The Method of Proof Analysis: Background, Developments, New Directions

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What is proof theory?

"The main concern of proof theory is to study and analyze structures of proofs. A typical question in it is 'what kind of proofs will a given formula A have, if it is provable?', or 'is there any *standard* proof of A?'. In proof theory, we want to derive some logical properties from the analysis of structures of proofs, by anticipating that these properties must be reflected in the structures of proofs. In most cases, the analysis will be based on combinatorial and constructive arguments. In this way, we can get sometimes much more information on the logical properties than with semantical methods, which will use set-theoretic notions like *models*,*interpretations* and *validity*."
(H. Ono, Proof-theoretic methods in nonclassical logic–an introduction, 1998)

Challenges in modal and non-classical logics

Difficulties in establishing analyticity and normalization/cut-elimination even for basic modal systems.

Extension of proof-theoretic semantics to non-classical logic.

Generality of model theory vs. goal directed developments in proof theory for non-classical logics.

Proliferation of calculi "beyond Gentzen systems".

Defeatist attitudes: "No proof procedure suffices for every normal modal logic determined by a class of frames." (M. Fitting, Modal Proof Theory, HML, 2007).

The method of proof analysis

- Basic goal: maximal extraction of information from the analysis of proofs in a formal inference system
- Analytic proof systems for pure logic: sequent calculus, natural deduction
- Extension to mathematical theories: generalized Hauptsatz requires cuts on axioms, so full analyticity is lost
- Conversion of axioms into rules allows full cut and contraction elimination for
  - theories with universal axioms (N and von Plato 1998)
  - geometric theories (N 2003)
  - a wide class of non-classical logics, including provability logic (N 2005), intermediate logics (Dyckhoff and N 2012), various logics in CS (conditional logics, description logics, etc), epistemic logics, ...

Design principles and properties

Conversion of axioms into rules of inference is obtained by a *uniform procedure* that has to respect the properties of the logical calculus to which the rules are added.

The resulting calculi are *complete* for the theory under consideration and have *the same structural properties* as the logical calculi one started with. The rules are designed in *harmony* with these requirements (e.g. context sharing/independent).

Our favorite system is *sequent calculus* but also *natural deduction* works nicely, with the condition that *general elimination rules* are used ("Proof Analysis" part I).

Separation of logical and mathematical parts in a derivation can be established for some extensions.

Full analyticity cannot be expected (not all theories are decidable!). No full subformula property but *subterm property* gives analyticity in the same sense as for first-order logic.

Hilbert-style systems

The most common way of presenting a logic.

Many axioms, one rule of inference

Axioms

1. $\perp \supset A$,
2. $A \supset (B \supset A \& B)$
3. $A \& B \supset A$
4. $A \& B \supset B$
5. $A \supset A \vee B$
6. $B \supset A \vee B$
7. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
8. $A \supset (B \supset A)$
9. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$.

Rule

$$\frac{A \quad A \supset B}{B} \text{ \textit{modus ponens}}$$

Hilbert-style systems (cont.)

The axioms and the rule are *schematic*, that is, any formula can be substituted in place of A, B, and C.

Despite their popularity, Hilbert systems are impossible to use in practice. One has to guess the appropriate instantiation of axioms to start a derivation.

Example: Derivation of $A \supset A$

$$\dfrac{\dfrac{(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A)) \quad A \supset ((A \supset A) \supset A)}{(A \supset (A \supset A)) \supset (A \supset A)} \qquad A \supset (A \supset A)}{A \supset A}$$

Hilbert systems have many axioms and few inference rules

The axioms are not natural!

One needs metatheorems to use them (e.g. the deduction theorem)

Unfriendly systems for humans, and even for machines.

Hilbert himself paved the way to a new system of deduction

Hilbert’s last (24th) problem

Hilbert’s famous list of 23 open mathematical problems at the international mathematical congress in Paris in 1900:

1. Cantor’s continuum problem, the question of the cardinality of the set of reals numbers.
2. Consistency of the arithmetic of real numbers, i.e., of analysis.

