Characterization, Definability and Separation via Saturated Models

Facundo Carreiro

Institute for Logic, Language and Computation
University of Amsterdam, Netherlands

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Outline

1. Classical results for the basic modal logic
   - Motivation
   - Characterization
   - Definability
   - Separation
2. Extending the results
   - Objectives
   - Basic definitions
   - Characterization
   - Definability
   - Separation
3. Conclusions and future work

Standard results for Basic Modal Logic

Motivation

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When designing a modal logic

- It is crucial to measure its *expressive power*.
- To be able to fine tune it and get the lowest possible computational complexity for a given task.
- Model equivalence relations (e.g., bisimulations) aid us in these tasks.

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What are bisimulations good for?

- Give a *structural* characterization of indistinguishability.
- Model/automata minimization.

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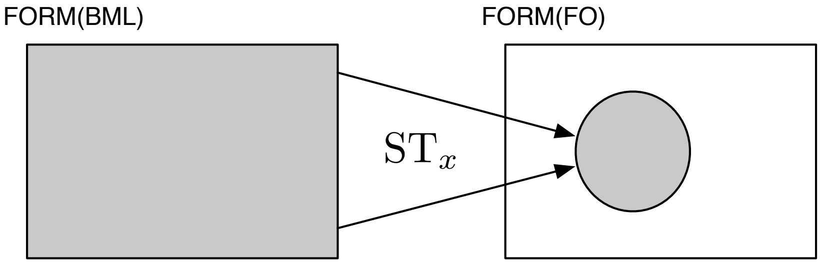
How can we choose the right ‘bisimulation’ notion for a given logic?

- There is no standard easy way!
- If we can develop the basic model theory that is a good hint of correctness.

Facts about BML

- Extends propositional logic with modalities $\Diamond, \Box$.
- Interpreted over Kripke models (directed labeled graphs).
- These models can also be seen as first order models.
- BML formulas have a translation to first order logic.
- BML can not distinguish between models which are *bisimilar*.

BML: Characterization



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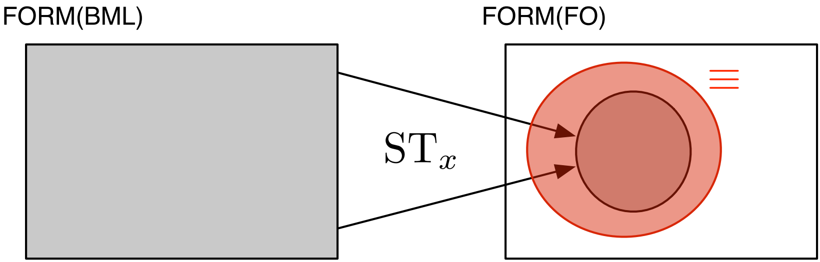

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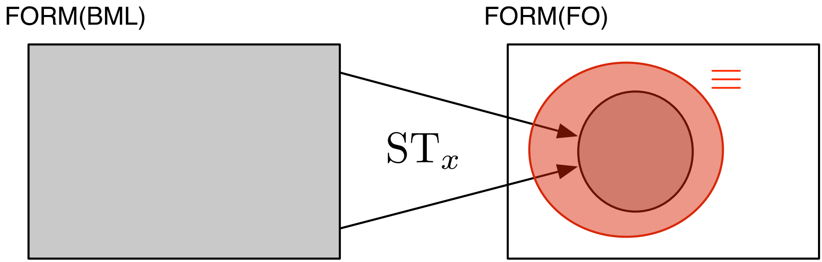
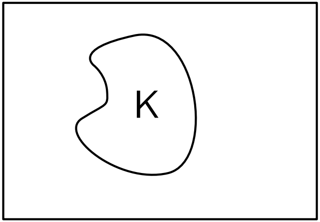


Theorem (van Benthem)

A first order formula $\alpha(x)$ with one free variable is equivalent to the translation of a BML-formula iff it is invariant under bisimulations.

