Coalgebraic Logics: Modalities Beyond Kripke Semantics

Part I: Modelling

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Modal Logic = Propositional Logic + Operators

Propositional Logic.

Syntax.

- Atomic propositions ('rains', 'snows', . . .)
- propositional connectives ($\wedge, \vee, \rightarrow, \neg$)

Example.

$$\mathsf{rains} \rightarrow (\mathsf{wet} \vee \mathsf{umbrella})$$

Modal Logic.

Syntax.

- propositional logic
- additional operators, e.g. 'possibly', 'probably', 'eventually', 'allowed', . . .

Example.

$$\mathsf{necessarily}(\mathsf{smoke} \rightarrow \mathsf{fire})$$

Algebraic Semantics

Propositional Logic.

Boolean Algebras: a set $\mathbb{A}$ with operations

$$\wedge, \vee, \rightarrow: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A} \qquad \neg : \mathbb{A} \rightarrow \mathbb{A} \qquad \top, \bot : \mathbb{A}$$

satisfying laws of propositional logic, e.g. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Modal Logic.

Boolean algebras with additional operators, e.g.

$$\textsf{possibly} : \mathbb{A} \rightarrow \mathbb{A}$$

Question. Which equations describe 'possibly', 'probably', 'eventually', 'allowed'?

Possible World Models

Propositional Connectives.

- boolean algebra $\mathcal{P}(W)$ over a set W of 'possible worlds'
- standard interpretation of connectives, e.g. $A \wedge B = A \cap B$

Modalities. Operators via extra structure, e.g. intended meaning, e.g.

- relations $R \subseteq W \times W$

  operator $\Diamond : \mathcal{P}(W) \to \mathcal{P}(W), A \mapsto \{w \in W \mid \exists w' \in A.R(w, w')\}$

- transition probabilities $\mu : W \times W \to [0, 1]$

  operator $L_p : \mathcal{P}(W) \to \mathcal{P}(W), A \mapsto \{w \in W \mid \sum_{w' \in A} \mu(w, w') \geq p\}$

Question. What are the 'right' equations for the operators?

Answer. Those that hold in all possible world models!

Why Modal Logics?

Non-deterministic computation. Computation as transition relation $R \subseteq W \times W$

$$\mathsf{start} \rightarrow \mathsf{always}(\neg \mathsf{failure})$$

Probabilistic Computation. Transition probabilities $\mu : W \times W \rightarrow [0, 1]$

$$\mathsf{request} \rightarrow L_{0.8}(\mathsf{acknowledgement})$$

Knowledge Representation. Concepts $C \subseteq W$ linked by relations $R \subseteq W \times W$

$$\mathsf{car} \rightarrow \langle \mathsf{has} \rangle \mathsf{wheel}$$

Multi-Agent Systems. Agents $a, b, c, \ldots$ form coalitions

$$[a](\mathsf{spy} \rightarrow \mathsf{capture}) \rightarrow [a, b](\mathsf{spy} \rightarrow \mathsf{extradiction})$$

Observations

Syntax vs **Semantics**

- two-way relationship (we don't take sides)
- syntax is *uniform*, semantics *varies wildly*

Questions.

- Can we find *general principles* that link syntax and semantics?
- Is there a *uniform* view on modal semantics?

(Partial) Answers.

- Yes, we can – stay tuned!
- The slogan is: *Modal Semantics is Co-Algebraic.*

What I want to say:

Lecture 1: Modelling

- first and foremost: examples
- basic definitions
- the Hennessy-Milner Property

Lecture 2: Reasoning

- one-step rules
- soundness, completeness
- the finite model property

Lecture 3: Deciding

- strictly complete rule-sets
- Sequent systems and complexity
- short demo

Lecture 4: Combining

- Composition, semantically
- Completeness and Complexity
- short demo

A Cook's Tour Through Modal Logics

Standard Modal Logic

- $\Diamond\phi$
- ϕ can be true

Conditional Logic

- $\phi \Rightarrow \psi$
- ψ if ϕ

Coalition Logic

- $[C]\phi$
- Agents C can force ϕ

Graded Modal Logic

- $\Diamond_k\phi$
- more than k successors validate ϕ

Probabilistic Modal Logic

- $L_p\phi$
- ϕ holds with probability $\geq p$

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Similarities

- all subject to the same *questions*: Completeness, decidability, complexity, ...
- arise in combination: probabilities and non-determinism, uncertainty in games

