is different from identity for exactly one argument, say $\tau(l) = k$. τ is then written as (l/k). A transformation τ is called *a retraction* if $\tau^2 = \tau$. It is a well-known mathematical fact that every bijection of α^+ onto itself is a permutation. Halmos generalized this fact as follows:

Lemma 10. *Every retraction is a product of replacements and every transformation is a product of a permutation and a retraction.*

In case of locally finite algebras of infinite dimension, we can restrict our attention only to retractions, i.e., products of replacements: this will be justified further on. A similar observation for locally finite polyadic algebras was made by Halmos [4]. Finally, a bit of notation. For τ a transformation, τ^l_k be the substitution defined as $\tau^l_k(j) = \tau(j)$ for $j \neq l$ and $\tau^l_k(l) = k$. Also, let $\tau - l$ be the transformation which is the same as τ except that it leaves l unchanged. Thus, τ^l_k is the composition of $\tau - l$ and (l/k).

4.2 Axioms for $\mathcal{H}(\downarrow, @)$

We present an axiom system for $\mathcal{H}(\downarrow, @)$ taken from Blackburn and ten Cate [8], [7]. A nominal variable i_k is called *bound* in a formula ϕ if it occurs within the scope of some $\downarrow^{i_k}$ and *free* otherwise.

Definition 2. *Let* $\tau : \alpha^+ \mapsto \alpha^+$ *be a transformation.* The nominal substitution associated with τ *of formulas of hybrid language is a function* $\psi \mapsto \psi^\tau$ *which replaces all free occurrences of* i_k *and* $@_{i_k}$ *with* $i_{\tau(k)}$ *and* $@_{i_{\tau(k)}}$*, respectively, in those places which are not in the scope of some* $\downarrow^{i_{\tau(k)}}$*.*

A $\mathcal{H}(\downarrow, @)$*-theory* is any set of hybrid formulas T containing all instances of

H1. classical tautologies,
H2. $\Box(\psi \to \phi) \to (\Box\psi \to \Box\phi)$,
H3. $@_{i_k}(\psi \to \phi) \to (@_{i_k}\psi \to @_{i_k}\phi)$,
H4. $@_{i_k}\psi \leftrightarrow \neg @_{i_k}\neg\psi$,
H5. $@_{i_k} i_k$,
H6. $@_{i_j}@_{i_k}\psi \leftrightarrow @_{i_k}\psi$,
H7. $i_k \to (\psi \leftrightarrow @_{i_k}\psi)$,
H8. $\Diamond @_{i_k}\psi \to @_{i_k}\psi$,
H9. $@_{i_k}(\downarrow^{i_l}.\psi \leftrightarrow \psi^{(l/k)})$,
H10. $\downarrow^{i_k}.(i_k \to \psi) \to \psi$, if i_k does not occur free in ψ,
H11. $@_{i_j}\Box \downarrow^{i_k}.@_{i_j}\Diamond i_k$

and closed under Modus Ponens, Substitution (i.e., if $\psi \in T$, then $\psi^\tau \in T$ for every substitution τ), and Generalization for all operators (i.e., if $\psi \in T$, then $@_{i_k}\psi \in T$, $\downarrow^{i_k}.\psi \in T$ and $\Box\psi \in T$). It is easy to see that our Ax1–Ax4 and Ax8 are direct translations of corresponding $\mathcal{H}(\downarrow, @)$ axioms. Those governing $\downarrow$ could not be translated straightforwardly into equations. See the concluding section for further comments on the relationship between these two axiomatizations.

Blackburn and ten Cate [8], [7] prove the following

Theorem 4 (Hybrid Completeness). *For every consistent $\mathcal{H}(\downarrow, @)$-theory T, there is a hybrid model $\mathfrak{M}_T$ such that T is exactly the set of all formulas whose value under all assignments is equal to the universe of $\mathfrak{M}_T$.*

In this work, we provide an algebraic counterpart of their result. First, let us characterize Lindenbaum-Tarski algebras of $\mathcal{H}(\downarrow, @)$-theories.

