Coalgebraic Logics: Modalities Beyond Kripke Semantics

Part III: Complexity

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Decidability and the FMP

Before the Break. Coalgebraic logics have the small model property.

However. Decidability is *not* automatic by FMP

Reason. If T maps finite sets to infinite sets, the set of transition structures

$$\{\gamma : C \to TC \mid \gamma \text{ a function}\}$$

may well be infinite.

Example. Game frames, probabilistic frames, multigraph frames

Recall. We know *nothing* about T and it may well encode the halting problem.

Wishful Thinking

Two Options for $\chi = \bigwedge_i \heartsuit_i \phi_i \to \bigvee_j \heartsuit_j \psi_k$:

Semantically

$\neg\chi$ satisfiable $\implies$

$\forall \phi/\psi \in \mathsf{R}, \sigma : V \to \mathcal{F}(\Lambda).$

$\psi\sigma \vdash_{\mathsf{PL}} \chi \implies \neg\phi\sigma$ satisfiable

$$\frac{\text{Countermodel of } \phi\sigma\text{'s}}{\text{Countermodel of } \chi}$$

Syntactically.

χ provable $\implies$

$\exists \phi/\psi \in \mathsf{R}, \sigma : V \to \mathcal{F}(\Lambda)$

$\phi\sigma$ provable and $\psi\sigma \vdash_{\mathsf{PL}} \chi$

$$\frac{\text{Proof of some } \phi\sigma}{\text{Proof of } \chi}$$

Problems.

1. checking single rules is insufficient, and
2. there may be infinitely many rules to check!

Example

Consequences of *multiple rules*

$$\dfrac{\dfrac{a \wedge (b \wedge c) \to a \wedge b \wedge c}{\Box a \wedge \Box(\mathbf{b} \wedge \mathbf{c}) \to \Box(a \wedge b \wedge c)} \qquad \dfrac{b \wedge c \to b \wedge c}{\Box b \wedge \Box c \to \Box(\mathbf{b} \wedge \mathbf{c})}}{\Box a \wedge \Box b \wedge \Box c \to \Box(a \wedge b \wedge c)}$$

Infinitely many *possible premises*

$$\dfrac{\vdash a \to b?}{\Box a \to \Box b \vdash \Box a \to \Box b} \qquad \dfrac{\vdash a \wedge a \to b?}{\Box a \wedge \Box a \to \Box b \vdash_{\mathsf{PL}} \Box a \to \Box b}$$

$$\dfrac{\vdash a \wedge a \wedge a \to b?}{\Box a \wedge \Box a \wedge \Box a \to \Box b \vdash_{\mathsf{PL}} \Box a \to \Box b}$$ etc.

Conceptually. Admissibility of Cut and Contraction in a Sequent Calculus.

Admissibility of Cut, or Combination of Conclusions

Stronger **Coherence Condition:** R is *strictly one-step complete* if

$$TX, \sigma \models \phi \implies \exists \phi/\psi \in \mathsf{R}, \tau : V \to V \text{ s.t. } X, \sigma \models \phi\tau \text{ and } \psi\tau \vdash_{\mathsf{PL}} \chi$$

whenever χ is a clause over $\heartsuit(\vec{p})(\heartsuit \in \Lambda)$ and $\sigma : V \to \mathcal{P}(X)$ is a valuation.

Intuition. Valid modalised formulas are derivable using a *single* rule.

Trivial Obervation. Strict Completeness implies completeness.

Strictly complete rule sets exist. The set of all one-step sound one-step rules is *strictly* one-step complete (as we had seen earlier).

Ongoing Assumption. Rule sets are one-step sound.

On the Structure of Proofs

Arbitrary rule sets: ϕ is R-derivable iff

- there exist $\phi_1/\psi_1, \ldots, \phi_n/\psi_n \in \mathsf{R}$ and $\rho_1, \ldots, \rho_n : V \to \mathcal{F}(\Lambda)$
- such that $\mathsf{R} \vdash \phi_i\rho_i$ for all $i = 1, \ldots, n$ and $\{\psi_1\rho_1, \ldots, \psi_n\rho_n\} \vdash_{\mathsf{PL}} \phi$

Strictly complete rule sets: A clause χ is R-derivable iff

- there exists $\phi/\psi \in \mathsf{R}$ and $\tau : V \to \mathcal{F}(\Lambda)$
- such that $\mathsf{R} \vdash \phi\tau$ and $\psi\tau \vdash_{\mathsf{PL}} \chi$

Proof Sketch.

1. reduce to clauses without propositional variables, i.e. assume that $\chi = \bigwedge_i \heartsuit_i\phi_i \to \bigvee_j \heartsuit_j\psi_j$
2. Apply upper result and argue by soundness and strict completeness

This deals with cut, i.e. propositional combinations of rule conclusions.

Strictly Complete Rule Sets: Beg, Steal or Borrow?