Graded Modal Logic

- $\Diamond_k\phi$
- more than k successors validate ϕ

Probabilistic Modal Logic

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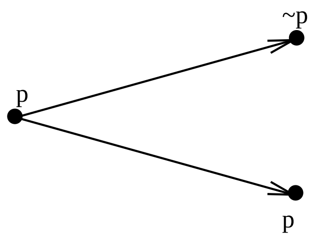
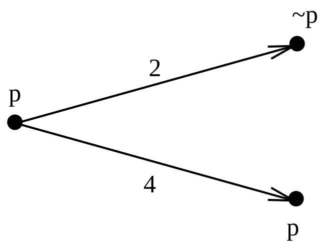
A Cook's Tour Through Modal Semantics

Kripke Frames.



$$C \to \mathcal{P}(C)$$

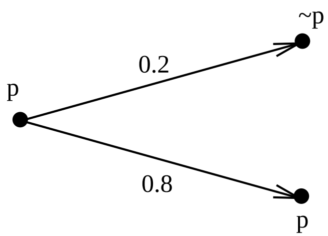
Multigraph Frames.



$$C \to \mathsf{B}(C)$$

$$\mathsf{B}(X) = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Probabilistic Frames.



$$C \to \mathsf{D}(C)$$

$$\mathsf{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

More Examples

Neighbourhood Frames.

$$C \to \mathcal{P}\mathcal{P}(C) = \mathsf{N}(C)$$

mapping each world $c \in C$ to a set of neighbourhoods

Game Frames over a set N of agents

$$C \to \{((S_n)_{n \in N}, f) \mid f : \prod_n S_n \to C\} = \mathsf{G}(C)$$

associating to each state $c \in C$ a *strategic game* with strategy sets S_n and outcome function f

Conditional Frames.

$$C \to \{f : \mathcal{P}(C) \to \mathcal{P}(C) \mid f \text{ a function}\} = \mathsf{C}(C)$$

where every state yields a *selection function* that assigns properties to conditions

Coalgebras and Modalites: A Non-Definition

Coalgebras are about *successors*. T-coalgebras are pairs (C, γ) where

$$\gamma : C \to TC$$

maps states to successors. Write $\mathsf{Coalg}(T)$ for the collection of T-coalgebras.

states $=$ elements $c \in C$	properties of states $=$ subsets $A \subseteq C$
successors $=$ elements $\gamma(c) \in TC$	properties of successors $=$ subsets $\heartsuit A \subseteq TC$

Modal Operators are about *properties* of successors, aka *predicate liftings*

$$[\![\heartsuit]\!]_C : \mathcal{P}(C) \to \mathcal{P}(TC)$$

with the intended interpretation $c \models \heartsuit\phi$ iff $\gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$.

Example: Kripke Frames

Intuition. In a Kripke frame $\gamma : C \to \mathcal{P}(C)$ think of $\gamma(c)$ as "the" successor. Then:

$$c \models \Box\phi \iff \text{all elements of "the" successor } \gamma(c) \text{ of } c \text{ satisfy } \phi$$
$$\iff \text{"the" successor } \gamma(c) \text{ of } c \text{ is a subset of } [\![\phi]\!]$$
$$\iff \gamma(c) \in \{B \subseteq C \mid B \subseteq [\![\phi]\!]\}$$

Associated **Predicate Lifting**

$$[\![\Box]\!]_C : \mathcal{P}(C) \to \mathcal{P}\mathcal{P}(C), A \mapsto \{B \subseteq C \mid A \subseteq B\}.$$

and ensuing notion of *semantic validity*

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!])$$

Another Example: Neighbourhood Frames

Intuition. In a N-frame $\gamma : C \to \mathcal{P}\mathcal{P}(C)$, think of $\gamma(c)$ as the neighbourhoods of c

$$\begin{aligned} c \models \Box\phi &\iff [\![\phi]\!] \in \gamma(w) \\ &\iff \gamma(c) \in \{N \in \mathsf{N}(W) \mid [\![\phi]\!] \in N\}. \end{aligned}$$

Associated **Predicate Lifting**

$$[\![\Box]\!]_C : \mathcal{P}(C) \to \mathcal{P}(\mathsf{N}C), \quad A \mapsto \{N \in \mathsf{N}(C) \mid A \in N\}$$

and ensuing notion of **semantic validity**

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!]_C)$$

(Recall the definition for Kripke Frames?)