4.3 Lindenbaum-Tarski Algebras

From now on, we always assume we work with $\alpha \geq \omega$. Fix a supply of propositional variables and denote the set of all hybrid formulas as $Form$. With every $\mathcal{H}(\downarrow, @)$-theory T we can associate an equivalence relation on the set of hybrid formulas: $[\psi]_T = \{\phi \mid \psi \leftrightarrow \phi \in T\}$. With every connective, we can associate a corresponding operator on equivalence classes, i.e., $\Diamond[\phi] = [\Diamond\phi]$, $@_k[\phi] = [@_{i_k}.\phi]$, $\downarrow^k [\phi] = [\downarrow^{i_k} \phi]$ etc. It is a matter of routine verification that this definition is correct, i.e., independent of the choice of representatives. We have to show that $Form/T$ with operators corresponding to logical connectives and constants is an element of $\mathbf{Prop}_\alpha$. Such a structure is called *the Lindenbaum-Tarski algebra* of T. Verification that Ax1–Ax4 and Ax8 hold does not pose any problems by the remark above. Verification of axioms governing $\downarrow^k$ can be done in uniform manner: first, use H9 and axioms governing $@_{i_j}$ to prove an instance of the axiom preceded by arbitrary $@_{i_j}$, e.g. $@_{i_j}(\neg \downarrow^{i_k} \phi \leftrightarrow \downarrow^{i_k} \neg\phi) \in T$. As i_j can be chosen such that i_j does not appear in ϕ, we can use H7, generalization rule for $\downarrow^{i_j}$ and H10 to get rid of initial $@_{i_j}$. The same strategy can be used to show that equivalence classes of propositional variables are zero-dimensional and thus the algebra is properly generated.

In the reverse direction, we use the same strategy as in the proof of Theorem 3. With every element $a \in A^{[0]}$, associate a distinct propositional variable p_a. Hybrid formulas ϕ in the language whose propositional variables are p_a's and nominal variables are in α^+ are in $1-1$ correspondence with constant terms in the language of SSA_α's extended with a name for every $a \in [A^0]$. And so, for every such formula ϕ, let Φ be the corresponding term. The substitution associated with τ for terms is defined in the same way as for hybrid formulas with $\mathbf{d}_k$, $\downarrow^k$ and $@_k$ replacing, respectively, i_k, $\downarrow^{i_k}$ and $@_{i_k}$. Define $T_{\mathfrak{A}} := \{\phi \in Form \mid \Phi = \top\}$. First, we show this is a $\mathcal{H}(\downarrow, @)$-theory. The only part which is not immediate is showing that all instances of H9 belong to T.

Lemma 11. *$\Psi^{(l/k)} =\downarrow^l \Psi^{(l/k)}$ for $l \neq k$.*

Proof: The only relevant information for the basic inductive step is that p_a's correspond to $a \in A^{[0]}$. The inductive steps are trivial for booleans and use Ax7, Ax6a and Ax5c for modal, satisfaction and substitution operators. ⊣

Lemma 12. *For every retraction τ, $@_k \Phi^{\tau^l_k} = @_k \downarrow^l \Phi^{\tau - l}$.*

Proof: To prove the lemma, it is enough to show that

$$@_l \mathbf{d}_k \leq \Psi \leftrightarrow \Psi^{(l/k)}. \tag{2}$$

For then we get that $\mathbf{d}_k \leq \downarrow^l \Psi \leftrightarrow \downarrow^l \Psi^{(l/k)}$. By Lemma 11, it is equivalent to $\mathbf{d}_k \leq \downarrow^l \Psi \leftrightarrow \Psi^{(l/k)}$. By laws of boolean algebras, it is equivalent to $\mathbf{d}_k \wedge \downarrow^l \Psi = \mathbf{d}_k \wedge \Psi^{(l/k)}$. But then

$$@_k \downarrow^l \Psi = @_k(\mathbf{d}_k \wedge \downarrow^l \Psi) = @_k(\mathbf{d}_k \wedge \Psi^{(l/k)}) = @_k \Psi^{(l/k)}.$$

Thus, let us prove 2 by induction on the complexity of Ψ. For nominals, it's a consequence of Ax3c and Ar5. For propositional variables, $p_a^{(l/k)} = p_a$. For booleans, the inductive step is trivial. For $\Diamond$, it follows from Ax4 and Ar1. For $@_j$, it follows from Ax3d. Finally, for $\downarrow^j$, we use the fact that either $j \notin \Delta(@_l \mathbf{d}_k)$ or $j \in \{k, l\}$. If the latter is the case, then $\downarrow^j \Psi^{(l/k)} =\downarrow^j \Psi$, by definition of nominal substitution. ⊣

We have proven that T is indeed a $\mathcal{H}(\downarrow, @)$-theory, but before proceeding with the proof that $\mathfrak{A}$ is isomorphic to $Form/T_{\mathfrak{A}}$ let us record two useful consequences of the Lemma just proven.

