SAT and Termination

Nao Hirokawa

Japan Advanced Institute of Science and Technology

Sudoku Puzzle

given 9×9-grid like

	1	8				7		
			3			2		
	7							
				7	1			
6							4	
3								
4			5					3
	2			8				
							6	

fill out numbers from 1 to 9 that each number appears exactly once in each {row, column, 3×3-subgrid}

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fill out numbers from 1 to 9 that each number appears exactly once in each $\{$row, column, 3×3-subgrid$\}$

how to solve? — difficult? — NP-complete

Solving Sudoku

x_{11}	**1**	**8**				**7**		
x_{21}	x_{22}	x_{23}	**3**			**2**		
x_{31}	**7**	x_{33}						
				7	**1**			
6							**4**	
3								
4			**5**					**3**
	2			**8**				
							6	

Solving Sudoku

x_{11}	**1**	**8**				**7**		
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x_{31}	**7**	x_{33}						
				7	**1**			
6							**4**	
3								
4			**5**					**3**
	2			**8**				
							6	

Lemma

$x_{33} = 3$

Proof

- $\{x_{11}, x_{21}, x_{22}, x_{23}, x_{31}, x_{33}\} = \{2, 3, 4, 5, 6, 9\}$ by subgrid constraint
- $\{x_{11}, x_{21}, x_{22}, x_{23}, x_{31}\} = \{2, 4, 5, 6, 9\}$ by row & column constraint

$\square$

how about x_{26} ?

Solving Sudoku

	1	**8**				**7**		
x_{21}	x_{22}	x_{23}	**3**	x_{25}	x_{26}	**2**	x_{28}	x_{29}
	7	**3**						
				7	**1**			
6							**4**	
3								
4			**5**					**3**
	2			**8**				
							6	

Solving Sudoku

	1	8				7		
x_{21}	x_{22}	x_{23}	3	x_{25}	x_{26}	2	x_{28}	x_{29}
	7	3						
				7	1			
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LEMMA

$x_{26} = 7$

Solving Sudoku

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				7	**1**			
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3								
4			**5**					**3**
	2			**8**				
							6	

Lemma

$x_{26} = 7$

Proof

- $\{x_{21}, x_{22}, x_{23}, x_{25}, x_{26}, x_{28}, x_{29}\} = \{1, 4, 5, 6, 7, 8, 9\}$
  by row constraint
- $7 \notin \{x_{21}, x_{22}, x_{23}, x_{25}, x_{28}, x_{29}\}$ by column & subgrid constraint

$\square$

Overview

- SAT

Overview

- SAT

- encoding techniques

Overview

- SAT
- encoding techniques
- termination analysis

SAT

DEFINITION

- syntax

$$\begin{array}{rcll} \ell & ::= & x \mid \neg x & \text{literal} \\ C & ::= & \ell_1 \vee \cdots \vee \ell_n & \text{clause} \\ \phi & ::= & C_1 \wedge \cdots \wedge C_n & \text{CNF} \end{array}$$

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- syntax

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\begin{aligned}
\ell &::= x \mid \neg x && \text{literal} \\
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\end{aligned}
$$

- SAT problem is decision problem of scheme:

instance:	CNF ϕ
question:	is ϕ satisfiable ?

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instance: CNF ϕ

question: is ϕ satisfiable ?

THEOREM Cook and Levin

SAT is NP-complete

Quiz

is next CNF ϕ satisfiable?

$$\bigwedge \left\{ \begin{array}{llll} & x_1 \vee \neg x_2, & & \\ \neg x_1 \vee & x_2 \vee & x_3, & \\ \neg x_2 \vee & x_3, & & \\ \neg x_3 & & & \end{array} \right\}$$

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ϕ is satisfiable because of next satisfiable assignment

$$x_1 \mapsto F \qquad x_2 \mapsto F \qquad x_3 \mapsto F$$

Quiz

is next CNF ϕ satisfiable?

$$\bigwedge \left\{ \begin{array}{l} \quad x_1 \vee \neg x_2, \\ \neg x_1 \vee \neg x_2 \vee \quad x_3, \\ \neg x_2 \vee \quad x_3, \\ \neg x_3 \end{array} \right\}$$

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is next CNF ϕ satisfiable?