Probabilistic Frames

Intuition. In a probabilistic frame, $\gamma(c)$ is the "successor distribution" of c.

$$c \models L_u \phi \iff \gamma(c)(\llbracket \phi \rrbracket) \geq u$$
$$\iff \gamma(c) \in \{\mu \in \mathsf{D}(C) \mid \mu(\llbracket \phi \rrbracket) \geq u\}.$$

Associated **Predicate Lifting**

$$\llbracket L_u \rrbracket_C : \mathcal{P}(C) \to \mathcal{P}(\mathsf{D}C), \quad A \mapsto \{\mu \in \mathsf{D}(C) \mid \mu(A) \geq u\}$$

and ensuing notion of **semantic validity**

$$c \models L_u \phi \iff \gamma(c) \in \llbracket L_u \rrbracket_C (\llbracket \phi \rrbracket_C)$$

(Recall Kripke frames and neighbourhood frames?)

Conditional Frames

Intuition. In a conditional frame $C \to (\mathcal{P}(C) \to \mathcal{P}(C))$, $\gamma(c)$ assigns properties to (non-monotonic) conditions

$$
\begin{aligned}
w \models \phi \Rightarrow \psi \iff & \gamma(w)(\llbracket\phi\rrbracket) \subseteq \llbracket\psi\rrbracket \\
\iff & \gamma(w) \in \{f \in \mathsf{C}W \mid f(\llbracket\phi\rrbracket) \subseteq \llbracket\psi\rrbracket\}.
\end{aligned}
$$

Associated **Predicate Lifting**

$$\llbracket\Rightarrow\rrbracket_W : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(\mathsf{C}W), \quad (A, B) \mapsto \{f \in \mathsf{C}W \mid f(A) \subseteq B\}$$

and ensuing notion of **semantic validity**

$$c \models \phi \Rightarrow \psi \iff \gamma(c) \in \llbracket\Rightarrow\rrbracket_C(\llbracket\phi\rrbracket_C, \llbracket\psi\rrbracket_C)$$

More Examples

Graded Modal Logic over **multigraph frames**

$$\gamma : C \to \mathsf{B}C = \{f : C \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Predicate Lifting for "more than k successors validate …"

$$[\![\Diamond_k]\!]_C(A) = \{f : C \to \mathbb{N} \mid \sum_{a \in A} f(a) \geq k\}$$

Coalition Logic over **game frames**

$$\gamma : C \to \mathsf{G}C = \{(f, (S_n)_{n \in N} \mid f : \prod_n S_n \to C\}$$

Predicate Lifting for "coalition $K \subseteq N$ can force …"

$$[\![[K]]\!]_C(A) = \{(f, (S_n)_{n \in N}) \in \mathsf{G}W \mid \exists \sigma \in (S_k)_{k \in K} \text{ s.t. } \forall \overline{\sigma} \in (S_k)_{k \notin K} \; (f(\sigma, \overline{\sigma}) \in A)\}$$

Modal Semantics in either case:

$$c \models \heartsuit \phi \iff \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$$

Towards Hennessy-Milner: Behavioural Equivalence

Wanted. When are two states of T-coalgebras behaviourally equivalent?

Idea. Consider *morphisms* of frames, e.g. functions $f : C \to D$ between (carriers of) Kripke frames (C, γ) and (D, δ) so that:

[zig] If $c' \in \gamma(c)$ then $f(c') \in \delta \circ f(c)$

[zag] If $d \in \delta \circ f(c)$ then there is $c' \in C$ such that $c' \in \gamma(c)$ and $f(c) = d$

That is, p-morphisms are functions that make the diagram

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & D \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathcal{P}(C) & \xrightarrow[\mathcal{P}(f)]{} & \mathcal{P}(D) \end{array}$$

commute, where $\mathcal{P}(f)(A) = \{f(a) \mid a \in A\}$.