Corollary 4. *If $\Psi = \top$, then for arbitrary retraction τ and arbitrary k in the range of τ, $@_k \Psi^\tau = \top$.*

We can also justify the observation made before: that in locally finite algebras of infinite dimension, the only kind of transformations which are relevant are retractions, i.e., products of replacements. In view of Lemma 10, it is enough to show the following.

Corollary 5. *For arbitrary transposition (k, l) and for every Φ, there exists a retraction τ s.t. $\Phi^{(k,l)} = \Phi^\tau$.*

Proof: Choose any $m \notin \Delta\Phi \cup \{k, l\}$ (here is where we use the fact that $\mathfrak{A}$ is locally finite and of infinite dimension). Define $\tau = (m/k)(k/l)(l/m)$. The only argument where τ can possibly differ from (k, l) is m, for $\tau(m) = k$ and $(k, l)(m) = m$. But then $\Psi^\tau =\downarrow^m \Psi^\tau =\downarrow^m @_m \downarrow^m \Psi^\tau =\downarrow^m @_m \Psi^{(k,l)} =\downarrow^m \Psi^{(k,l)} = \Psi^{(k,l)}$. ⊣

Now, arbitrary $a \in \mathfrak{A}$ is named by a certain term Ψ. Thus, for arbitrary a we can arbitrarily choose one Ψ_a and define a mapping $f(a) = [\Psi_a]$. It is straightforward to observe this mapping is correctly defined, $1 - 1$ and onto. Hence, we have shown

Theorem 5. *For every $\mathcal{H}(\downarrow, @)$-theory T, $Form/T$ with operators corresponding to logical connectives and constants is an element of $\mathbf{Prop}_\alpha$. Conversely, for every $\mathfrak{A} \in \mathbf{Prop}_\alpha$ there is a $\mathcal{H}(\downarrow, @)$-theory $T_\mathfrak{A}$ s.t. $Form/T_\mathfrak{A}$ is isomorphic to $\mathfrak{A}$.*

4.4 The Main Result

In this section, we finally prove the main result of the paper: a representation theorem for SSA's after the manner of Andréka and Németi [6].

Definition and Lemma 13 *Let $\alpha \geq \omega$, $\mathfrak{A} \in \mathrm{SSA}_\alpha$ and F be any filter. Define $\sim_F$ on α^+ as $k \sim_F l$ if $@_k\mathbf{d}_l \in F$. This is a congruence relation. Define also R_F on $\alpha^+/\sim_F$ as $[k]_F R_F [l]_F$ if $@_k\Diamond\mathbf{d}_l \in F$. This is a correct definition.*

Proof: That $\sim_F$ is a congruence relation follows from Ax3c, Ar4 and Ar5. Correctness of the definition of R_F follows from Ar4 and Ar7. ⊣

Theorem 6 (Countable Representation). *Let $\alpha \geq \omega$, $\mathfrak{A} \in \mathbf{Prop}_\alpha$ be a subdirectly irreducible algebra, $\#\mathfrak{A} \leq \omega$. $\mathfrak{A}$ is embeddable in a FSSA_α. More specifically, let H be an elegant ultrafilter containing an opremum element of $\mathfrak{A}$. $\mathfrak{A}$ is embeddable in the full set algebra with base $\mathfrak{F}_H := \langle \alpha^+/H, R_H \rangle$.*

Proof: Just like in Section 4.3, associate with elements of $\mathfrak{A}$ formulas of the language whose propositional variables are $\{p_a \mid a \in A^{[0]}\}$, so that every formula ψ corresponds to a term Ψ in the extended language and every element $a \in \mathfrak{A}$ is named by such a term. Define a valuation V_H of propositional variables in $\mathfrak{F}_H$ by

