Constructive and Classical Reasonings

Hajime Ishihara

School of Information Science
Japan Advanced Institute of Science and Technology
Nomi, Ishikawa 923-1292, Japan

JAIST Spring School 2012
– Formal Reasoning: Theorey and Application –
March 5 – 9, 2012

Contents

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- (individual) variables $v_0, v_1, \ldots$;
- (individual) constants $c_0, c_1, \ldots$;
- predicate (relation) symbols $R_0, R_1, \ldots$;
- function symbols $f_0, f_1, \ldots$;
- primitive logical operators $\wedge, \vee, \to, \bot, \forall, \exists$.

Terms

Terms are defined inductively by

- variables and constants are terms;
- if $t_1, \ldots, t_n$ are terms and f is an (n-ary) function symbol, then $f(t_1, \ldots, t_n)$ is a term.

The set $\mathrm{FV}(t)$ of free variables of a term t is defined inductively by

- $\mathrm{FV}(x) := \{x\}$ and $\mathrm{FV}(c) := \emptyset$;
- $\mathrm{FV}(f(t_1, \ldots, t_n)) := \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$.

Formulas

Formulas are defined inductively by

- $\perp$ is a formula;
- if $t_1, \ldots, t_n$ are terms and R is an (n-ary) predicate symbol, then $R(t_1, \ldots, t_n)$ is an (atomic) formula;
- if A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulas;
- if A is a formula and x is a variable, then $(\forall x A)$ and $(\exists x A)$ are formulas.

We introduce the abbreviations

- $\neg A \equiv A \rightarrow \perp$;
- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

Formulas

The set $\mathrm{FV}(A)$ of free variables of a formula A is defined inductively by

- $\mathrm{FV}(\bot) := \emptyset$;
- $\mathrm{FV}(R(t_1, \ldots, t_n)) := \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$;
- $\mathrm{FV}(A \circ B) := \mathrm{FV}(A) \cup \mathrm{FV}(B)$, where $\circ \in \{\wedge, \vee, \to\}$;
- $\mathrm{FV}(\forall x A) := \mathrm{FV}(\exists x A) := \mathrm{FV}(A) \setminus \{x\}$.

For a set Γ of formulas, let $\mathrm{FV}(\Gamma) := \bigcup\{\mathrm{FV}(A) \mid A \in \Gamma\}$.

Substitution (1)

Let s and t be terms, and let x be a variable. Then define a term $s[x/t]$ by

- $x[x/t] \equiv t$, $y[x/t] \equiv y$ $(x \not\equiv y)$, and $c[x/t] \equiv c$;
- $(f(t_1, \ldots, t_n))[x/t] \equiv f(t_1[x/t], \ldots, t_n[x/t])$.

Let A be a formula, let t be a term, and let x be a variable. Then define a formula $A[x/t]$ by

- $\bot[x/t] \equiv \bot$;
- $R(t_1, \ldots, t_n)[x/t] \equiv R(t_1[x/t], \ldots, t_n[x/t])$;
- $(A \circ B)[x/t] \equiv (A[x/t] \circ B[x/t])$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- $(\forall y A)[x/t] \equiv \forall y(A[x/t])$ and $(\exists y A)[x/t] \equiv \exists y(A[x/t])$, if $x \not\equiv y$, and $(\forall y A)[x/t] \equiv \forall y A$ and $(\exists y A)[x/t] \equiv \exists y A$, otherwise.

Free for (1)

Let A be a formula, let t be a term, and let x be a variable. Then define a predicate t is free for x in A by

- t is free for x in $\bot$;
- t is free for x in $R(t_1, \ldots, t_n)$;
- if t is free for x in A and B, then t is free for x in $(A \circ B)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- if t is free for x in A, $x \not\equiv y$ and $y \notin \mathrm{FV}(t)$, then t is free for x in $\forall y A$ and $\exists y A$.

Substitution (2)

We introduce

- a proposition symbol (0-ary predicate symbol) $*$ acting as a place holder.
- an abbreviation $\neg_* A \equiv A \to *$.

Let A and C be formulas. Then define a formula $A[*/C]$ by

- $\bot[*/C] \equiv \bot$;
- $*[*/C] \equiv C$ and $(R(t_1, \ldots, t_n))[*/C] \equiv R(t_1, \ldots, t_n)$;
- $(A \circ B)[*/C] \equiv (A[*/C] \circ B[*/C])$, where $\circ \in \{\wedge, \vee, \to\}$;
- $(\forall x A)[*/C] \equiv \forall x(A[*/C])$ and $(\exists x A)[*/C] \equiv \exists x(A[*/C])$,

Free for (2)

Let A and C be formulas. Then define a predicate C is free for $*$ in A by

- C is free for $*$ in $\perp$;
- C is free for $*$ in $*$ and $R(t_1, \ldots, t_n)$;
- if C is free for $*$ in A and B, then C is free for $*$ in $(A \circ B)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- if C is free for $*$ in A and $x \notin \mathrm{FV}(C)$, then C is free for $*$ in $\forall x A$ and $\exists x A$.

Natural Deduction System

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Minimal logic

Deductions are inductively defined as follows.