BML: Definability

PMODS(BML)



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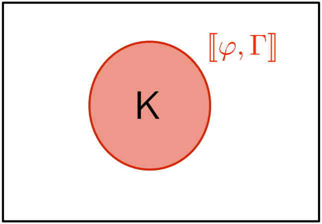
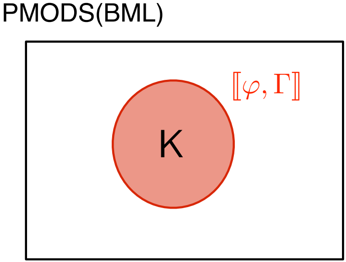


$\exists?$ formula φ

$\exists?$ set Γ

BML: Definability



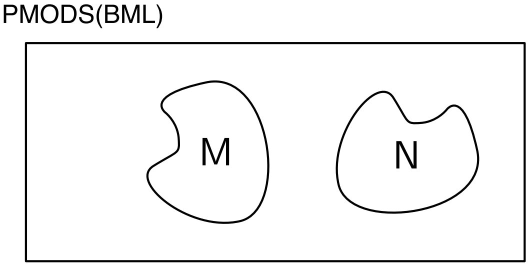
$\exists?$	formula	φ
$\exists?$	set	Γ

Theorem (de Rijke)

Definability by a set of formulas. A class K is definable by a *set* of BML-formulas iff K is closed under ultraproducts, $\overline{\mathsf{K}}$ is closed under ultrapowers and both K and $\overline{\mathsf{K}}$ are closed under bisimulations.

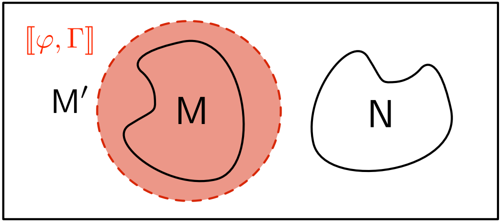
Definability by a formula. A class K is definable by a BML-*formula* iff both K and $\overline{\mathsf{K}}$ are closed under bisimulations and ultraproducts.

BML: Separation



BML: Separation

PMODS(BML)

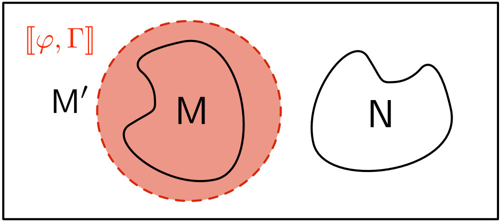


$\exists \varphi$?

$\exists \Gamma$?

BML: Separation

PMODS(BML)



$\exists \varphi$?

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Theorem (de Rijke)

Let M and N be such that $\mathsf{M} \cap \mathsf{N} = \emptyset$.

Separation by a set of formulas. If M is closed under bisimulations and ultraproducts, and N is closed under bisimulations and ultrapowers, then there exists M′ definable by a set of formulas such that $\mathsf{M} \subseteq \mathsf{M}'$ and $\mathsf{N} \cap \mathsf{M}' = \emptyset$.

Separation by a formula. If both M and N are closed under bisimulations and ultraproducts, then M and N are separable by a singleton set.

Extending the results

(joint work with Carlos Areces and Santiago Figueira)

Objectives

We will extend previous results in the following two directions

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Arbitrary modal logic

- There are many other logics below first order
- Is there an uniform proof of these theorems that covers them all?
- The following problems arise:
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  2. Interpreted over variations of Kripke models
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Restriction to a particular class of models

- For example: tree models, linear orders, finite models, etc.
- The amount of valid formulas increases
- The amount of bisimulation-invariant formulas increases
- Does a characterization-like theorem hold in this case?
- How does this impact the definability/separation theorems?