⋮

23. Problem about the calculus of variations.

In 2000 Rüdiger Thiele found from old archives in Göttingen some notes in Hilbert’s hand that begin with:

> *As a 24th problem of my Paris talk I wanted to pose the problem: criteria for the simplicity of proofs, or, to show that certain proofs are simpler than any others. In general, to develop a theory of proof methods in mathematics.*

Hilbert systems are inadequate for this. A beginning for a solution arrived 30 years later.

Gentzen

- Introduced natural deduction systems and sequent calculi
- Trivial axioms, “natural rules”
- The rules formalize informal rules of reasoning
- Symmetry of the rules: Introduction/Elimination
- A methodology of permuting the order of application of rules led to normalization, cut elimination, and a consistency proof

2. Sequent calculus

Rules of sequent calculus can have **independent** or **shared** contexts.
For instance the right rule for conjunction with independent contexts is

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A\&B}\ R\&_{ind}$$

With shared contexts it is

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A\&B}\ R\&_{sh}$$

Fact: The two styles are equivalent in the presence of the structural rules.
Shared contexts add determinism to proof search.

With the cut rule the **subformula property** is no longer guaranteed. Thus one of the main tasks of structural proof theory is the design of sequent calculi where cut is an **eliminable** or **admissible** rule.

Contraction can be as “bad” as cut, as concerns a root-first search for a derivation of a given sequent: Formulas in antecedents can be multiplied with no end.

Weakening is easily avoided: modify the axiom $A \to A$ to the form $A, \Gamma \to A$.
Invertible rules are needed for decomposing root first a sequent to be proved.
So, for instance, the single rule

$$\frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}$$

is better than the two equivalent rules

$$\frac{A, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta} \qquad \frac{B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}$$

Summing up the desiderata for our sequent calculus are:

multi-succedent,

context sharing rules,

admissible structural rules,

invertible logical rules.

G3 sequent calculi

- Introduced by Ketonen and successively modified and extended by Kleene, Dragalin, Troelstra
- The rules are invertible
- Not only cut but also weakening and contraction are admissible
- Shared context
- Suited for root-first proof search
- Multisuccedent sequents allow uniform treatment of classical and intuitionistic logic

The calculus **G3c**

Initial sequents:

$P, \Gamma \to \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \to \Delta}{A \& B, \Gamma \to \Delta}\, L\&$$

$$\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \& B}\, R\&$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\, L\vee$$

$$\frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, L\supset$$

$$\frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\, R\supset$$

$$\frac{}{\bot, \Gamma \to \Delta}\, L\bot$$

Theorem: Height-preserving inversion. *All rules of* **G3c** *are invertible, with height-preserving inversion*.
E.g.: If $\vdash_n \Gamma \rightarrow \Delta, A\&B$, then $\vdash_n \Gamma \rightarrow \Delta, A$ and $\vdash_n \Gamma \rightarrow \Delta, B$.
Proof **by induction on** n: If $\Gamma \rightarrow \Delta, A\&B$ is an axiom or conclusion of $L\bot$, then, $A\&B$ not being atomic, also $\Gamma \rightarrow \Delta, A$ and $\Gamma \rightarrow \Delta, B$ are axioms or conclusions of $L\bot$. Assume height preserving inversion up to height n, and let $\vdash_{n+1} \Gamma \rightarrow \Delta, A\&B$.
There are two cases:
If $A\&B$ is not principal in the last rule, it has one or two premisses $\Gamma' \rightarrow \Delta', A\&B$ and $\Gamma'' \rightarrow \Delta'', A\&B$, of derivation height $\leqslant n$, so by inductive hypothesis, $\vdash_n \Gamma' \rightarrow \Delta', A$ and $\vdash_n \Gamma' \rightarrow \Delta', B$ and $\vdash_n \Gamma'' \rightarrow \Delta'', A$ and $\vdash_n \Gamma'' \rightarrow \Delta'', B$. Now apply the last rule to these premisses to conclude $\Gamma \rightarrow \Delta, A$ and $\Gamma \rightarrow \Delta, B$ with a height of derivation $\leqslant n+1$.
If $A\&B$ is principal in the last rule, the premisses $\Gamma \rightarrow \Delta, A$ and $\Gamma \rightarrow \Delta, B$ have derivations of height $\leqslant n$.