Lemma: Two states are bisimilar if and only if they can be identified by p-morphisms.

Behavioural Equivalence, Coalgebraically

Defn. A function $f : C \to D$ between T-coalgebras (C, γ) and (D, δ) is a *coalgebra homomorphism* if

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & D \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ TC & \xrightarrow[\;Tf\;]{} & TD \end{array}$$

commutes. A pair $(c, d) \in C \times D$ is *behaviourally equivalent* (in symbols $c \simeq d$) if c and d can be identified by a coalgebra-morphism.

Ooops! How is $T(f)$ defined *in general*?

Answer: We require that T be a *functor*, i.e. $Tf : TA \to TB$ if $f : A \to B$ and

$$T(\mathrm{id}_A) = \mathrm{id}_{TA} \quad T(g \circ f) = Tg \circ Tf$$

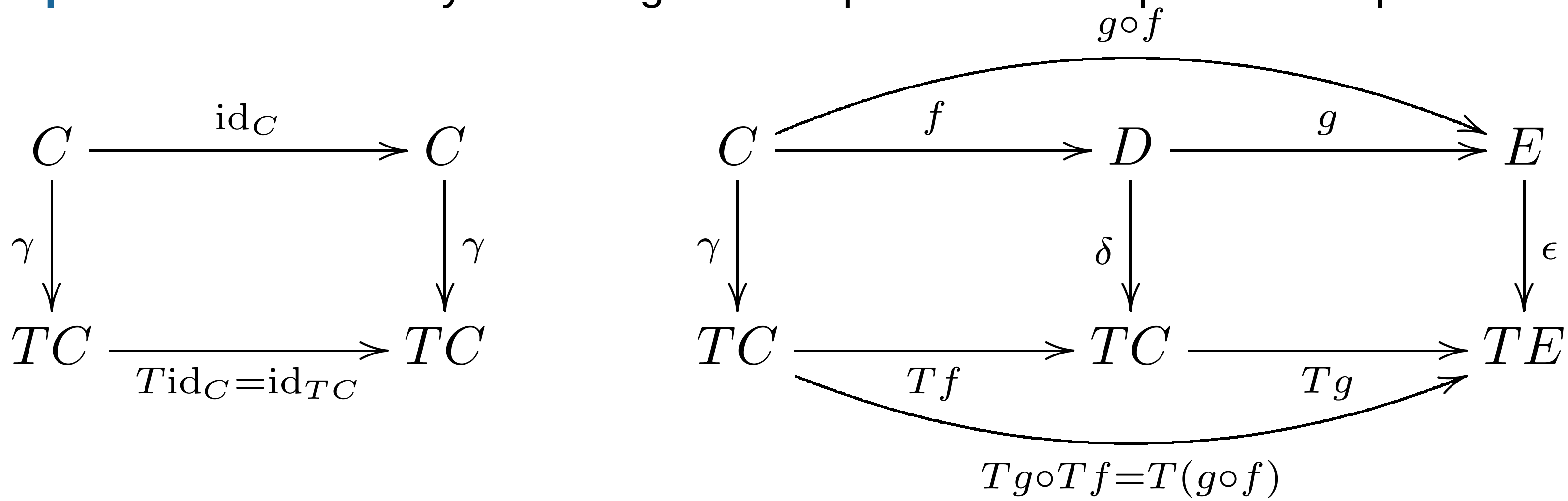
for $A \in \mathsf{Set}$ and composable functions f, g.

Lucklily. There is only ever one way to extend T to functions in a meaningful way.

