Coalgebraic Logics: Modalities Beyond Kripke Semantics

Part II: Reasoning

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Modal Rules

Coalgebras $\gamma : C \to TC$ describe dynamics between *states* and *successors*

Structures $[\![\heartsuit]\!] : \mathcal{P}(X) \to \mathcal{P}(TX)$ describe *properties of successors* in terms of *properties of states*

states $=$ elements $c \in C$	properties of states $=$ subsets $A \subseteq C$
successors $=$ elements $\gamma(c) \in TC$	properties of successors $=$ subsets $\heartsuit A \subseteq TC$

One-Step Rules are the syntactic side of the states/successors interplay

$$\frac{\bigwedge_{i_1} p_{i_1} \to \bigvee_{j_1} p_{j_1} \wedge \cdots \wedge \bigwedge_{i_k} p_{i_k} \to \bigvee_{j_k} p_{j_k}}{\bigwedge_i \heartsuit_i \vec{p_i} \to \bigvee_j \heartsuit_j \vec{p_j}} \quad \sim \quad \frac{\text{property of } states}{\text{property of } successors}$$

provide the capability to *reason* about *properties of successors* in terms of *properties of states*

Methodological Digression

On Structure and Properties

- Clearly, one-step rules are not of the most general kind, and preclude e.g. $T : \Box p \to p$ or $4 : \Box\Box p \to \Box p$
- However, both T and 4 describe *properties* of frames, rather than the *structure* of being a frame
- Compared to Sara's lecture: *more* model classes, *fewer* properties

Methodology: Put *structure* in the front seat, worry about *properties* later

- *one-step rules* axiomatise the *structure* of, e.g. being a probability distribution
- frame conditions are extra axioms of arbitrary nature, e.g. transitivity

Examples

Modal Logic K

$$\frac{p}{\Box p} \qquad \frac{p \wedge q \to r}{\Box p \wedge \Box q \to \Box r}$$

Neighbourhood Frames

$$\frac{p \leftrightarrow q}{\Box p \to \Box q}$$

Probabilistic Modal Logic

$$\frac{}{L_0 p} \qquad \frac{p}{L_u p} \qquad \frac{\neg p \vee \neg q}{\neg L_u p \vee \neg L_v b}(u + v > 1) \frac{p \vee q}{L_u p \vee L_v q}\ (u + v = 1)$$

$$\frac{\sum_{i=1}^{r} 1_{p_i} = \sum_{j=1}^{s} 1_{\bar{q}_j}}{\bigwedge_{i=1}^{r} L_{u_i} p_i \wedge \bigwedge_{j=2}^{s} L_{(1-v_j)} q_j \to L_{v_1} q_1} (\sum_{j=1}^{s} v_j = \sum_{i=1}^{r} u_i)$$

where $\bar{d}_1 = d_1$ and $\bar{d}_j = \neg d_j$ for $j \geq 2$.

Observation. One-step rules appear to be enough to axiomatise the class of *all* T-coalgebras.

More Examples

Coalition Logic for pairwise disjoint sets C_i of coalitions:

$$\frac{\bigvee_{i=1,\dots,n} \neg p_i}{\bigvee_{i=1,\dots,n} \neg [C_i] p_i} \qquad \frac{p}{[C]p} \qquad \frac{p \vee q}{[\emptyset]p \vee [N]q} \qquad \frac{\bigwedge_{i=1,\dots,n} p_i \to q}{\bigwedge_{i=1,\dots,n} [C_i] p_i \to [\bigcup C_i] q}$$

Graded Modal Logic

$$\frac{p \to q}{\Diamond_{n+1} p \to \Diamond_n q} \qquad \frac{r \to p \vee q}{\Diamond_{n+k} r \to \Diamond_n p \vee \Diamond_k q} \qquad \frac{p \leftrightarrow q}{\Diamond_k p \to \Diamond_k q}$$

$$\frac{(p \vee q \to r) \wedge (p \wedge q \to s)}{\Diamond_n p \wedge \Diamond_k q \to \Diamond_{n+k} r \vee \Diamond_0 s} \qquad \frac{\neg p}{\neg \Diamond_0 p}$$

Conditional Logic

$$\frac{q}{p \Rightarrow q} \qquad \frac{p_1 \wedge p_2 \to p_0}{(p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \to (p_0 \Rightarrow q)} \qquad \frac{q_1 \leftrightarrow q_2}{(p \Rightarrow q_1) \to (p \Rightarrow q_2)}$$

Observation. Indeed, one-step rules seem to be enough.

Soundness and Completeness

We now assume a set R of one-step rules.

Logical Consequence. The set of R-*derivable formulas*

- contains all propositional tautologies and is closed under modus ponens
- contains $\psi\sigma$ whenever $\phi/\psi \in$ R and R $\vdash \phi\sigma$

where R $\vdash \phi$ if ϕ is R-derivable and $\sigma : V \to \mathcal{F}(\Lambda)$ ranges over all substitutions.