Idea. Absorb propositional reasoning in the rule set

Resolvents. If $\phi = \bigvee_{i \in I} \phi_i, \psi = \bigvee_{j \in J} \psi_j$ are clauses, a clause ρ is a *resolvent* of ϕ, ψ at a literal $\lambda \in \mathsf{At}(\Lambda)$ provided $\rho = \bigvee_{i \neq i_0} \phi_i \vee \bigvee_{j \neq j_0} \psi_j$ and either

- $\phi_{i_0} = \lambda$ and $\psi_{j_0} = \neg\lambda$, or
- $\phi_{i_0} = \neg\lambda$ and $\psi_{j_0} = \lambda$.

Saturated Rulesets. A rule set R *absorbs cut*, if all rules $\phi_1/\psi_1, \phi_2/\psi_2$ and renamings $\tau_1, \tau_2 : V \to V$ that admit a resolvent ρ of $\psi_1\tau_1$ and $\psi_2\tau_2$, there exists a rule $\phi/\psi \in$ R and a renaming $\tau : V \to V$ such that

- $\psi\tau \vdash_{\mathsf{PL}} \rho$
- $\phi_1\tau_1 \wedge \phi_2\tau_2 \vdash_{\mathsf{PL}} \phi\tau$

Informally. Every resolution step / instance of cut can be replaced by a rule application with *weaker* premise and *stronger* conclusion.

Propn. Absorption of cut and strict completeness are equivalent.

Examples of Strictly Complete Rule Sets

Modal Logic E.

$$\frac{p \leftrightarrow q}{\Box p \to \Box q}$$

Modal Logic K.

$$\frac{\bigwedge_{i=1,\dots,n} p_i \to q}{\bigwedge_{i=1,\dots,n} \Box p_i \to \Box q}$$

Graded Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} \Diamond_{k_i} p_i \to \bigvee_{j=1}^{m} \Diamond l_j q_j},$$

Probabilistic Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} L_{u_i} p_i \, to \bigvee_{j=1}^{m} L_{v_j} q_j},$$

Conditional Logic.

$$\frac{\bigwedge_{i=1,\dots,n} p_i \to p_0 \wedge \bigwedge_{i=1,\dots,n} q_i \leftrightarrow q_0}{\bigwedge_{i=1,\dots,n} (p_i \Rightarrow q_i) \to (p_0 \Rightarrow q_0)}$$

Coalition Logic.

$$\frac{\bigvee_{i=1}^{n} \neg p_i}{\bigvee_{i=1}^{n} \neg [C_i] p_i} \qquad \frac{\bigwedge_{i=1}^{n} p_i \to q \vee \bigvee_{j=1}^{m} r_j}{\bigwedge_{i=1}^{n} [C_i] p_i \to [D] q \vee \bigvee_{j=1}^{m} [N] r_j}$$

Reduction to a Finite Number of Rules

Contraction: Duplication of Literals may lead to infinitely many possible premises.

Example (as before): check $\vdash a \to b?$, $\vdash a \wedge a \to b?$, $\vdash a \wedge a \wedge a \to b? \dots$

Idea. Absorb contraction into the rule set.

Defn. R is *contraction closed*, if, for all $\phi/\psi \in$ R and renamings $\rho : V \to V$ there exists an *injective* renaming ρ_i and a rule $\phi_i/\psi_i \in$ R such that

- $\psi_i \rho_i \vdash_{\mathsf{PL}} \psi\rho$, and
- $\phi\rho \vdash_{\mathsf{PL}} \phi_i \rho_i$

Informally. Application of contraction to a rule conclusion are not neccessary.

Algorithmic Flavour. Get an absorbing and contraction closed rule set by

1. add instances of cut until saturation is reached
2. add rule instances that duplicate literals in the conclusion

Decidability and Complexity

Algorithm to decide $\vdash \phi$:

- (universal) guess component χ of the cnf of ϕ
- (existential) guess a rule ϕ/ψ and a substitution σ s.t. $\psi\sigma \vdash_{\mathsf{PL}} \chi$
- recursively check $\vdash \phi\sigma$

Termination. Subject to finitiely many “matching” rule conclusions

Conceptually. Cut-Free Proof Search

Sequent Calculi for Coalgebraic Logics

Sequents are multisets of formulas. Write Γ, Δ for $\Gamma \cup \Delta$ and Γ, A for $\Gamma, \{A\}$

Propositional Rules

$$\frac{\Gamma, A}{\Gamma, \neg\neg A} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

Modal Rules from a one-step rule ϕ/ψ where σ ranges over substitutions

$$\frac{\mathsf{Lit}(\phi_1)\sigma \quad \ldots \mathsf{Lit}(\phi_n)\sigma}{\mathsf{Lit}(\psi)\sigma, \Delta}$$

and $\mathsf{cnf}(\phi) = \phi_1 \wedge \cdots \wedge \phi_n$ and $\mathsf{Lit}(\cdot)$ is the set of literals occuring in a clause.