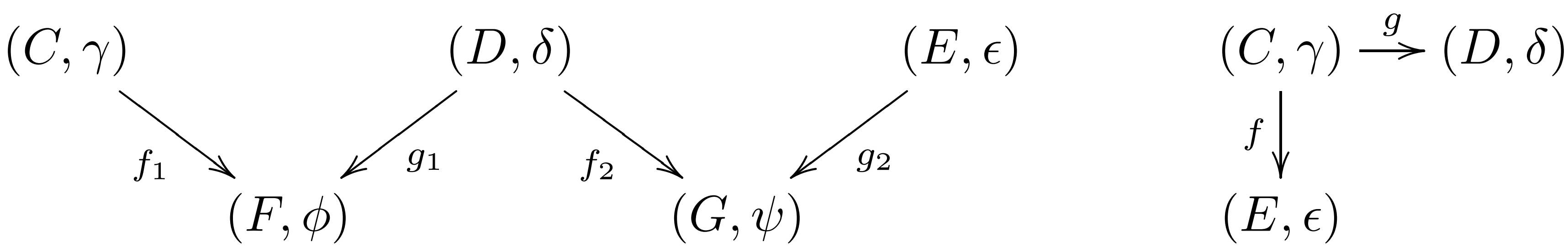
A Wee Bit of Structure Theory

Goal. To prove statements about T-coalgebras *without knowing* how T is defined.

Simple Stuff The identity is a coalgebra morphism and morphisms compose.



Harder Stuff Behavioural equivalence is transitive and preserved by morphisms



Modalities and Behavioural Equivalence

Goal. Modalities and Morphisms need to interact.

"Good" Modalities are **compatible**. This one is from hell:

$$[\![\Box]\!]_C(A) = \begin{cases} \emptyset & C = \mathbb{N} \\ TC & \text{o/w} \end{cases}$$

Defn. Suppose $T : \mathsf{Set} \to \mathsf{Set}$ is a functor. An *n-ary predicate lifting* for T is a set-indexed family $(\lambda_X)_{X \in \mathsf{Set}}$ of functions $\lambda_X : \mathcal{P}(X)^n \to \mathcal{P}(TX)$ such that

$$\begin{array}{ccc} (\mathcal{P}X)^n & \xrightarrow{\lambda_X} & \mathcal{P}(TX) \\ {\scriptstyle (f^{-1})^n}\uparrow & & \uparrow{\scriptstyle (Tf)^{-1}} \\ (\mathcal{P}Y)^n & \xrightarrow[\lambda_Y]{} & \mathcal{P}(TY) \end{array}$$

commutes for all $f : X \to Y$.

Finally: Proper Definitions

Suppose $T : \mathsf{Set} \to \mathsf{Set}$ is a functor.

Defn. A *modal signature* Λ is a set of modal operators with assocated arities.
Λ-*formulas* are given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \heartsuit(\phi_1, \ldots, \phi_n) \qquad (p \in V, \heartsuit\ n\text{-ary})$$

A *sentence* is a formula without propositional variables.

A Λ-*structure* over T assigns an n-ary predicate lifting $[\![\heartsuit]\!]$ to each n-ary $\heartsuit \in \Lambda$.

Modal Semantics with respect to $(C, \gamma) \in \mathsf{Coalg}(T)$ and $\sigma : V \to \mathcal{P}(C)$:

$$C, c, \sigma \models p \text{ iff } c \in \sigma(p)$$

$$C, c, \sigma \models \heartsuit(\phi_1, \ldots, \phi_n) \text{ iff } \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]_{C,\sigma}, \ldots, [\![\phi_n]\!]_{C,\sigma})$$

where $[\![\phi]\!]_{C,\sigma} = \{c \in C \mid C, c, \sigma \models \phi\}$ denotes truth sets.

Logical Equivalence vs Behavioural Equivalence

Suppose throughout that $T : \mathsf{Set} \to \mathsf{Set}$ comes with a Λ-stucture.

Defn. If $(C, \gamma) \in \mathsf{Coalg}(T)$ we call a pair $c, c' \in C$ of states *logically equivalent* if

$$c \models \phi \iff c' \models \phi$$

for all sentences $\phi \in \mathcal{F}(\Lambda)$.

The easy part of the Hennessy-Milner Property:

Lemma. Morphisms preserve semantics, that is

$$C, c, f^{-1} \circ \sigma \models \phi \iff D, f(c), \sigma \models \phi$$

where $f : (C, \gamma) \to (D, \delta)$, $\sigma : V \to \mathcal{P}(D)$ and $\phi \in \mathcal{F}(\Lambda)$.

Cor. Semantics is invariant under behavioural equivalece, that is

$$C, c \models \phi \iff D, d \models \phi$$

whenever ϕ is a sentence and $c \simeq d$.