Basis: For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Induction step:

Minimal logic

- if $\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ A\end{array}$ and $\begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ B\end{array}$ are deductions, then

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{array}}{A \wedge B}\wedge\mathrm{I}$$

is a deduction with conclusion $A \wedge B$ and assumptions $\Gamma_1 \cup \Gamma_2$;

Minimal logic

- if $\begin{matrix}\Gamma\\ \mathcal{D}\\ A \wedge B\end{matrix}$ is a deduction, then

$$\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ A \wedge B\end{matrix}}{A}\ \wedge\mathrm{E}_r \qquad \dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ A \wedge B\end{matrix}}{B}\ \wedge\mathrm{E}_l$$

  are deductions with conclusions A and B, respectively, and assumptions Γ;

Minimal logic

- if $\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$ is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}}{A \vee B}\ \vee\mathrm{I}_r \qquad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}}{B \vee A}\ \vee\mathrm{I}_l$$

  are deductions with conclusions $A \vee B$ and $B \vee A$, respectively, and assumptions Γ;

Minimal logic

- if $\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ A \vee B\end{array}$, $\begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ C\end{array}$ and $\begin{array}{c}\Gamma_3\\ \mathcal{D}_3\\ C\end{array}$ are deductions, then

$$\dfrac{\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}\Gamma_3\\ \mathcal{D}_3\\ C\end{array}}{C}\ \vee\mathrm{E}$$

  is a deduction with conclusion C and assumptions $\Gamma_1 \cup (\Gamma_2 \setminus \{A\}) \cup (\Gamma_3 \setminus \{B\})$;

Minimal logic

- if $\begin{matrix}\Gamma\\ \mathcal{D}\\ B\end{matrix}$ is a deduction, then

$$\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ B\end{matrix}}{A \to B}\ \to\mathrm{I}$$

  is a deduction with conclusion $A \to B$ and assumptions $\Gamma \setminus \{A\}$.

Minimal logic

- if $\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ A \to B\end{array}$ and $\begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ A\end{array}$ are deductions, then

$$\dfrac{\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ A \to B\end{array} \quad \begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ A\end{array}}{B} \to\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Minimal logic

- if $\begin{matrix}\Gamma\\ \mathcal{D}\\ A\end{matrix}$ is a deduction, $x \notin \mathrm{FV}(\Gamma)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$, then

$$\frac{\begin{matrix}\Gamma\\ \mathcal{D}\\ A\end{matrix}}{\forall y A[x/y]}\ \forall\mathrm{I}$$

is a deduction with conclusion $\forall y A[x/y]$ and assumptions Γ.

Minimal logic

- if $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \forall x A\end{array}$ is a deduction and t is free for x in A, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \forall x A\end{array}}{A[x/t]}\ \forall\mathrm{E}$$

  is a deduction with conclusion $A[x/t]$ and assumptions Γ.

Minimal logic

- if $\begin{matrix}\Gamma \\ \mathcal{D} \\ A[x/t]\end{matrix}$ is a deduction, then

$$\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ A[x/t]\end{matrix}}{\exists x A}\ \exists\mathrm{I}$$

  is a deduction with conclusion $\exists x A$ and assumptions Γ.

Minimal logic

- if $\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ \exists y A[x/y]\end{array}$ and $\begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ C\end{array}$ are deductions, $x \notin \mathrm{FV}(C)$, $x \notin \mathrm{FV}(\Gamma_2 \setminus \{A\})$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$, then

$$\dfrac{\begin{array}{c}\Gamma_1\\ \mathcal{D}_1\\ \exists y A[x/y]\end{array} \quad \begin{array}{c}\Gamma_2\\ \mathcal{D}_2\\ C\end{array}}{C}\ \exists\mathrm{E}$$

is a deduction with conclusion C and assumptions $\Gamma_1 \cup (\Gamma_2 \setminus \{A\})$.

Minimal logic

We denote by

$$\Gamma \vdash_m A$$

that there is a deduction in minimal logic with conclusion A and assumptions Δ which is a subset of Γ.

Example (1)

$$
\dfrac{\dfrac{\dfrac{[\neg\neg A] \qquad \dfrac{\dfrac{[\neg\neg(A \to B)] \qquad \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg(A \to B)}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{\dfrac{\bot}{\neg\neg B}\,{\to}\mathrm{I}}\,{\to}\mathrm{E}}{\neg\neg A \to \neg\neg B}\,{\to}\mathrm{I}}{\neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)}\,{\to}\mathrm{I}
$$

Example (2)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg_*\neg_* A] \qquad \dfrac{\dfrac{[\neg_*\neg_*(A \to B)] \qquad \dfrac{\dfrac{[\neg_* B] \qquad \dfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}}{*}\,{\to}\mathrm{E}}{\neg_*(A \to B)}\,{\to}\mathrm{I}}{*}\,{\to}\mathrm{E}}{\neg_* A}\,{\to}\mathrm{I}}{*}\,{\to}\mathrm{E}}{\neg_*\neg_* B}\,{\to}\mathrm{I}}{\neg_*\neg_* A \to \neg_*\neg_* B}\,{\to}\mathrm{I}}{\neg_*\neg_*(A \to B) \to (\neg_*\neg_* A \to \neg_*\neg_* B)}\,{\to}\mathrm{I}
$$

Example (3)

$$
\dfrac{\dfrac{\dfrac{[\neg_* \neg_* A] \qquad \dfrac{\dfrac{[\neg_* \neg(A \rightarrow B)] \qquad \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A \rightarrow B] \quad [A]}{B}\,{\rightarrow}\mathrm{E}}{\bot}\,{\rightarrow}\mathrm{E}}{\neg(A \rightarrow B)}\,{\rightarrow}\mathrm{I}}{*}\,{\rightarrow}\mathrm{E}}{\neg_* A}\,{\rightarrow}\mathrm{I}}{*}\,{\rightarrow}\mathrm{E}}{\dfrac{\neg_* \neg B}{\neg_* \neg_* A \rightarrow \neg_* \neg B}\,{\rightarrow}\mathrm{I}}\,{\rightarrow}\mathrm{I}}{\neg_* \neg(A \rightarrow B) \rightarrow (\neg_* \neg_* A \rightarrow \neg_* \neg B)}\,{\rightarrow}\mathrm{I}
$$

Intuitionistc logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule.