$$V_H(p_a) := \{[k] \mid @_k a \in H\}.$$

Let $\mathfrak{M} = \langle \mathfrak{F}_H, V_H \rangle$. We are going to show that $\mathfrak{A}$ is isomorphic to $\mathbf{Ss}\mathfrak{M}$. By Corollary 5, we can restrict our attention only to those τ's which are retractions. Thus, by Lemma 10 it is enough to formulate all claims and proofs only for replacements. For a mapping $\tau : \alpha \mapsto \alpha^+$, let $\tau^+ := \tau|_{\alpha^+}$. For arbitrary term Ψ, define auxiliary mapping g' as

$$g'(\Psi) := \{\tau' : \alpha \mapsto \alpha^+ \mid @_{\tau'(0)}\Psi^{\tau'^+} \in H\},$$

and then

$$g(\Psi) := \{\tau : \alpha \mapsto \alpha^+/H | \exists \tau' \in g'(\Psi). \forall i \in \alpha. \tau'(i) \in \tau(i)\}$$

Now, for arbitrary a choose ψ_a to be arbitrary formula s.t. $\Psi_a = a$ and define $f(a) := g(\Psi_a)$. We have to show that this is a correct definition, i.e., that f is independent of the choice of ψ. It is enough to prove that for arbitrary Ψ, Φ, $\Psi \to \Phi = \top$ implies $g(\Psi) \leq g(\Phi)$. Assume $g(\Psi) \not\leq g(\Phi)$. Let τ be such that $@_{\tau(0)}\Psi^{\tau^+} \in H$ and $@_{\tau(0)}\Phi^{\tau^+} \notin H$. It means that $@_{\tau(0)}(\Psi \wedge \neg\Phi)^{\tau^+} \in H$, i.e, $@_{\tau(0)}(\Psi \wedge \neg\Phi)^{\tau^+} \neq \bot$ and hence $(\Psi \wedge \neg\Phi)^{\tau^+} \neq \bot$. Choose arbitrary $k \notin \Delta(\Psi \wedge \neg\Phi)$ in the range of τ. By Fact 6, $@_k(\Psi \wedge \neg\Phi)^{\tau^+} \neq \bot$ and hence by Corollary 4, $\Psi \not\leq \Phi$.

Let us prove that f is a homomorphism. We don't need the assumption of subdirect irreducibility here, only the fact that H is an elegant ultrafilter. Subdirect irreducibility will be used only to show f is an embedding.

Given $\tau : \alpha \mapsto \alpha^+$, let $[\tau] : \alpha \mapsto \alpha^+/H$ be a mapping defined as $[\tau](i) = [\tau(i)]$. Thus, g can be redefined as $g(\Psi) = \{[\tau'] | \tau' \in g'(\Psi)\}$. With every $\sigma : \alpha^+ \mapsto \alpha^+$, we can associate a nominal assignment v^σ in $\mathfrak{M}$ defined as $v^\sigma(i_k) := [\sigma(k)]$. $v^\sigma(l/k)(i_l) := [k]$ and $v^\sigma(l/k)(i_j) := v^\sigma(i_j)$ for $j \neq l$. Sometimes, we denote by v the mapping $v(i_k) = [k]$, i.e., v^{id}. As we are interested only in finitary retractions, every v^σ is of the form $v(l_1/k_1)\dots(l_n/k_n)$ for some $l_1, \dots, l_n, k_1, \dots, k_n$. By $[k] \in v^\sigma(\psi)$ we mean that ψ holds at $[k]$ in $\mathfrak{M}$ under v^σ.

Claim 1: $[k] \in v^\sigma(\downarrow^l \psi)$ iff $[k] \in v^\sigma(l/k)(\psi)$. In fact, this is just a clause from definition of satisfaction, as $v^\sigma(l/k) = (v^\sigma)^l_{[k]}$. We just use more elegant notation to avoid clumsiness.

Claim 2: $v(l/k)(\psi) = v(\psi^{(l/k)})$. Thus, for every σ, $v^\sigma(\psi) = v(\psi^\sigma)$ and $v^\sigma(l/k)(\psi) = v(\psi^{\sigma^l_k})$.

Claim 3: $[j] \in v(\psi^\sigma)$ iff $@_j\Psi^\sigma \in H$.