Wanted. Coherence Conditions that guarantee soundness and completeness

Coherence Conditions

One-Step Rules over sets X and valuations $\sigma : V \to \mathcal{P}(X)$

$$\frac{[\![\text{propositional premise}]\!]_{X,\sigma} \subseteq X}{[\![\text{modalised conclusion}]\!]_{TX,\sigma} \subseteq TX}$$

via propositional logic and predicate liftings

$$[\![p]\!]_{X,\sigma} = \sigma(p) \subseteq X \qquad \text{etc., and} \qquad [\![\heartsuit\phi]\!]_{TX,\sigma} = [\![\heartsuit]\!]_X([\![\phi]\!]_{X,\sigma}) \subseteq TX$$

Notation.

$$X, \sigma \models \phi \iff [\![\phi]\!]_{X,\sigma} = X \text{ and } TX, \sigma \models \psi \iff [\![\psi]\!]_{TX,\sigma} \models \psi$$

Coherence. $[\![\phi]\!]_{X,\sigma} \sim [\![\psi]\!]_{TX,\sigma}$ for $\phi/\psi \in \mathsf{R}$ and $\sigma : V \to \mathcal{P}(X)$

The Easy Part: Soundness

Defn. A rule ϕ/ψ is *one-step sound* if

$$X, \sigma \models \phi \implies TX, \sigma \models \psi$$

for all sets X and all valuations $\sigma : V \to \mathcal{P}(X)$.

Note. One-step soundness replaces quantification over *models* by quantification over *sets*.

Propn. (Soundness) $\mathsf{Coalg}(T) \models \phi$ whenever $\mathsf{R} \vdash \phi$ if every rule in R is one-step sound.

Proof. Induction on the definition of provability where one-step soundness accounts for the modal steps.

Completeness and the Small Model Property

Defn. R is *one-step complete* if

$$TX, \sigma \models \chi \implies \{\psi\tau \mid X, \sigma \models \phi\tau, \tau : V \to \mathsf{Prop}(V), \phi/\psi \in \mathsf{R}\} \vdash_{\mathsf{PL}} \chi$$

whenever χ is a clause over $\heartsuit(\vec{p})(\heartsuit \in \Lambda)$ and $\sigma : V \to \mathcal{P}(X)$ is a valuation.

Goal. Every consistent formula is satisfiable in a model of exponential size.

Terminology. A set $\Sigma \subseteq \mathcal{F}(\Lambda)$ is closed if

- $\phi \in \Sigma, \psi \in \mathsf{Sf}(\phi) \implies \psi \in \Sigma$
- $\phi \in \Sigma$ and $\phi \neq \neg\phi' \implies \neg\phi \in \Sigma$

where $\mathsf{Sf}(\phi)$ is the set of subformulas of ϕ.

Roadmap. Relative to a closed set Σ

Existence Lemma $\rightsquigarrow$ Truth Lemma $\rightsquigarrow$ Small Model Theorem

Completeness

Fix $\Sigma \subseteq \mathcal{F}(\Lambda)$ closed and finite, and $S = \{M \subseteq \Sigma \mid M \text{maximally consistent}\}$.

Existence Lemma. (Schröder 2006) For all $M \in S$ there exists $t \in TS$ such that

$$t \in [\![\heartsuit]\!]_S(\Sigma_{\phi_1}, \ldots, \Sigma_{\phi_n}) \iff \heartsuit(\phi_1, \ldots, \phi_n) \in M$$

for all $\heartsuit(\phi_1, \ldots, \phi_n) \in \Sigma$ where $\Sigma_\phi = \{M \in S \mid \phi \in M\}$.

Proof. If the above fails for M, we have that

$$TS, \sigma \models \bigvee_{\heartsuit(\phi_1,\ldots,\phi_n) \in M} \neg\heartsuit(p_{\phi_1}, \ldots, p_{\phi_n}) \vee \bigvee_{\heartsuit(\phi_1,\ldots,\phi_n) \in \Sigma \setminus M} \neg\heartsuit(p_{\phi_1}, \ldots, p_{\phi_n})$$

for $\sigma : V \to \mathcal{P}(S)$ satisfying $\sigma(p_\phi) = \Sigma_\phi = \{M \in S \mid \phi \in M\}$.

This clause is derivable by one-step completeness, contradicting consistency of M.

Corollary. Every consistent formula ϕ is satisfiable in a model of size $\leq 2^{\mathsf{Sf}(\phi)}$.

Examples

Trick. One-step completeness is usually *much* easier to check than soundness.

Example. The rule set R_K is one-step complete: if

$$TX, \sigma \models \bigwedge_i \Box p_i \to \bigvee_j \Box q_j$$

then

$$\bigcap_i \sigma(p_i) \in \bigcap_i [\![\Box]\!]_X(\sigma(p_i)) \subseteq \bigcup_j [\![\Box]\!]_X(\sigma(q_j))$$

and therefore we can find j such that

$$\bigcap_i \sigma(p_i) \subseteq q_j$$

which we use as rule premises in a one-step deduction.

More Examples. The rule sets seen previously (graded / probabilistic / coalition / conditional logic) are one-step complete.

Are One-Step Rules Really Enough?

Propn. One-step rules suffice, or: the set of all one-step sound one-step rules is one-step complete.

Proof Sketch. Suppose $TX, \sigma \models \psi$ for a clause ψ over variables $V_0 \subseteq V$. Pick

$$\phi = \bigwedge \{\chi \mid \chi \text{ propositional clause over } V_0 \text{ and } X, \sigma \models \chi\}$$

It follows that ϕ / ψ is one-step sound.

Remarks.

- Note that ψ can be derived using a *single* rule
- but the set of al rules is too large to be practically useful.