- if $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$ is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_i$$

  is a deduction with conclusion A and assumptions Γ.

Intuitionistc logic

We denote by

$$\Gamma \vdash_i A$$

that there is a deduction in intuitionistic logic with conclusion A and assumptions in Γ.

Note that

$$\Gamma \vdash_m A \Rightarrow \Gamma \vdash_i A.$$

Example (4)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \qquad [A]}{\bot}}{B}\;\bot_i}{A \to B}}{\bot}}{\neg\neg A}}{\neg\neg B} \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{[B]}{A \to B}}{\bot}}{\neg B}}{\bot}}{\neg\neg(A \to B)}}{(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)}
$$

Example (5)

$$
\dfrac{\dfrac{[\neg_* \neg A \to \neg_* \neg_* B] \qquad \dfrac{\dfrac{[\neg_*(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \qquad [A]}{\bot}}{B}\,\bot_i}{A \to B}}{*}}{\neg_* \neg A}}{\neg_* \neg_* B} \qquad \dfrac{\dfrac{\neg_*(A \to B)] \qquad \dfrac{[B]}{A \to B}}{*}}{\neg_* B}}{\dfrac{*}{\dfrac{\neg_* \neg_* (A \to B)}{(\neg_* \neg A \to \neg_* \neg_* B) \to \neg_* \neg_* (A \to B)}}}
$$

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule.

- if $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$ is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

is a deduction with conclusion A and assumption $\Gamma \setminus \{\neg A\}$.

Classical logic

We denote by

$$\Gamma \vdash_c A$$

that there is a deduction in classical logic with the conclusion A and the assumptions in Γ.

Note that

$$\Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A.$$

Examples (6)

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\,{\to}\mathrm{E}}{A}\,\bot_c}{\neg\neg A \to A}\,{\to}\mathrm{I}$$

$$\dfrac{\dfrac{[\neg(A\vee\neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee\neg A)] \quad \dfrac{[A]}{A\vee\neg A}\,\vee\mathrm{I}_r}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{A\vee\neg A}\,\vee\mathrm{I}_l}{\bot}\,{\to}\mathrm{E}}{A\vee\neg A}\,\bot_c$$

The Gödel-Gentzen negative translation

Definition

The Gödel-Gentzen negative translation $(\cdot)^g$ on the formulas of predicate logic is defined inductively by

- $\bot^g \equiv \bot$;
- $P^g \equiv \neg\neg P$ for P atomic;
- $(A \wedge B)^g \equiv A^g \wedge B^g$;
- $(A \vee B)^g \equiv \neg(\neg A^g \wedge \neg B^g)$;
- $(A \to B)^g \equiv A^g \to B^g$;
- $(\forall x A)^g \equiv \forall x A^g$;
- $(\exists x A)^g \equiv \neg\forall x \neg A^g$.

The Gödel-Gentzen negative translation

Lemma

- $\vdash_m \neg A \leftrightarrow \neg\neg\neg A$,
- $\vdash_m \neg\neg A \wedge \neg\neg B \leftrightarrow \neg\neg(A \wedge B)$,
- $\vdash_m \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$,
- $\vdash_m \forall x \neg\neg A \leftrightarrow \neg\neg \forall x \neg\neg A$.

The Gödel-Gentzen negative translation

Lemma

$\vdash_m A^g \leftrightarrow \neg\neg A^g$.

Proof.

By induction on the complexity of A.

Basis: $\vdash_m \bot \leftrightarrow \neg\neg\bot$ and $\vdash_m \neg\neg P \leftrightarrow \neg\neg P$.

Induction step:

- $\vdash_m A^g \wedge B^g \leftrightarrow \neg\neg A^g \wedge \neg\neg B^g \leftrightarrow \neg\neg(A^g \wedge B^g)$.
- $\vdash_m \neg(\neg A^g \wedge \neg B^g) \leftrightarrow \neg\neg\neg(\neg A^g \wedge B^g)$.
- $\vdash_m (A^g \to B^g) \to \neg\neg(A^g \to B^g) \to (\neg\neg A^g \to \neg\neg B^g) \leftrightarrow (A^g \to B^g)$.
- $\vdash_m \forall x A^g \leftrightarrow \forall x \neg\neg A^g \leftrightarrow \neg\neg\forall x \neg\neg A^g \leftrightarrow \neg\neg\forall x A^g$.
- $\vdash_m \neg\forall\neg A^g \leftrightarrow \neg\neg\neg\forall x \neg A^g$.

□

The Gödel-Gentzen negative translation

Proposition

If $\Gamma \vdash_c A$, *then* $\Gamma^g \vdash_m A^g$, *where* $\Gamma^g = \{B^g \mid B \in \Gamma\}$.