Proof of claim: For $\psi = p_a$ it follows from the definition of V_H. For $\psi = i_k$ — from the definition of $\sim_H$. For booleans: from distributivity of $@_k$ over boolean connectives and the fact that H is an ultrafilter. The clause for $\Diamond$ is the one where we use the fact that H is elegant: $[j] \in v(\Diamond\phi^\sigma)$ iff (by definition of a valuation in a hybrid model) exists $[k]$ s.t. $[j]R_H[k]$ and $k \in v(\phi^\sigma)$ iff (by definition of R_H and IH) there is $[k]$ s.t. $@_j\Diamond\mathbf{d}_k \in H$ and $@_k\Phi^\sigma \in H$ iff (by assumption on H) $@_j\Diamond\Phi^\sigma \in H$.
Assume now $\psi = @_k\phi$. Then $[j] \in v((@_k\phi)^\sigma)$ iff $[j] \in v(@_{\sigma(k)}\phi^\sigma)$ iff $[\sigma(k)] \in v(\phi^\sigma)$ iff (by IH) $@_{\sigma(k)}\Phi^\sigma \in H$ iff $\Psi^\sigma \in H$ iff (by Ax3d) $@_j\Psi^\sigma \in H$.

Finally, assume $\psi =\downarrow^k \phi$. Then $[j] \in v((\downarrow^k \phi)^\sigma)$ iff $[j] \in v(\downarrow^k \phi^{\sigma-k})$ iff (by Claim 1) $[j] \in v(k/j)(\phi^{\sigma-k})$ iff (by Claim 2) $[j] \in v(\phi^{\sigma^k_j})$ iff (by IH) $@_j \Phi^{\sigma^k_j} \in H$ iff (by Lemma 12) $@_j \downarrow^k \Phi^{\sigma-k} \in H$ iff $@_j \Psi^\sigma \in H$. ⊣

Claim 3 immediately implies that f is a homomorphism: for every $\tau : \alpha \mapsto \alpha^+$ and every $a \in \mathfrak{A}$, $\tau \in f(a)$ iff $[\tau(0)] \in v^{\tau^+}(\psi_a)$.

In order to show f is an embedding we finally use assumption of subdirect irreducibility and the fact that H contains opremum. We want to show that $a \not\leq b$ implies $f(a) \not\leq f(b)$. By Lemma 3 $\Psi_a \to \Psi_b \neq \top$ implies there is $\blacklozenge_1, \ldots, \blacklozenge_n \in \mathbf{Mod}_\alpha$ s.t. $@_k\blacklozenge_1(\Psi_a \wedge \neg\Psi_b) \vee \cdots \vee @_k\blacklozenge_n(\Psi_a \wedge \neg\Psi_b) \in H$. By the fact that H is an ultrafilter, we obtain that there is $\blacklozenge \in \mathbf{Mod}_\alpha$ s.t. $@_k\blacklozenge(\Psi_a \wedge \neg\Psi_b) \in H$. Because of Ax3d and the fact that H is elegant, if $@_k\blacklozenge(\Psi_a \wedge \neg\Psi_b) = @_k\blacklozenge_1\blacklozenge_2(\Psi_a \wedge \neg\Psi_b)$ for some $\blacklozenge_1$ consisting only of diamonds and satisfaction operators, then for some $l \in \alpha^+$, $@_l\blacklozenge_2(\Psi_a \wedge \neg\Psi_b) \in H$. In other words, we can get rid of initial diamonds and satisfaction operators. $@_l \downarrow^m (\blacklozenge_2(\Psi_a \wedge \neg\Psi_b)) \in H$ can be rewritten as $@_l(\blacklozenge_2(\Psi_a \wedge \neg\Psi_b))^{(m/l)} \in H$. Proceeding in this way, we finally obtain that for some j and some σ, $@_j(\Psi_a^\sigma \wedge \neg\Psi_b^\sigma) \in H$. Reasoning the same way as in the proof that f is correctly defined, we finally obtain that $\Psi_a \to \Psi_b \neq \top$, i.e., $a \not\leq b$.

f is in fact an isomorphism onto $\mathbf{Ss}\mathfrak{M}$, cf. the proof of Theorem 3. ⊣

Theorem 7 (Representation). *For $\alpha \geq \omega$, every subdirectly irreducible $\mathfrak{A} \in \mathbf{Prop}_\alpha$ is isomorphic to a* FSSA_α. *Consequently,* $\mathbf{Prop}_\alpha = \mathrm{RSSA}_\alpha$.