Admissibility of structural rules

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\,LW \quad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\,RW$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\,LC \quad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\,RC$$

Theorem: Height-preserving weakening.
If $\vdash_n \Gamma \to \Delta$*, then* $\vdash_n A, \Gamma \to \Delta$.
If $\vdash_n \Gamma \to \Delta$*, then* $\vdash_n \Gamma \to \Delta, A$.

Admissibility of structural rules of G3c (cont.)

Theorem: Height-preserving contraction.
If $\vdash_n C, C, \Gamma \to \Delta$*, then* $\vdash_n C, \Gamma \to \Delta$*.*
If $\vdash_n \Gamma \to \Delta, C, C$*, then* $\vdash_n \Gamma \to \Delta, C$*.*
Proof by simultaneus induction on the height of the derivation for left and right contraction, using height-preserving invertibility of the rules.

Theorem *The rule of cut,*

$$\frac{\Gamma \to \Delta, D \quad D, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,\mathit{Cut}$$

is admissible in **G3c**.

Here only a sketch. For all the details see proof of theorem 3.2.3 of *Structural Proof Theory*.

Advantage of a contraction-free calculus: no need of **multicut**

$$\frac{\Gamma \to \Delta, \overbrace{D, \ldots, D}^{n\times} \quad \overbrace{D, \ldots, D}^{m\times}, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,\mathit{Multicut}$$

(However see Jan von Plato's *A proof of Gentzen's Hauptsatz without multicut*, Archive for Mathematical Logic, vol. 40 (2001), pp. 9-18.).

Assume only one cut as the last step.

Induction on the weight of the cut formula with sub-induction on the sum of the heights of the two premisses of cut.

The proof is organized as follows:

1,2: At least one premiss in a cut is an axiom or conclusion of $L\perp$ and show how cut is eliminated.

Otherwise there are the cases:

3. The cut formula is not principal in either premiss of cut.
4. The cut formula is principal in just one premiss of cut.
5. The cut formula is principal in both premisses of cut.

1. The **left premiss** $\Gamma \to \Delta, D$ of cut is an axiom or concl. of $L\bot$. There are three subcases:

1.1. The cut formula D is in Γ. In this case we derive $\Gamma, \Gamma' \to \Delta, \Delta'$ from the right premiss $D, \Gamma' \to \Delta'$ by weakening.

1.2. Γ and Δ have a common atom. Then $\Gamma, \Gamma' \to \Delta, \Delta'$ is an axiom.

1.3. $\bot$ is a formula in Γ. Then $\Gamma, \Gamma' \to \Delta, \Delta'$ is a concl. of $L\bot$.

2. The **right premiss** $D, \Gamma' \to \Delta'$ is an axiom or concl. of $L\bot$.

2.1. D is in Δ'. Then $\Gamma, \Gamma' \to \Delta, \Delta'$ follows from the first premiss by weakening.

2.2. $\Gamma' \to \Delta'$ is an axiom. Then also $\Gamma, \Gamma' \to \Delta, \Delta'$ is an axiom.

2.3. $\bot$ is in Γ'. Then $\Gamma, \Gamma' \to \Delta, \Delta'$ is a concl. of $L\bot$.

2.4. $D = \bot$. Then either the first premiss is an axiom or concl. of $L\bot$ and $\Gamma, \Gamma' \to \Delta, \Delta'$ follows as in case 1, or $\Gamma \to \Delta, \bot$ has been derived. Six cases according to the rule used. Since $\bot$ is never principal in a rule, these are special cases of 3.1–3.6 below.

Cut with neither premiss an axiom: We have three cases:

3. Cut formula D is **not principal** in the left premiss. We have six subcases according to the rule used to derive the left premiss. For $L\&$ and $L\vee$, the transformations are analogous to cases 3.1 and 3.2 of theorem 2.4.3. For implication, we have

3.3. $L\supset$, with $\Gamma = A \supset B, \Gamma''$. The derivation

$$\dfrac{\dfrac{\Gamma'' \to \Delta, D, A \quad B, \Gamma'' \to \Delta, D}{A \supset B, \Gamma'' \to \Delta, D}{\scriptstyle L\supset} \qquad D, \Gamma' \to \Delta'}{A \supset B, \Gamma'', \Gamma' \to \Delta, \Delta'}{\scriptstyle Cut}$$

is transformed into the derivation

$$\dfrac{\dfrac{\Gamma'' \to \Delta, D, A \quad D, \Gamma' \to \Delta'}{\Gamma'', \Gamma' \to \Delta, \Delta', A}{\scriptstyle Cut} \quad \dfrac{B, \Gamma'' \to \Delta, D \quad D, \Gamma' \to \Delta'}{B, \Gamma'', \Gamma' \to \Delta, \Delta'}{\scriptstyle Cut}}{A \supset B, \Gamma'', \Gamma' \to \Delta, \Delta'}{\scriptstyle L\supset}$$

with two cuts of lower cut-height.