Proof.

By induction on the depth of a deduction of $\Gamma \vdash_c A$.
Basis: A is translated into A^g.
Induction step:

□

The Gödel-Gentzen negative translation

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{A \vee B}\ \vee\mathrm{I}_r$$

is transfered into

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}^g\\ A^g\end{matrix} \qquad \dfrac{[\neg A^g \wedge \neg B^g]}{\neg A^g}}{\perp}}{(A \vee B)^g}\ \to\mathrm{I}_r$$

The Gödel-Gentzen negative translation

$$\dfrac{\begin{matrix} \mathcal{D}_1 \\ A \vee B \end{matrix} \qquad \begin{matrix} [A] \\ \mathcal{D}_2 \\ C \end{matrix} \qquad \begin{matrix} [B] \\ \mathcal{D}_3 \\ C \end{matrix}}{C}\ \vee\mathrm{E}$$

is translated into

$$\dfrac{\begin{matrix} \vdots \\ \neg\neg C^g \to C^g \end{matrix} \qquad \dfrac{\dfrac{\begin{matrix} \mathcal{D}_1^g \\ (A \vee B)^g \end{matrix} \qquad \dfrac{\dfrac{\dfrac{[\neg C^g] \qquad \begin{matrix} [A^g] \\ \mathcal{D}_2^g \\ C^g \end{matrix}}{\bot}}{\neg A^g} \qquad \dfrac{\dfrac{[\neg C^g] \qquad \begin{matrix} [B^g] \\ \mathcal{D}_3^g \\ C^g \end{matrix}}{\bot}}{\neg B^g}}{\neg A^g \wedge \neg B^g}\ \wedge\mathrm{I}}{\bot}}{\neg\neg C^g}}{C^g}$$

The Gödel-Gentzen negative translation

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A[x/t]\end{array}}{\exists x A}\ \exists\mathrm{I}$$

is transfered into

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}^g\\ (A[x/t])^g\end{array} \quad \dfrac{[\forall x \neg A^g]}{\neg (A[x/t])^g}}{\bot}}{(\exists x A)^g}\ \to\mathrm{I}$$

The Gödel-Gentzen negative translation

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \exists y A[x/y]\end{matrix} \qquad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix}}{C}\ \exists\mathrm{E}$$

is translated into

$$\dfrac{\begin{matrix}\vdots\\ \neg\neg C^g \to C^g\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^g\\ (\exists y A[x/y])^g\end{matrix} \qquad \dfrac{\dfrac{\dfrac{[\neg C^g] \qquad \begin{matrix}[A^g]\\ \mathcal{D}_2^g\\ C^g\end{matrix}}{\bot}}{\neg A^g}}{\forall y \neg (A[x/y])^g}\ \forall\mathrm{I}}{\bot}}{\neg\neg C^g}}{C^g}$$

The Gödel-Gentzen negative translation

$$\cfrac{\begin{array}{c}[\neg A]\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

is translated into

$$\cfrac{\begin{array}{c}\vdots\\ \neg\neg A^g \to A^g\end{array} \qquad \cfrac{\begin{array}{c}[\neg A^g]\\ \mathcal{D}^g\\ \bot\end{array}}{\neg\neg A^g}\ {\to}\mathrm{I}}{A^g}\ {\to}\mathrm{E}$$

Negative formulas

Definition

We define the class $\mathcal{N}$ of negative formulas as follows. Let P range over atomic formulas, and N and N' over $\mathcal{N}$. Then $\mathcal{N}$ is inductively generated by the clause

$$\bot, \neg P, N \wedge N', N \to N', \forall x N \in \mathcal{N}.$$

Negative formulas

Lemma

If $N \in \mathcal{N}$, then $\vdash_m N \leftrightarrow N^g$.

Proof.

By induction on the definition of $\mathcal{N}$.

Basis: $\vdash_m \bot \leftrightarrow \bot$ and $\vdash_m \neg P \leftrightarrow \neg\neg\neg P$.

Induction step:

- $\vdash_m N \wedge N' \leftrightarrow N^g \wedge N'^g$.
- $\vdash_m (N \to N') \leftrightarrow (N^g \to N'^g)$.
- $\vdash_m \forall x N \leftrightarrow \forall x N^g$.

□

The conservative extension result

Theorem
If $\Gamma \subseteq \mathcal{N}$ and $A \in \mathcal{N}$, then $\Gamma \vdash_c A$ implies $\Gamma \vdash_m A$.

Proof.
Suppose that $\Gamma \subseteq \mathcal{N}$ and $A \in \mathcal{N}$. Then $\Gamma \vdash_m B^g$ for each $B \in \Gamma$ and $A^g \vdash_m A$ by Lemma. Therefore, if $\Gamma \vdash_c A$, then $\Gamma^g \vdash_m A^g$, and so $\Gamma \vdash_m A$. □

Leivant's conservative extension result

Definition

We define simultaneously classes $\mathcal{S}$ (spreading), $\mathcal{W}$ (wiping) and $\mathcal{I}$ (isolating) of formulas as follows. Let P range over atomic formulas, S and S' over $\mathcal{S}$, W and W' over $\mathcal{W}$, and I and I' over $\mathcal{I}$. Then $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{I}$ are inductively generated by the clauses

- $\bot, P, S \wedge S', S \vee S', \forall x S, \exists x S, I \to S \in \mathcal{S}$;
- $\bot, W \wedge W', \forall x W, S \to W \in \mathcal{W}$;
- $P, W, I \wedge I', I \vee I', \exists x I, S \to I \in \mathcal{I}$.