Notation. $\mathsf{GR} \vdash \Gamma$ if Γ can be derived using the propositional rules and the "imported" modal rules.

Sequent Proofs vs Hilbert Proofs

Easy Lemma. $\mathsf{R} \vdash \bigvee \Gamma$ whenever $\mathsf{GR} \vdash \Gamma$.

Towards Cut free completeness

Lemma. Suppose R is strictly complete and contraction closed.

- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

Thm. Suppose R is strictly complete and contraction closed. Then

$$\mathsf{Coalg}(T) \models \bigvee \Gamma \text{ iff } \mathsf{GR} \vdash \Gamma.$$

Proof Sketch. If $\mathsf{Coalg}(T) \models \bigvee \Gamma$ then $\mathsf{R} \vdash \bigvee \Gamma$. Consequently:

- for every component χ of the cnf of $\bigvee \Gamma$
- we can find $\phi/\psi \in \mathsf{R}$ and $\sigma : V \to \mathcal{F}(\Lambda)$ s.t. $\psi\sigma \vdash_{\mathsf{PL}} \chi$ and $\mathsf{R} \vdash \phi\sigma$

All these steps can be simulated in GR.

Complexity

PSPACE Bounds via proof search:

- polynomial bound on the height of the proof tree
- for every sequent Γ, the (codes of) rules that entails Γ can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

Formally. R is *NPMV* if there exists a finite alphabet Σ such that all sequents can be represented in Σ and a pair

$$f : \Sigma \to \mathcal{P}(\Sigma) \quad g : \Sigma \to \mathcal{P}(\Sigma)$$

of nondeterministic polytime functions such that

$$\{\{\Gamma_1, \dots, \Gamma_n\} \mid \frac{\Gamma_1, \dots, \Gamma_n}{\Gamma} \in \mathsf{GR}\} = \{g(x) \mid x \in f(\Gamma)\}$$

for all sequents Γ.

Thm. If R is NPMV, sound and strictlh complete, then R-satisfiability is in PSPACE.

Examples

Modal Logics K, E, coalition logic and conditional logic:

- earlier rule sets are already contraction closed and NPMV

Graded Modal Logic. closure under contraction affects rule premises

$$\frac{\sum_{i=1}^{n} r_i p_i \geq 0}{\bigvee_{i=1}^{n} \mathsf{sgn}(r_i) \Diamond_{k_i} p_i},$$

where $n \geq 1$ and $r_1, \ldots, r_n \in \mathbb{Z} - \{0\}$ plus side condition

Probabilistic Modal Logic. Similar and one gets

$$\frac{\sum_{i=1}^{n} r_i p_i \geq u}{\bigvee_{1 \leq i \leq n} \mathsf{sgn}(r_i) L_{u_i} p_i} \; (u \in \mathbb{Z})$$

where $n \geq 1$ and $r_1, \ldots, r_n \in \mathbb{Z} - \{0\}$ plus side condition

NPMV-ness of graded and probabilistic modal logic via polysize solutions of linear inequalities / linear programming

Semantic Approach

Syntactically. Completeness via finite model theorem

Semantic Alternative. Direct construction on top of a tableau induced by dualising the sequent rules.

Shallow Model Theorem.

If ϕ has a tableau, then ϕ is satisfiable in a model (C, γ) whose carrier consists of the modal nodes of the tableau.

Proof Sketch. Similar to the completeness proof, but using *strict* completeness: We construct

$$\gamma : C \to TC \text{ s.t. } \gamma(c) \in T\{c' \mid c' \text{ sub-node of } c\}$$

by contradiction.

Implementation of Satisfiability / Provability

Parametric Formulas.

```
data L a
  =  F | T | Atom Int
    | Neg (L a) | And (L a) (L a) | M a (L a)
```

Example. The logic K and graded modal logic

```
data K = K deriving (Eq,Show)
data G = G Int
```

Logic. Type-class that supports matching

```
class (Eq a,Show a) => Logic a where
  match :: Clause a -> [[L a]]
```

(double lists as rule premises are generally in cnf)

Matching and Provability

Example. Syntax of K (again)

```
data K = K
```

Matching: representation of resolution closed rule sets

```
instance Logic K where
  match (Clause (pl,nl)) =
    let (nls,pls) = (map neg (stripany nl), stripany pl)
    in map disjlst (map (\x -> x:nls) pls)
```

Generic Provability Predicate.

```
provable :: (Logic a) => L a -> Bool
provable phi = all (\c -> any (all provable) ( match c)) (cn
```

(lazyness of Haskell guarantees polynomial space)

Other Lines of Research

Frame Conditions of a more general type

- coherence conditions for completeness and complexity for rank 0/1

Proof Theoretic Aspects

- interpolation and general frame conditions

Extensions

- like a global modality, nominals or fixpoints