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ϕ is unsatisfiable. why?

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ϕ is unsatisfiable. why?

because

- exhaustive search shows that any assignment is unsatisfiable, or

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because

- exhaustive search shows that any assignment is unsatisfiable, or
- deduction derives ϕ is $\perp$

SAT Solver and DIMACS Format

is $\bigwedge \left\{ \begin{array}{l} x_1 \vee \neg x_2, \\ \neg x_1 \vee x_2 \vee \neg x_3, \\ x_2 \vee \neg x_3, \\ \neg x_3 \end{array} \right\}$ satisfiable ?

SAT Solver and DIMACS Format

is $\bigwedge \left\{ \begin{array}{l} \quad x_1 \vee \neg x_2, \\ \neg x_1 \vee \quad x_2 \vee \neg x_3, \\ \quad x_2 \vee \neg x_3, \\ \neg x_3 \end{array} \right\}$ satisfiable ?

```
$ cat a.cnf
p cnf 3 #variables 4 #clauses
 1 -2  0
-1  2 -3  0
 3 -2  0
-3  0
```

```
$ minisat a.cnf a.ans
$ cat a.ans
SAT
1 -2 -3 0
```

Sudoku Constraints

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constraints

- each cell is number from 1 to 9
- each number appears at most once in each row
- each number appears at most once in each column
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☞ use $s_{111}, \ldots, s_{999}$ with $s_{ijk} = T \iff x_{ij} = k$

Sudoku Constraints in Boolean

- each cell is number from 1 to 9

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} (s_{ij1} \vee \cdots \vee s_{ij9})$$

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- each number appears at most once in each 3×3 subgrid

$$\bigwedge_{G:\text{subgrid}}\bigwedge_{(i,j)\neq(i',j')}\bigwedge_{k=1}^{9}\neg s_{ijk} \vee \neg s_{i'j'k}$$

Uniqueness of Solution

QUESTION

how to check whether there is exactly one solution

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SOLUTION

check whether there is no other solution

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THEOREM

let ϕ be CNF encoding, and α its satisfiable assignment

solution is unique if and only if

$$\phi \wedge \left(\Big(\bigvee_{\alpha(x)=T} \neg x \Big) \vee \Big(\bigvee_{\alpha(x)=F} x \Big) \right)$$

is satisfiable

Encoding Techniques

SAT Encoding

- modern SAT solvers are extremely fast

SAT Encoding

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- how to translate problem to CNF?

Quiz

find equivalent CNFs

- $(x \wedge y) \ \vee \ \neg(u \wedge v)$

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how to avoid exponential blow up? ☞ Tseitin conversion

Tseitin's transformation

$x \leftrightarrow (y \wedge z) = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z \vee x)$ CNF

$x \leftrightarrow (y \vee z) = (x \vee \neg y) \wedge (x \vee \neg z) \wedge (y \vee z \vee \neg x)$ CNF

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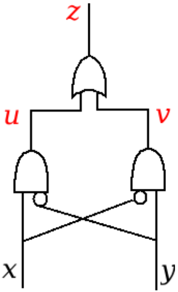

$$\begin{array}{rl} & \models (x \wedge \neg y) \vee (\neg x \wedge y) \\ \Leftrightarrow & \models z \text{ where } \begin{cases} z = u \vee v \\ u = x \wedge \neg y \\ v = \neg x \wedge y \end{cases} \\ \Leftrightarrow & \models z \wedge \bigwedge \left\{ \begin{array}{l} z \leftrightarrow (u \vee v), \\ u \leftrightarrow (x \wedge \neg y), \\ v \leftrightarrow (\neg x \wedge y) \end{array} \right\} \end{array}$$