Note that

$$\mathcal{N} \subseteq \mathcal{S} \cap \mathcal{W}.$$

Leivant's conservative extension result

Lemma

- $\vdash_m \neg(\neg A \wedge \neg B) \leftrightarrow \neg\neg(A \vee B)$,
- $\vdash_m (\neg\neg A \rightarrow \neg\neg B) \leftrightarrow (A \rightarrow \neg\neg B)$,
- $\vdash_i \neg\neg(A \rightarrow B) \leftrightarrow (\neg\neg A \rightarrow \neg\neg B)$,
- $\vdash_m \neg\forall x \neg A \leftrightarrow \neg\neg\exists x A$.

Leivant's conservative extension result

Proposition

- *If* $A \in \mathcal{S}$*, then* $\vdash_i A \to A^g$*;*
- *If* $A \in \mathcal{W}$*, then* $\vdash_i A^g \to A$*;*
- *If* $A \in \mathcal{I}$*, then* $\vdash_i A^g \to \neg\neg A$*.*

Leivant's conservative extension result

Proof.

By simultaneous induction on the definition of $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{I}$.

Basis: $\vdash_m \bot \to \neg\neg\bot$ and $\vdash_m P \to \neg\neg P$.

Induction step:

- $\vdash_i S \vee S' \to \neg\neg(S^g \vee S'^g) \leftrightarrow \neg(\neg S^g \wedge \neg S'^g)$.
- $\vdash_i \exists x S \to \neg\neg\exists x S^g \leftrightarrow \neg\forall x \neg S^g$.
- $\vdash_i (I \to S) \to (\neg\neg I \to \neg\neg S) \to (I^g \to \neg\neg S^g) \leftrightarrow (I^g \to S^g)$.
- $\vdash_i \neg(\neg I^g \wedge \neg I'^g) \to \neg(\neg I \wedge \neg I') \leftrightarrow \neg\neg(I \vee I')$.
- $\vdash_i \neg\forall x \neg I^g \to \neg\forall x \neg I \leftrightarrow \neg\neg\exists x I$.
- $\vdash_i (S^g \to I^g) \to (S \to \neg\neg I) \leftrightarrow \neg\neg(S \to I)$.

□

Leivant's conservative extension result

Theorem (Leivant 1985)

If $\Gamma \subseteq \mathcal{S}$ and $A \in \mathcal{W}$, then $\Gamma \vdash_c A$ implies $\Gamma \vdash_i A$.

Proof.

Suppose that $\Gamma \subseteq \mathcal{S}$ and $A \in \mathcal{W}$. Then $\Gamma \vdash_i B^g$ for each $B \in \Gamma$ and $A^g \vdash_i A$ by Proposition. Therefore, if $\Gamma \vdash_c A$, then $\Gamma^g \vdash_m A^g$, and so $\Gamma \vdash_i A$. □

A variant of the Gödel-Gentzen translation

Definition

The $*$-negative translation $(\cdot)^*$ on the formulas of predicate logic is defined by $A^* \equiv A^g[\bot/*]$, that is,

- $\bot^* \equiv *$;
- $P^* \equiv \neg_* \neg_* P$ for P atomic;
- $(A \wedge B)^* \equiv A^* \wedge B^*$;
- $(A \vee B)^* \equiv \neg_*(\neg_* A^* \wedge \neg_* B^*)$;
- $(A \to B)^* \equiv A^* \to B^*$;
- $(\forall x A)^* \equiv \forall x A^*$;
- $(\exists x A)^* \equiv \neg_* \forall x \neg_* A^*$.

A variant of the Gödel-Gentzen translation

Lemma

$\vdash_m A^* \leftrightarrow \neg_* \neg_* A^*$.

Proof.

Note that $\bot$ is treated as an arbitrary proposition letter in minimal logic and $A^* \leftrightarrow \neg_* \neg_* A^* \equiv (A^g \leftrightarrow \neg\neg A^g)[\bot/*]$. Since $\vdash_m A^g \leftrightarrow \neg\neg A^g$, we have $\vdash_m A^* \leftrightarrow \neg_* \neg_* A^*$. □

Proposition

If $\Gamma \vdash_c A$, *then* $\Gamma^* \vdash_m A^*$, *where* $\Gamma^* = \{B^* \mid B \in \Gamma\}$.

Proof.

Since $\Gamma^* \equiv \Gamma^g[\bot/*]$ and $A^* \equiv A^g[\bot/*]$, if $\Gamma \vdash_c A$, then $\Gamma^g \vdash_m A^g$, and hence $\Gamma^* \vdash_m A^*$. □

Another conservative extension result

Definition

We define simultaneously classes $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{K}$ of formulas as follows. Let P range over atomic formulas, Q and Q' over $\mathcal{Q}$, R and R' over $\mathcal{R}$, J and J' over $\mathcal{J}$, and K and K' over $\mathcal{K}$. Then $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{K}$ are inductively generated by the clauses

- $\bot, P, Q \wedge Q', Q \vee Q', \forall x Q, \exists x Q, J \to Q \in \mathcal{Q}$;
- $\bot, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}$;
- $\bot, P, J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}$;
- $J, K \wedge K', \forall x K, Q \to K \in \mathcal{K}$.