Basic definitions

Definition (base logic)

1. $\mathfrak{L}$ be a (modal) language extending $\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \top \mid \bot$
2. $\mathrm{MODS}(\mathfrak{L})$ be the (set-based) class of $\mathfrak{L}$-models under consideration
3. $\mathrm{PMODS}(\mathfrak{L}) := \{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \in \mathrm{MODS}(\mathfrak{L})$ and $w \in |\mathcal{M}|\}$

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Observe that

- We do not require negation nor any specific modality
- $\mathrm{MODS}(\mathfrak{L})$ may be different from the class of all models of the signature of $\mathfrak{L}$ (e.g., only trees, linear orders, etc.)

Basic definitions

We say that $\mathfrak{L}$ is adequately below first order if there is

1. A formula translation $\mathsf{Tf}_x : \mathrm{FORM}(\mathfrak{L}) \to \mathrm{FORM}_1(\mathsf{FO})$
2. A class of FO-pointed models $\mathsf{K} \subseteq \mathrm{PMODS}(\mathsf{FO})$
3. A *model translation*: a bijective function $\mathsf{Tm} : \mathrm{PMODS}(\mathfrak{L}) \to \mathsf{K}$

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such that

1. $\mathsf{Tf}_x(\varphi \wedge \psi) = \mathsf{Tf}_x(\varphi) \wedge \mathsf{Tf}_x(\psi)$ and $\mathsf{Tf}_x(\varphi \vee \psi) = \mathsf{Tf}_x(\varphi) \vee \mathsf{Tf}_x(\psi)$
2. K is closed under ultraproducts
3. The translations are *truth-preserving*: for all $\varphi \in \mathrm{FORM}(\mathfrak{L})$ and all $\langle \mathcal{M}, w \rangle \in \mathrm{PMODS}(\mathfrak{L})$ they satisfy

$$\mathcal{M}, w \Vdash \varphi \textit{ iff } \mathsf{Tm}(\mathcal{M}, w) \models \mathsf{Tf}_x(\varphi).$$

Basic definitions

Every logic has an associated notion of observational equivalence, e.g,

Logic	*Notion*	*Clauses*
Basic modal logic	bisimulation	atom, zig, zag
Negation free BML	simulation	atom', zig
Graded modal logic	counting bisimulation	atom, zig, zag, bij-succ
Hybrid logic	hybrid bisimulation	atom, nominals, zig, zag
First order	potential isomorphisms	...
Tense logic w/S+U	...	...

What do they have in common? we abstract it in the following definition

Definition

An *$\mathfrak{L}$-similarity* is a relation $\rightrightarrows_{\mathfrak{L}} \subseteq \mathrm{PMODS}(\mathfrak{L}) \times \mathrm{PMODS}(\mathfrak{L})$ such that if $\mathcal{M}, w \rightrightarrows_{\mathfrak{L}} \mathcal{N}, v$ then $\mathcal{M}, w \Rrightarrow_{\mathfrak{L}} \mathcal{N}, v$.

Notation: $\mathcal{M}, w \Rrightarrow_{\mathfrak{L}} \mathcal{N}, v$ iff for all $\varphi \in \mathrm{FORM}(\mathfrak{L})$; $\mathcal{M}, w \Vdash \varphi$ implies $\mathcal{N}, v \Vdash \varphi$

Basic definitions

We know that $\rightleftarrows \subseteq \Rrightarrow$ but the converse does **not** hold in general!

Definition

A class of models K has the *Hennessy-Milner* property if for each $\langle \mathcal{M}, w \rangle$, $\langle \mathcal{N}, v \rangle \in$ K it holds that $\mathcal{M}, w \Rrightarrow \mathcal{N}, v$ implies $\mathcal{M}, w \rightleftarrows \mathcal{N}, v$.

Logic	*Notion*	*Class with HM*
Basic modal logic	bisimulation	finite models finitely branching models modally-saturated
First order	pot. iso.	recursively saturated

These are all examples of ω-saturated models!