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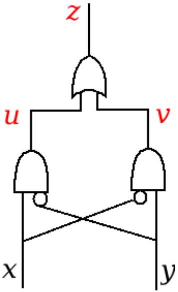

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\end{aligned}
$$

NOTE

given formula of size n, converted formula is of size $O(n)$

Arithmetic

$\vec{x}_k = (a_k, \ldots, a_1)$ is binary representation of $x < 2^k$

DEFINITION bit encoding

$$
\begin{aligned}
\vec{x}_k = \vec{y}_k \quad &= \quad \bigwedge_{i=1}^{k} (x_k \leftrightarrow y_k) \\
\vec{x}_k > \vec{y}_k \quad &= \quad \begin{cases} x_1 \wedge y_1 & \text{if } k = 1 \\ (x_k \wedge \neg y_k) \vee \big((x_k \leftrightarrow y_k) \wedge \vec{x}_{k-1} > \vec{y}_{k-1}\big) & \text{if } k > 1 \end{cases} \\
\vec{x}_k + \vec{y}_k \quad &= \quad (c_k, s_k, \ldots, s_1)
\end{aligned}
$$

where

$$
c_0 = \bot \qquad
\begin{aligned}
c_i &= (x_i \wedge y_i) \vee (x_i \wedge c_{i-1}) \vee (y_i \wedge c_{i-1}) && \text{for } i \geqslant 1 \\
s_i &= x_i \oplus y_i \oplus c_{i-1} && \text{for } i \geqslant 1
\end{aligned}
$$

More...

many encodable mathematical objects:

- finite sets

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- order $>$ on finite set

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DPLL Algorithm

Implementation

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- decision, BCP, conflict analysis, clause learning

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- decision, BCP, conflict analysis, clause learning
- two-watched literal (for BCP)

Boolean Constraint Propagation

current assignment:

$$x \mapsto T, y \mapsto F$$

current decision: $z \mapsto T$

$$\neg x \vee z \vee b \vee c$$

$$\neg x \vee y \vee \neg z \vee \neg w$$

$$w \vee y \vee a$$

$$\neg a \vee z \vee b \vee y$$

$$\neg x \vee c \vee d \vee e$$

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$\neg a \vee z \vee b \vee y$	–
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Two Watch Literals

current assignment:

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$$\neg x \vee y \vee \boxed{\neg z} \vee \boxed{\neg w}$$

$$\boxed{w} \vee \boxed{y} \vee a$$

$$\boxed{\neg a} \vee z \vee \boxed{b} \vee y$$

$$\neg x \vee \boxed{c} \vee \boxed{d} \vee e$$

Conflict-Directed Backtracking

current decision: $x_1 \mapsto T@6$ (T is assigned to x_1 at 6th decision)

$$
\begin{tikzcd}
 & & x_{10} \mapsto F@3 \arrow[dr] & & \\
 & x_2 \mapsto T@6 \arrow[dr] & & x_5 \mapsto T@6 \arrow[dr] & \\
x_1 \mapsto T@6 \arrow[ur] \arrow[dr] & & x_4 \mapsto T@6 \arrow[ur] \arrow[dr] & & \bot \\
 & x_3 \mapsto T@6 \arrow[ur] & & x_5 \mapsto F@6 \arrow[ur] & \\
x_9 \mapsto F@1 \arrow[ur] & & x_{11} \mapsto F@3 \arrow[ur] & &
\end{tikzcd}
$$

- if $x_1 \neq \bot$, back to level 3
- learned clause: $\neg x_1 \vee x_9 \vee x_{10} \vee x_{11}$

Implementation II

many SAT solvers use DPLL algorithm

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- restart rather than backtrack

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- quick restart with e.g. 32 decisions