Another conservative extension result

Lemma

- $\vdash_m (A \to B) \to (\neg_* \neg_* A \to \neg_* \neg_* B)$,
- $\vdash_m (\neg_* \neg_* A \to \neg_* \neg_* B) \leftrightarrow (A \to \neg_* \neg_* B)$,
- $\vdash_m \neg_* \neg (A \to B) \to (\neg_* \neg_* A \to \neg_* \neg B)$,
- $\vdash_i (\neg_* \neg A \to \neg_* \neg_* B) \to \neg_* \neg_* (A \to B)$.

Another conservative extension result

Proposition

- *If* $A \in \mathcal{Q}$, *then* $\vdash_i A \to A^*$;
- *If* $A \in \mathcal{R}$, *then* $\vdash_i \neg_* \neg A \to A^*$;
- *If* $A \in \mathcal{J}$, *then* $\vdash_i A^* \to \neg_* \neg_* A$.

Another conservative extension result

Proof.
By simultaneous induction on the definition of $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{J}$.
Basis: $\vdash_i \bot \rightarrow *$, $\vdash_m P \rightarrow \neg_*\neg_* P$, $\vdash_m \neg_*\neg\bot \rightarrow *$, and $\vdash_m * \rightarrow \neg_*\neg_*\bot$.
Induction step:

- $\vdash_i (J \rightarrow Q) \rightarrow (\neg_*\neg_* J \rightarrow \neg_*\neg_* Q) \leftrightarrow (J^* \rightarrow \neg_*\neg_* Q^*) \leftrightarrow (J^* \rightarrow Q^*)$,
- $\vdash_i \neg_*\neg(J \rightarrow R) \rightarrow (\neg_*\neg_* J \rightarrow \neg_*\neg R) \rightarrow (J^* \rightarrow R^*)$,
- $\vdash_i (R^* \rightarrow J^*) \rightarrow (\neg_*\neg R \rightarrow \neg_*\neg_*) \rightarrow \neg_*\neg_*(R \rightarrow J)$.

□

Another conservative extension result

A set Γ of formulas is closed under $(\cdot)^*$ if $\Gamma \vdash_i A^*[*/C]$ for each $A \in \Gamma$ and C being free for $*$ in A^*.

Theorem (I 2000)

If Γ is a set of formulas closed under $(\cdot)^*$ and $A \in \mathcal{K}$, then $\Gamma \vdash_c A$ implies $\Gamma \vdash_i A$.

Corollary

If $\Gamma \subseteq \mathcal{Q}$ and $A \in \mathcal{K}$, then $\Gamma \vdash_c A$ implies $\Gamma \vdash_i A$.

Another conservative extension result

Proof of Theorem.

By induction on the definition of $\mathcal{K}$.

Basis: Suppose that $\Gamma \vdash_c J$ and $J \in \mathcal{J}$. Then $\Gamma^* \vdash_m J^*$, and hence $\Gamma^* \vdash_i \neg_* \neg_* J$. Therefore $\Gamma^*[*/J] \vdash_i (\neg_* \neg_* J)[*/J] \equiv (J \to J) \to J$, and, since Γ is closed under $(\cdot)^*$, we have $\Gamma \vdash_i J$.

Induction step:

- Suppose that $\Gamma \vdash_c K \wedge K'$. Then $\Gamma \vdash_c K$ and $\Gamma \vdash_c K'$, and hence $\Gamma \vdash_i K$ and $\Gamma \vdash_i K'$ by induction hypothesis. Thus $\Gamma_i \vdash K \wedge K'$.
- Suppose that $\Gamma \vdash_c \forall x K$. Then $\Gamma \vdash_c K$, and hence $\Gamma \vdash_i K$ by induction hypothesis. Thus $\Gamma_i \vdash \forall x K$.
- Suppose that $\Gamma \vdash_c Q \to K$. Then $\Gamma \cup \{Q\} \vdash_c K$, and therefore, since $\Gamma \cup \{Q\}$ is closed under $(\cdot)^*$, we have $\Gamma \cup \{Q\} \vdash_i K$ by induction hypothesis. Thus $\Gamma \vdash_i Q \to K$.

□

Application (Barr's theorem)

Definition

We define classes $\mathcal{G}$ and $\mathcal{G}_I$ of geometric formulas and geometiric implications, respectively, as follows. Let P range over atomic formulas, G and G' over $\mathcal{G}$ and G_I over $\mathcal{G}_I$. Then $\mathcal{G}$ and $\mathcal{G}_I$ are inductively generated by the clauses

- $\bot, \top, P, G \wedge G', G \vee G', \exists x G \in \mathcal{G}$;
- $G \to G', \forall x G_I \in \mathcal{G}_I$,

where $\top \equiv \bot \to \bot$.

Theorem (Barr's thoerem)

If $\Gamma \subseteq \mathcal{G}_I$ *and* $A \in \mathcal{G}_I$*, then* $\Gamma \vdash_c A$ *implies* $\Gamma \vdash_i A$*.*

Proof.

Note that $\mathcal{G} \subseteq \mathcal{Q} \cap \mathcal{J}$, and hence $\mathcal{G}_I \subseteq \mathcal{Q} \cap \mathcal{K}$. □

Application (first-order arithmetic)

Theorem

If $A \in \mathcal{K}$, then $\mathbf{PA} \vdash A$ *implies* $\mathbf{HA} \vdash A$.