Proof: For countable algebras, this already follows from Theorem 6. For uncountable $\mathfrak{A}$, we can prove it very similarly to Lemma 3 in [6]. Namely, let $\{\mathfrak{B}_l\}_{l\in\beta}$ be a directed system of s.i. algebras in $I(\mathrm{CSSA}_\alpha)$ sharing a common opremum element. Then $\bigcup_{l\in\beta} \mathfrak{B}_l \in I(\mathrm{CSSA}_\alpha)$. Lack of space (i.e., LNCS 15 pages limit) prevents us from proving the theorem in detail. ⊣

5 Open Problems and Further Developments

The algebraic axiomatization of Section 3 suggests there should exist a $\mathcal{H}(\downarrow, @)$ analogue of Tarski's axiomatization of first-order logic which uses neither the notion of a free variable nor the notion of proper substitution of a variable in a formula. [10] Also, our Andréka-Németi style proof of The Representation Theorem employs a slightly different strategy from the one used by Henkin, based on the notion of *thin elements* and *rich algebras* — cf. [11] or [12]. It could be interesting to prove the representation theorem for SSA's also in this way.

Another path open for exploration: Monk [13] shows that significant part of algebraic model theory can be presented by focusing on set-theoretical algebras, without any axiomatic definition of abstract cylindric algebras. By analogy, it could be tempting to develop a part of algebraic $\mathcal{H}(\downarrow, @)$-model theory or algebraic bounded model theory by means of set SSA's.

The referee of the present paper posed two interesting questions, which are currently investigated by the author. First, the bounded fragment is known to be a *conservative reduction class* for first-order logic. That suggests that known results about cylindric algebras may be derivable from theorems concerning SSA's. Second, both bounded fragment and hybrid logic with binders are model-theoretically well-behaved: one example is the interpolation property. This should translate into nice algebraic characteristics of SSA's (e.g., the amalgamation property).

Finally, the possible connection with computer science, which was in fact a motivation to present these results to the computer science community. It is known that cylindric algebras capture exactly those database queries which are first-order expressible, cf. [14] for details. Is there an a related interpretation for SSA's — for example, in terms of databases where the user is allowed to ask questions concerning only *accessible* entries?

References

1. Areces, C., Blackburn, P., Marx, M.: Hybrid logic is the bounded fragment of first order logic. In de Queiroz, R., Carnielli, W., eds.: Proceedings of 6th Workshop on Logic, Language , Information and Computation, WOLLIC99, Rio de Janeiro, Brazil (1999) 33–50
2. Feferman, S., Kreisel, G.: Persistent and invariant formulas relative to theories of higher order. Bulletin of the American Mathematical Society **72** (1966) 480–485 Research Announcement.
3. Feferman, S.: Persistent and invariant formulas for outer extensions. Compositio Mathematica **20** (1968) 29–52
4. Halmos, P.: Algebraic Logic. Chelsea Publishing Company (1962)
5. Pinter, C.: A simple algebra of first order logic. Notre Dame Journal of Formal Logic **1** (1973) 361–366
6. Andréka, H., Németi, I.: A simple, purely algebraic proof of the completeness of some first order logics. Algebra Universalis **5** (1975) 8–15
7. ten Cate, B.: Model theory for extended modal languages. PhD thesis, University of Amsterdam (2005) ILLC Dissertation Series DS-2005-01.
8. Blackburn, P., Cate, B.: Pure extensions, proof rules, and hybrid axiomatics. In Schmidt, R., Pratt-Hartmann, I., Reynolds, M., Wansing, H., eds.: Preliminary proceedings of Advances in Modal Logic (AiML 2004), Manchester (2004)
9. Koppelberg, S.: Handbook of boolean algebras. Volume I. Elsevier, North-Holland (1989)
10. Tarski, A.: A simplified formalization of predicate logic with identity. Archiv für Mathematische Logik und Grundlagenforschung **7** (1965)
11. Henkin, L., Monk, J., Tarski, A.: Cylindric algebras, Part II. North Holland, Amsterdam (1985)
12. Andréka, H., Givant, S., Mikulás, S., Németi, I., Simon, A.: Notions of density that imply representability in algebraic logic. Annals of Pure and Applied Logic **91** (1998) 93–190
13. Monk, D.: An introduction to cylindric set algebras (with an appendix by H. Andréka). Logic Journal of the IGPL **8** (2000) 451–506
14. Van den Bussche, J.: Applications of Alfred Tarski's ideas in database theory. Lecture Notes in Computer Science **2142** (2001) 20–37