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many SAT solvers use DPLL algorithm

- restart rather than backtrack
- quick restart with e.g. 32 decisions
- glue clauses (generalisation of unit clauses)

Termination

Term Rewriting

- pair of terms $\ell \to r$ is rewrite rule if ℓ is non-variable and $\mathsf{Var}(r) \subseteq \mathsf{Var}(\ell)$

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Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

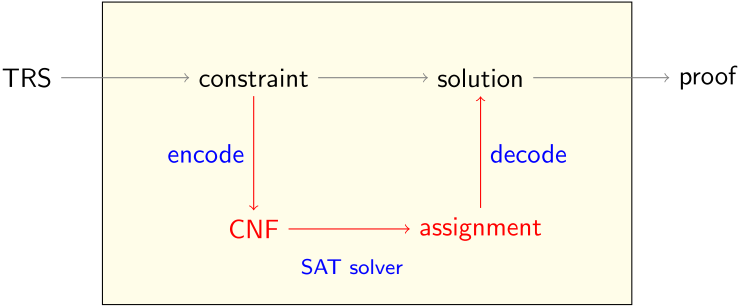
rewriting

$$\begin{aligned} \mathsf{s}(0) \times \mathsf{s}(0) &\to_{\mathcal{R}} \mathsf{s}(0) \times 0 + \mathsf{s}(0) \\ &\to_{\mathcal{R}} 0 + \mathsf{s}(0) \\ &\to_{\mathcal{R}} \mathsf{s}(0 + 0) \\ &\to_{\mathcal{R}} \mathsf{s}(0) \qquad \text{normal form} \end{aligned}$$

APPLICATIONS

- verification for functional programming
- theorem proving
- code optimization in compilers
- symbolic computation in mathematics
- ...

Implementation of Termination Tools



Precedence Termination

precedence $>$ is strict order on function symbols

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Definition

$\ell >_{\mathsf{prec}} r$ if $\mathsf{Var}(\ell) \supseteq \mathsf{Var}(r)$, $\ell = f(\ldots)$, and $f > g$ for all functions g in r

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Theorem

finite TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq >_{\mathsf{prec}}$ for some precedence $>$

Encoding of Precedence Termination Problem

Encoding of Precedence Termination Problem

Corollary

assume $f > g$ stands for propositional variable.

finite TRS $\mathcal{R}$ is terminating if $\vDash O \wedge I \wedge T$, where

$$O = \bigwedge_{\ell \to r \in \mathcal{R}} (\ell >_{\mathsf{prec}} r)$$

$$I = \bigwedge_{f \in \mathcal{F}} \neg(f > f)$$

$$T = \bigwedge_{f,g,h \in \mathcal{F}} ((f > g) \wedge (g > h) \to (f > h))$$

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NOTE

size of T is $O(n^3)$, where $n = |\mathcal{F}|$ (number of function symbols)

Example

prove termination of TRS

$$\begin{aligned}
\mathsf{not}(x) &\to \mathsf{if}(x, \mathsf{false}, \mathsf{true}) \\
\mathsf{and}(x, y) &\to \mathsf{if}(x, y, \mathsf{false}) \\
\mathsf{or}(x, y) &\to \mathsf{if}(x, \mathsf{true}, y) \\
\mathsf{equiv}(x, y) &\to \mathsf{if}(x, y, \mathsf{not}(y))
\end{aligned}
\qquad
\begin{aligned}
\mathsf{if}(\mathsf{true}, x, y) &\to x \\
\mathsf{if}(\mathsf{false}, x, y) &\to y
\end{aligned}$$

$$
O = \bigwedge \left\{ \begin{array}{rlrl}
\mathsf{not}(x) & >_{\mathsf{prec}} \mathsf{if}(x, \mathsf{false}, \mathsf{true}), & \mathsf{if}(\mathsf{true}, x, y) & >_{\mathsf{prec}} x, \\
\mathsf{and}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, y, \mathsf{false}), & \mathsf{if}(\mathsf{false}, x, y) & >_{\mathsf{prec}} y, \\
\mathsf{or}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, \mathsf{true}, y), & & \\
\mathsf{equiv}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, y, \mathsf{not}(y)) & &
\end{array} \right\}
$$