Proof.

The axioms and the axiom schema of first-order arithmetic are closed under $(\cdot)^*$. □

Corollary

PA *is conservative over* **HA** *with respect to* Π^0_2 *formulas, and, moreover, the following form of formulas.*

$$\forall x[\forall u_1 \exists v_1 \ldots \forall u_n \exists v_n (s(\vec{u}, \vec{v}, x) = 0) \to \exists y(t(x, y) = 0)].$$

Proof.

Π^0_2 formulas and the formulas of the above form are in $\mathcal{K}$. □

Application (first-order arithmetic)

Moreover, we can extend the class $\mathcal{R}$ (and hence the classes $\mathcal{J}$, $\mathcal{Q}$ and $\mathcal{K}$) by the clause

$$\bot, P, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R},$$

because, for atomic P, since $\mathbf{HA} \vdash P \vee \neg P$, we have $\mathbf{HA} \vdash \neg_* \neg P \to P^*$, and the following proposition holds for the extended classes in $\mathbf{HA}$.

Proposition

- *If* $A \in \mathcal{Q}$, *then* $\mathbf{HA} \vdash A \to A^*$;
- *If* $A \in \mathcal{R}$, *then* $\mathbf{HA} \vdash \neg_* \neg A \to A^*$;
- *If* $A \in \mathcal{J}$, *then* $\mathbf{HA} \vdash A^* \to \neg_* \neg_* A$.

Schwichtenberg's question

Helmut Schwichtenberg has asked about a possibility of extending the classes $\mathcal{R}$ and $\mathcal{J}$, defined by the clauses

- $\bot, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}$;
- $\bot, P, J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}$,

by introducing $\exists$ and $\forall$ in the clauses, respectively, to the classes $\mathcal{R}_0$ and $\mathcal{J}_0$, defined by

- $\bot, R \wedge R', R \vee R', \forall x R, \exists x R, J \to R \in \mathcal{R}_0$;
- $\bot, P, J \wedge J', J \vee J', \forall x J, \exists x J, R \to J \in \mathcal{J}_0$.

Intuitionistic sequent calculus **G3i**

$$P, \Gamma \Rightarrow P \quad \mathrm{Ax} \qquad\qquad \bot, \Gamma \Rightarrow A \quad \mathrm{L}\bot$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \mathrm{L}\wedge \qquad\qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ \mathrm{L}\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ \mathrm{R}\vee_1 \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ \mathrm{R}\vee_2$$

$$\frac{A \rightarrow B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C}\ \mathrm{L}\rightarrow \qquad\qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ \mathrm{R}\rightarrow$$

where in Ax, P is atomic.

Intuitionistic sequent calculus **G3i**

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ \mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}\ \mathrm{R}\forall$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ \mathrm{L}\exists \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\ \mathrm{R}\exists$$

where in $\mathrm{R}\forall$, $y \notin \mathrm{FV}(\Gamma)$, $y \equiv x$ or $y \notin \mathrm{FV}(A)$, and in $\mathrm{L}\exists$, $y \notin \mathrm{FV}(\Gamma, C)$, $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

We denote by

$$\vdash_i \Gamma \Rightarrow A$$

that there is a deduction of the sequent $\Gamma \Rightarrow A$ in **G3i**.
Note that

$$\vdash_i \Gamma \Rightarrow A \text{ if and only if } \Gamma \vdash_i A.$$

Classical sequent calculus **G3c**

$$P, \Gamma \Rightarrow \Delta, P \quad \mathrm{Ax} \qquad\qquad \bot, \Gamma \Rightarrow \Delta \quad \mathrm{L}\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\wedge \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\rightarrow \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathrm{R}\rightarrow$$

where in Ax, P is atomic.

Classical sequent calculus **G3c**

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ \mathrm{R}\forall$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ \mathrm{L}\exists \qquad \frac{\Gamma \Rightarrow \Delta, A[x/t], \exists x A}{\Gamma \Rightarrow \Delta, \exists x A}\ \mathrm{R}\exists$$

where in $\mathrm{R}\forall$ and $\mathrm{L}\exists$, $y \notin \mathrm{FV}(\Gamma, \Delta)$, $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

We denote by

$$\vdash_c \Gamma \Rightarrow \Delta$$

that there is a deduction of the sequent $\Gamma \Rightarrow \Delta$ in **G3c**.
Note that

$$\vdash_c \Gamma \Rightarrow A \text{ if and only if } \Gamma \vdash_c A.$$

Structural rules

The structural rules (weakening, contraction and cut) are admissible in **G3c** and in **G3i**.