Separating Sets

Essential Ingredients for the Hennessy-Milner Property:

- "enough" modal operators
- restriction to finite branching models

Defn. A system $S \subseteq \mathcal{P}(Y)$ is a *separating system of subsets of Y* if the function

$$x \mapsto \{A \in S \mid x \in A\}$$

is injective. A Λ-structure for T has the *one-step Hennessy-Milner property* if

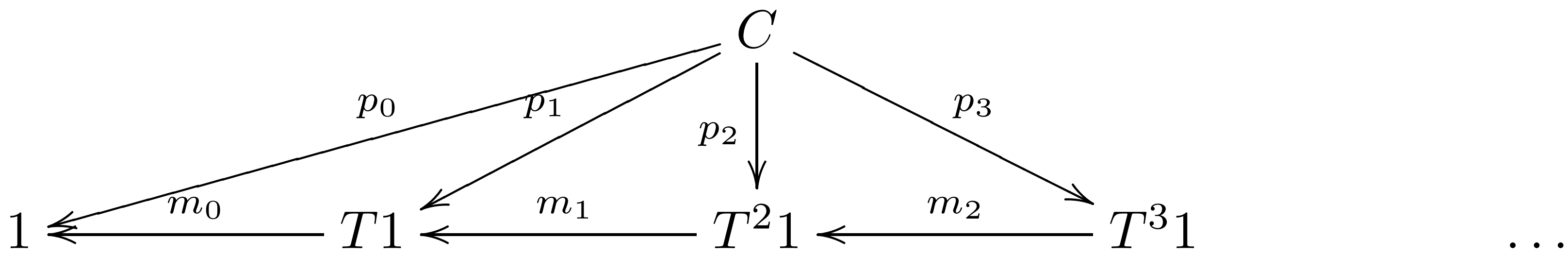
$$\{[\![\heartsuit]\!]_X(A_1, \dots, A_n) \mid \heartsuit \in \Lambda \; n\text{-ary}, A_1, \dots, A_n \subseteq X\}$$

is a separating system of subsets of TX, for all sets X.

The finite Hennessy-Milner Property

Goal. If (C, γ) is *finite*, then logical and behavioural equivalence on C coincide.

Behavioural Equivalence. Let $p_{n+1} = Tp_n \circ \gamma$ and $m_{n+1} = Tm_n$



Lemma. Let $c \sim_n d$ if $p_n(c) = p_n(d)$.

1. If C is *finite*, then $c \simeq d$ iff $c \sim_n d$ for all $n \in \mathbb{N}$.
2. for all $A \subseteq T^n 1$ there exists a sentence ϕ such that $[\![\phi]\!]_C = p_n^{-1}(A)$

Cor. If T comes with a separating structure, then logical and behavioural equivalence coincide on finite T-coalgebras.

The Hennessy-Milner Property

Second Ingredient: finite branching.

Defn. T is *finitary* if, for all $x \in TX$ there exists $Y \subseteq X$ finite such that $x = Ti(y)$ for some $y \in Y$ where $i : Y \to X$ is the inclusion.

Intuition. The action of T on any set can be reconstrunstructed from T's action on finite sets.

Structure Theoretic Result. If T is finitary and $(C, \gamma) \in \mathsf{Coalg}(T)$, then every finite subset of C is contained in a finite subcoalgebra of (C, γ).

Logical Reading. Submodels generated by finite sets are finite.

Theorem. Suppose T is finitary and comes with a separating structure. Then logical and behavioural equivalence conincide for T-coalgebras.

Examples

Kripke Frames. The functor $T = \mathcal{P}$ is *not* finitary – but its cousin

$$\mathcal{P}_f(X) = \{Y \subseteq X \mid Y \text{ finite }\}$$

is finitary. $\mathcal{P}_f$ can be equipped with the same (separating) structure as $\mathcal{P}$.

Neighbourhood Frames. The functor $\mathsf{N}(X) = \mathcal{P}\mathcal{P}X$ is not finitary and its structure cannot be separating. (why?)

Probabilistic Frames. The functor D is finitary (due to finite support) and the structure $[\![L_u]\!]$ is separating.

Multigraph Frames. The functor B is finitary and comes with a separating structure.