4. Cut formula D is **principal in the left premiss only**, and the derivation is transformed in one with a cut of lower cut-height according to derivation of the right premiss. We have six subcases according to the rule used. Only the cases of $L\supset$ and $R\vee$ are significantly different from the cases of theorem 2.4.3:

4.5. $R\vee$, with $\Delta = A \vee B, \Delta''$. The derivation

$$\dfrac{\Gamma \rightarrow \Delta, D \quad \dfrac{D, \Gamma' \rightarrow A, B, \Delta}{D, \Gamma' \rightarrow A \vee B, \Delta''}\,R\vee}{\Gamma, \Gamma' \rightarrow \Delta, A \vee B, \Delta''}\,Cut$$

is transformed into the derivation with a cut of lower cut-height

$$\dfrac{\dfrac{\Gamma \rightarrow \Delta, D \quad D, \Gamma' \rightarrow A, B, \Delta''}{\Gamma, \Gamma' \rightarrow \Delta, A, B, \Delta''}\,Cut}{\Gamma, \Gamma' \rightarrow \Delta, A \vee B, \Delta''}\,R\vee$$

5. Cut formula D is **principal in both premisses**, and we have three subcases, of which conjunction is very similar to case 5.1 of theorem 2.4.3.

5.2. $D = A \vee B$, and the derivation

$$\dfrac{\dfrac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\,R\vee \qquad \dfrac{A, \Gamma' \to \Delta' \quad B, \Gamma' \to \Delta'}{A \vee B, \Gamma' \to \Delta'}\,L\vee}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, A, B \quad A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta', B}\,Cut \qquad B, \Gamma' \to \Delta'}{\Gamma, \Gamma', \Gamma' \to \Delta, \Delta', \Delta'}\,Cut}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Ctr$$

with two cuts of lower cut-height.

Corollary *Each formula in the derivation of* $\Gamma \to \Delta$ *in* **G3c** *is a subformula of* Γ, Δ.

Corollary: Consistence *The sequent* $\to$ *is not derivable*.
By admissibility of weakening, if $\Gamma \to$ is derivable, then also $\Gamma \to \bot$ is derivable. The converse is obtained by applying cut to $\Gamma \to \bot$ and $\bot \to$, thus, an empty succedent behaves like $\bot$.

Beyond classical propositional logic

We want to extend proof analysis to

1. Theories with axioms: Need a way to add axioms to sequent calculus while maintaining the structural properties.

2. Non-classical logics: Need rules for the modalities that respect the guidelines of proof-theoretic semantics.

We have to be careful...

A parenthesis

- Our project of Proof Analysis started in 1997 when ...

I presented a solution to the following problem:

Given a formula A derivable by the axioms of an apartness relation, if all its atoms are negated, it is derivable by the axioms of equality defined as the negation of apartness

- The axioms: $\neg a \neq a, \quad a \neq b \supset a \neq c \vee b \neq c$

Example of a "negatomic" formula:

$$\neg a \neq b \,\&\, \neg b \neq c \supset \neg a \neq c$$

- With $a = b \equiv \neg a \neq b$ this follows from the axioms of equality, i.e., belongs to the "equality fragment" of the theory of apartness

• conservativity proved by analyzing derivations in sequent calculus extended by rules that correspond to the apartness axioms:

$$\frac{a \neq a, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathit{Irr} \qquad \frac{a \neq c, \Gamma \to \Delta \quad b \neq c, \Gamma \to \Delta}{a \neq b, \Gamma \to \Delta}\,\mathit{Split}$$