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Definition (adequate $\mathfrak{L}$-similarity)

Let $\mathfrak{L}$ be adequately below first order. An $\mathfrak{L}$-similarity is an *adequate similarity* for $\mathfrak{L}$ if the class of ω-saturated models in K has the Hennessy-Milner property.

Characterization

From now on we fix

1. a logic $\mathfrak{L}$ adequately below first order, and
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Definition ($\mathfrak{L}$-similarity K-invariance)

A formula $\alpha(x) \in \mathrm{FORM}_1(\mathsf{FO})$ is K-*invariant for $\mathfrak{L}$-similarity* if for all $\mathfrak{L}$-pointed models $\mathcal{M}, w$ and $\mathcal{N}, v$, such that $\mathcal{M}, w \rightrightarrows \mathcal{N}, v$, if $\mathsf{Tm}(\mathcal{M}, w) \models \alpha(x)$ then $\mathsf{Tm}(\mathcal{N}, v) \models \alpha(x)$.

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Main Theorem (Characterization)

A formula $\alpha(x) \in \mathrm{FORM}_1(\mathsf{FO})$ is K-equivalent to the translation of an $\mathfrak{L}$-formula iff $\alpha(x)$ is K-invariant for $\mathfrak{L}$-similarity.

Definability

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Main Theorem (Definability by a set)

A class $\mathsf{M} \subseteq \mathrm{PMODS}(\mathfrak{L})$ is definable by a set of $\mathfrak{L}$-formulas iff

1. M is closed under $\mathfrak{L}$-similarity,
2. $\mathsf{Tm}(\mathsf{M})$ is closed under ultraproducts and,
3. $\mathsf{Tm}(\overline{\mathsf{M}})$ is closed under ultrapowers.

Main Theorem (Definability by a single formula)

A class $\mathsf{M} \subseteq \mathrm{PMODS}(\mathfrak{L})$ is definable by a single $\mathfrak{L}$-formula iff

1. M is closed under $\mathfrak{L}$-similarity and,
2. both $\mathsf{Tm}(\mathsf{M})$ and $\mathsf{Tm}(\overline{\mathsf{M}})$ are closed under ultraproducts.

Separation

Separation

Main Theorem (Separation by a set of formulas)

Let $\mathsf{M}, \mathsf{N} \subseteq \mathrm{PMODS}(\mathfrak{L})$ be such that $\mathsf{M} \cap \mathsf{N} = \emptyset$,

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then there exists a class $\mathsf{M}' \supseteq \mathsf{M}$ such that it is definable by a set of $\mathfrak{L}$-formulas and $\mathsf{M}' \cap \mathsf{N} = \emptyset$.

Main Theorem (Separation by a formula)

Let $\mathsf{M}, \mathsf{N} \subseteq \mathrm{PMODS}(\mathfrak{L})$ be such that $\mathsf{M} \cap \mathsf{N} = \emptyset$,

1. M and N are closed under $\mathfrak{L}$-similarity,
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then M and N are separable by a singleton set.

Conclusions

- This result can be applied for many logics
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- About the requirements
  - Most of them are trivially satisfied by logics below FO
  - The requirement about the class of ω-saturated models is non-trivial
  - It isolates the specific logic-related aspects
- If we want to study Characterization, Definability and Separation it is important to study ω-saturated classes

Future work

1. Adapt the results for logics without disjunction
2. Adapt the results for the class of *finite models*
3. Concentrate in the study of classes of ω-saturated models
   - Prove the Hennessy-Milner property for families of modal logics
4. Given a logic $\mathfrak{L}$, try to give 'canonical' bisimulation notions
   - Using relation liftings?
   - Using behavioural equivalence of coalgebras?
5. Study connections with Interpolation
   - *Sometimes*, Craig interpolation follows from Separation
   - When? Can it be incorporated to this framework?

Questions?