$$
= \left(\begin{array}{rcrcrc}
\mathsf{not} > \mathsf{if} & \wedge & \mathsf{not} > \mathsf{false} & \wedge & \mathsf{not} > \mathsf{true} & \wedge \\
\mathsf{and} > \mathsf{if} & \wedge & \mathsf{and} > \mathsf{false} & \wedge & & \\
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$O \wedge I \wedge T$ is satisfiable:

$$\mathsf{and} > \mathsf{or} > \mathsf{equiv} > \mathsf{not} > \mathsf{false} > \mathsf{true} > \mathsf{if}$$

hence TRS is terminating

Example

prove termination of TRS $\mathcal{R}$

$$
\begin{aligned}
x + \mathsf{0} &\to x & \qquad x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\
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precedence termination does not hold:

$$\begin{aligned} & \mathcal{R} \subseteq >_{\mathsf{prec}} \\ \Leftrightarrow\ & \mathsf{s}(x) + y >_{\mathsf{prec}} \mathsf{s}(x + y) \ \wedge\ \cdots \\ \Leftrightarrow\ & + > \mathsf{s}\ \wedge\ + > +\ \wedge\ \cdots \end{aligned}$$

is unsatisfiable

Lexicographic Path Order

Definition given precedence $>$

$s >_{\mathsf{lpo}} t$ if $s = f(s_1, \ldots, s_m)$, and either $t \in \mathsf{Var}(s)$ or $t = g(t_1, \ldots, t_n)$ and

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- $f > g$ and $s >_{\mathsf{lpo}} t_i$ for all $1 \leqslant i \leqslant n$, or
- $f = g$ and there is $1 \leqslant i \leqslant n$ with

  $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$, $s_i >_{\mathsf{lpo}} t_i$, and $s >_{\mathsf{lpo}} t_{i+1}, \ldots s >_{\mathsf{lpo}} t_n$

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finite TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq >_{\mathsf{lpo}}$ for some precedence $>$

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SAT encoding is similar to precedence termination

Example

TRS $\mathcal{R}$

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SAT solver finds precedence that fulfils $\mathcal{R} \subseteq >_{\mathsf{lpo}}$:

$$\times > + > \mathsf{s} > \mathsf{0}$$

hence $\mathcal{R}$ is terminating

Optimization

Annov, Codish, and Stuckey, RTA 2006

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transitivity constraint is $O(n^3)$

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total precedence $>$ can be represented by weight assignment

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- size of constraint is $O(N \log n)$, where N is size of TRS

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- weight assignment is admissible for precedence $>$ if

$$w_f > 0 \qquad \text{or} \qquad f \geqslant g$$

  for all unary functions f and all functions g

Definition Knuth-Bendix order; given weight w and precedence $>$

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Theorem Knuth and Bendix, 1970; Dershowitz, 1979

finite TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq >_{\mathsf{kbo}}$ for some weight $(w_0, w_f, w_g, \ldots)$ over $\mathbb{R}$ and precedence $>$

THEOREM Hirokawa, Zankl, Middeldorp, JAR 2010

next statements are equivalent

- $\mathcal{R} \subseteq >_{\mathsf{kbo}}$ for some weight w over $\mathbb{R}$ and precedence $>$
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here N is size of $\mathcal{R}$

Theorem Hirokawa, Zankl, Middeldorp, JAR 2010

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NOTE

large weights (> 15) are hardly required

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- complexity analysers (AProVE, CaT, TCT, ...) use same way

  POP* by Avanzini and Moser (2008)

Summary

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thank you for your kind attention!