- rules give axioms:

$$\frac{a \neq a \rightarrow a \neq a}{\rightarrow a \neq a}\,\mathit{Irr}$$

$$\cfrac{\cfrac{\cfrac{a \neq c \rightarrow a \neq c, b \neq c \quad b \neq c \rightarrow a \neq c, b \neq c}{a \neq b \rightarrow a \neq c, b \neq c}\,\mathit{Split}}{a \neq b \rightarrow a \neq c \vee b \neq c}\,R\vee}{\rightarrow a \neq b \supset a \neq c \vee b \neq c}\,R\supset$$

- axioms give rules by the use of cuts

• The conservativity result about apartness over equality for negatomic formulas was proved through:

- a sequent calculus with added mathematical rules

- a proof of admissibility of the structural rules

- a reduction of derivations to a normal form, where the apartness rules can be converted to rules for defined equality. The result is not obvious because premisses of rules with negatomic conclusion need not be negatomic.

• The common belief was that

"cut elimination fails in the presence of axioms"

1. Theories with axioms

Cut elimination fails in the presence of proper axioms:
A simple example is given in Girard (1987, p. 125): Let the axioms have the forms

$$A \supset B\ (Ax_1), \quad A\ (Ax_2)$$

These are represented by the “axiomatic sequents”

$$\rightarrow A \supset B, \quad \rightarrow A$$

The sequent $\rightarrow B$ is derived from these axiomatic sequents, as in:

$$\dfrac{\rightarrow A \qquad \dfrac{\rightarrow A \supset B \qquad \dfrac{A \rightarrow A \quad B \rightarrow B}{A, A \supset B \rightarrow B}\,{\scriptstyle L\supset}}{A \rightarrow B}\,{\scriptstyle Cut}}{\rightarrow B}\,{\scriptstyle Cut}$$

However, there is no cut-free derivation of $\rightarrow B$.

1. Theories with axioms (cont.)

If the axioms $A \supset B$ and A are converted into the equivalent inference rules

$$\frac{B, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}\ Ax_1\text{-}R \qquad \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ Ax_2\text{-}R$$

we have the following cut-free derivation of $\rightarrow B$:

$$\cfrac{\cfrac{B \rightarrow B}{A \rightarrow B}\ Ax_1\text{-}R}{\rightarrow B}\ Ax_2\text{-}R$$

Criteria for a good extension

- Added rules must guarantee that the extension is **conservative** (non-creative definition).
- In sequent calculus this follows from **cut elimination**, in natural deduction from **normalization** (and subsequent **subformula property**).
- **Structural properties** of the inference relation have to be maintained. In sequent calculus admissibility of weakening, contraction, cut, and reduction to atomic initial sequents.
- **Analyticity** (not always a consequence of cut elimination) has to be maintained.

Four types of extensions (corresponding to parts I–IV of *Proof Analysis*)

1. Natural deduction:

- Applies to universal axioms that do not have essential disjunctions

2. Multisuccedent sequent calculus:

- Applies to universal axioms

3. Systems of rules with eigenvariables:

- Applies to geometric and co-geometric implications

4. Systems of labelled calculi

-Applies also to axioms that are not first-order and to modalities

1. Natural deduction

Write all E-rules in the style of $\vee E, \exists E$:

$$\dfrac{A\&B \qquad \begin{matrix} \overset{1}{A}, \overset{1}{B} \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \&E,1} \qquad \dfrac{A \supset B \quad A \qquad \begin{matrix} \overset{1}{B} \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \supset E,1} \qquad \dfrac{\forall x A \qquad \begin{matrix} \overset{1}{A(t/x)} \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \forall E,1}$$

Rules follow from an
Inversion principle: (N and von Plato 2001)

> *Whatever follows from the* direct grounds *for asserting a proposition must follow from that proposition.*

Derivations are converted to **normal form**: all major premisses of elimination rules are assumptions (von Plato 2001).

Mathematical rules

A typical axiom is of the form, with the P_i, Q atomic

$$P_1 \& \ldots \& P_m \supset Q$$

The conversion into a rule is

$$\frac{P_1 \quad \ldots \quad P_m}{Q}$$

With $m = 0$ we have

$$\frac{}{Q}$$

Another limiting case is

$$\neg(P_1 \& \ldots \& P_m)$$

The corresponding rule has $\perp$ in place of the formula Q:

$$\frac{P_1 \quad \ldots \quad P_m}{\perp}$$

Mathematical rules

Derivations by mathematical rules are finitely branching trees with atoms at the nodes. An extension of natural deduction **NI** by such rules is denoted by **NI***.