Those structural rules are formulated in **G3i** as follows:

$$\frac{\Gamma \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \mathrm{LW} \quad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \mathrm{LC}$$

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\ \mathrm{Cut}.$$

Some conservative extension results

Definition

We define simultaneously classes $\mathcal{R}_0$, $\mathcal{J}_0$, $\mathcal{Q}_m$ and $\mathcal{K}_m$ ($m = 1, 2$) of formulas as follows. Let P range over atomic formulas and $*$, R and R' over $\mathcal{R}_0$, J and J' over $\mathcal{J}_0$, Q_m and Q'_m over $\mathcal{Q}_m$, and K_m and K'_m over $\mathcal{K}_m$ ($m = 1, 2$). Then $\mathcal{R}_0$, $\mathcal{J}_0$, $\mathcal{Q}_m$ and $\mathcal{K}_m$ ($m = 1, 2$) are inductively generated by the clauses

- $\bot, R \wedge R', R \vee R', \forall x R, \exists x R, J \to R \in \mathcal{R}_0$;
- $\bot, P, J \wedge J', J \vee J', \forall x J, \exists x J, R \to J \in \mathcal{J}_0$;
- $P, R, Q_1 \wedge Q'_1, Q_1 \vee Q'_1, \exists x Q_1, J \to Q_1 \in \mathcal{Q}_1$;
- $P, R, Q_2 \wedge Q'_2, \forall x Q_2, \exists x Q_2, J \to Q_2 \in \mathcal{Q}_2$;
- $J, K_m \wedge K'_m, \forall x K_m, Q_m \to K_m \in \mathcal{K}_m$ ($m = 1, 2$).

Some conservative extension results

Proposition

If either $\Gamma \subseteq \mathcal{Q}_1$ *or* $\Gamma \subseteq \mathcal{Q}_2$, $\Delta \subseteq \mathcal{R}_0$ *and* $\Sigma \subseteq \mathcal{J}_0$, *then* $\vdash_c \Gamma, \Delta \Rightarrow \Sigma$ *implies* $\vdash_i \Gamma, \neg_* \neg \Delta, \neg_* \Sigma \Rightarrow *$.

Proof.

By induction on the depth of a deduction of $\vdash_c \Gamma, \Delta \Rightarrow \Sigma$. □

Some conservative extension results

Theorem (I 2011)

For each $m = 1, 2$*, if* $\Gamma \subseteq \mathcal{Q}_m$ *and* $A \in \mathcal{K}_m$*, then* $\vdash_c \Gamma \Rightarrow A$ *implies* $\vdash_i \Gamma \Rightarrow A$.

Proof.

By induction on the definition of $\mathcal{K}_m$.
Suppose that $A \in \mathcal{J}_0$ and $\vdash_c \Gamma \Rightarrow A$. Then $\vdash_i \Gamma, \neg_* A \Rightarrow *$, by Proposition. Therefore, since A is free for $*$ in $\Gamma, \neg_* A, *$, we have

$$\vdash_i \Gamma, A \to A \Rightarrow A,$$

and so $\vdash_i \Gamma \Rightarrow A$. □

Positive and negative occurrences

- C occurs positively in C;
- if C occurs positively and negatively in A or in B, then C occurs positively and negatively, respectively, in $A \wedge B$ and in $A \vee B$;
- if C occurs negatively in A or positively in B, and positively in A or negatively in B, then C occurs positively, and negatively, respectively, in $A \rightarrow B$;
- if C occurs positively and negatively in A, then C occurs positively and negatively, respectively, in $\forall x A$ and in $\exists x A$.

Strictly positive occurrences

- C occurs strict positively in C;
- if C occurs strict positively in A or in B, then C occurs strict positively in $A \wedge B$ and in $A \vee B$;
- if C occurs strict positively in B, then C occurs strict positively in $A \to B$;
- if C occurs strict positively in A, then C occurs strict positively in $\forall x A$ and in $\exists x A$.

Some conservative extension results

Lemma

If $\vdash_i *^n, \Gamma, \neg_* \Delta \Rightarrow A$*, where* $*^n$ *stands for* n *copies of* $*$*, and* $*$ *does not occur in* Γ *negatively nor positively in* A*, then* $\vdash_i \Gamma \Rightarrow A$*.*

Proof.

By induction on the depth of a deduction of
$\vdash_i *^n, \Gamma, \neg_* \Delta \Rightarrow A$. □

Lemma

If $\vdash_i \Gamma, \neg_* A[x/y], \neg_* \Delta \Rightarrow *$*, where* $*$ *does not occur in the antecedent negatively, there is no strictly positive occurrence of* $\forall$ *in* Γ*, and* $y \notin \mathrm{FV}(\Gamma)$*,* $y \equiv x$ *or* $y \notin \mathrm{FV}(A)$*, then*
$\vdash_i \Gamma, \neg_* \forall x A, \neg_* \Delta \Rightarrow *$*.*

Proof.

By induction on the depth of a deduction of
$\vdash_i \Gamma, \neg_* A[x/y], \neg_* \Delta \Rightarrow *$. □

Some conservative extension results

Lemma

If $\vdash_i \Gamma, \neg_* A, \neg_* \Delta \Rightarrow *$*, where* $*$ *does not occur in the antecedent negatively, and there is no strictly positive occurrence of* $\vee$ *in* Γ*, then either* $\vdash_i \Gamma \Rightarrow A$*, or* $\vdash_i \Gamma, \neg_* \Delta \Rightarrow *$*.*

Proof.

By induction on the depth of a deduction of
$\vdash_i \Gamma, \neg_* A, \neg_* \Delta \Rightarrow *$. □

Corollary

If $\vdash_i \Gamma, \neg_* A[x/y], \neg_* \Delta \Rightarrow *$*, where* $*$ *does not occur in the antecedent negatively, there is no strictly positive occurrence of* $\vee$ *in* Γ*, and* $y \notin \mathrm{FV}(\Gamma)$*,* $y \equiv x$ *or* $y \notin \mathrm{FV}(A)$*, then*
$\vdash_i \Gamma, \neg_* \forall x A, \neg_* \Delta \Rightarrow *$*.*