With the NI-rules in general form, logical rules permute to below the mathematical ones:

I-rules have compound formulas as conclusions, so the only possible rules are E-rules. If the major premiss is $A\&B$ and minor premiss P_1 in rule $\&E$ followed by rule R, we have the part of derivation and its permutation:

$$\dfrac{\dfrac{A\&B \qquad \begin{array}{c}[A,B]\\ \vdots \\ P_1\end{array}}{P_1}\,{\scriptstyle \&E} \qquad P_2 \quad \dots \quad P_n}{Q}\,{\scriptstyle R} \qquad\qquad \dfrac{A\&B \qquad \dfrac{\begin{array}{c}[A,B]\\ \vdots \\ P_1\end{array} \qquad P_2 \quad \dots \quad P_n}{Q}\,{\scriptstyle R}}{Q}\,{\scriptstyle \&E}$$

The combinatorial possibilities of an axiom system can be studied in a pure form

A simple example:

logical derivation of $d = a$ from $a = b, c = b$, and $c = d$ by the standard axioms of equality

$$
\dfrac{a = d \supset d = a \qquad \dfrac{a = c \mathbin{\&} c = d \supset a = d \qquad \dfrac{\dfrac{a = b \mathbin{\&} b = c \supset a = c \qquad \dfrac{a = b \qquad \dfrac{c = b \supset b = c \qquad c = b}{b = c}{\supset}E}{a = b \mathbin{\&} b = c}\,\&\,I}{a = c}{\supset}E \qquad c = d}{a = c \mathbin{\&} c = d}\,\&\,I}{a = d}{\supset}E}{d = a}{\supset}E
$$

Equality axioms as rules of inference

$$\frac{}{a = a}\,Ref \qquad \frac{a = b}{b = a}\,Sym \qquad \frac{a = b \quad b = c}{a = c}\,Tr$$

Our example derivation becomes:

$$\dfrac{\dfrac{a = b \quad \dfrac{c = b}{b = c}\,Sym}{a = c}\,Tr \qquad c = d}{\dfrac{a = d}{d = a}\,Sym}\,Tr$$

Note that normalization extends to **NI***: Major premisses of *E*-rules turn into assumptions

In general, trace of atoms can be lost but

In many cases, proof search can be limited to **known terms**

An example: rule system **NDLT** for lattice theory

$$\frac{}{a \leqslant a}\,Ref \qquad \frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,Tr$$

$$\frac{}{a \wedge b \leqslant a}\,L\wedge_1 \qquad \frac{}{a \wedge b \leqslant b}\,L\wedge_2 \qquad \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge$$

$$\frac{}{a \leqslant a \vee b}\,R\vee_1 \qquad \frac{}{b \leqslant a \vee b}\,R\vee_2 \qquad \frac{a \leqslant c \quad b \leqslant c}{a \vee b \leqslant c}\,L\vee$$

Theorem. Subterm property for NDLT. *If an atom is derivable from atomic assumptions in* **NDLT***, it has a derivation with no new terms.*

Analogy: subterm property $\sim$ subformula property

In a typical case, there is by assumption some new term $b \wedge c$ that gets removed by a step of *Tr*.

Permutations of rules lead to the critical case

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ a \leqslant b\end{array} \quad \begin{array}{c}\vdots \\ a \leqslant c\end{array}}{a \leqslant b \wedge c}\,R\wedge \qquad \dfrac{}{b \wedge c \leqslant b}\,L\wedge_1}{\begin{array}{c}a \leqslant b \\ \vdots\end{array}}\,Tr
$$

The derivation has a **loop** and is transformed into

$$
\begin{array}{c}\vdots \\ a \leqslant b \\ \vdots\end{array}
$$

A bounded number of terms gives a bounded number of loop-free derivations

Note the analogy to a detour conversion on *B&C*

The result gives the simplest solution to the word problem for freely generated lattices.

See N and von Plato (2002) for the system for lattice theory presented here and N and von Plato (2004) and von Plato (2007) for other systems.

By the same methods, a solution to the uniform word problem for **ortholattices** has been given by A. Meinander (2010).

2. The calculus **G3c**

Initial sequents:

$P, \Gamma \rightarrow \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \rightarrow \Delta}{A\&B, \Gamma \rightarrow \Delta}\,L\& \qquad \frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A\&B}\,R\&$$

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma \rightarrow \Delta, A \quad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}\,L\supset \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}\,R\supset$$

$$\frac{}{\bot, \Gamma \rightarrow \Delta}\,L\bot$$

Mathematical rules in sequent calculus

Conversion of axioms to **conjunctive normal form**: Every quantifier-free formula can be converted to a conjunction of disjunctions of **literals**, that is, of atomic formulas and negations of atomic formulas.

Equivalently: Every quantifier-free formula can be converted to a conjunction of implications of the form

$$P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$$

Every implication of the form $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$ can be converted to a rule that can be added to the calculus without losing the structural rules.

Every quantifier-free formula can be converted to a finite number of rules that can be added to the calculus without losing the structural rules.

Mathematical rules in sequent calculus (cont.)

Take the contraction- and cut-free calculus **G3c** and rules that correspond to axioms

$$P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$$

formulation as a **left** rule:

$$\frac{Q_1, \Gamma \to \Delta \quad \dots \quad Q_n, \Gamma \to \Delta}{P_1, \dots, P_m, \Gamma, \to \Delta}\,R$$

formulation as a **right** rule:

$$\frac{\Gamma \to \Delta, P_1 \quad \dots \quad \Gamma \to \Delta, P_m}{\Gamma, \to \Delta, Q_1, \dots, Q_n}\,R$$

Mathematical rules in sequent calculus (cont.)

The axioms are derivable from the corresponding rules

The rules are derivable from the corresponding axioms, using cuts

Examples

Axiom	**Rule**
$\forall x\ xRx$ reflexivity	$\dfrac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}$
$\forall xyz(xRy \mathbin{\&} yRz \supset xRz)$ trans.	$\dfrac{xRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$
$\forall xyz(xRy \mathbin{\&} xRz \supset yRz)$ euclid.	$\dfrac{yRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}$
$\forall xyz(xRy \mathbin{\&} xRz \supset yRz \vee zRy)$ connectedness	$\dfrac{yRz,\Gamma \to \Delta \quad zRy,\Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$

Adequacy of the extension with rules

Prerequisites of extensions are satisfied:
Cut elimination without compromises thanks to the form of the rules:

- rules act only on one side of sequents

- rules act only on atomic formulas

Typical conversion:

$$\dfrac{\Gamma \to \Delta, P \qquad \dfrac{\vdots}{P, \Gamma' \to \Delta'}\,R}{\Gamma, \Gamma' \to \Delta, \Delta'}\,\mathit{Cut}$$

If R is a left rule with P principal, P is not principal in the left premiss of cut and cut can be permuted

Adequacy of the extension with rules

Admissibility of weakening: thanks to arbitrary contexts

Admissibility of contraction: Analyze proof of admissibility of contraction. 3 cases:
1. None of the contraction formulas is principal in the mathematical rule: OK
2. Only one contraction formula is principal in the mathematical rule: Principal formulas copied in the premisses.
3. Both contraction formulas are principal in the mathematical rule: Closure condition.

Reduction to atomic initial sequents: maintained by monotonicity of extensions (obs. that instead admissibility is not necessarily maintained in extensions)

Examples revisited after the conditions for contraction

Axiom	**Rule**
$\forall x\; xRx$ reflexivity	$\dfrac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}$
$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, xRy, yRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$
$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, xRy, xRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, xRy, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}$
$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness	$\dfrac{yRz, xRy, xRz, \Gamma \to \Delta \quad zRy, xRy}{xRy, xRz, \Gamma \to \Delta}$
closure cond. for trans., euclid, connect.	$\dfrac{xRx, xRx, \Gamma \to \Delta}{xRx, \Gamma \to \Delta}$

Contraction and closure condition

Example of addition imposed by the closure condition:
Take the axiom of *asymmetry* $\neg(a < b \,\&\, b < a)$. As a rule, it is

$$\frac{}{a < b, b < a, \Gamma \to \Delta}$$

The addition imposed by the closure condition is the rule

$$\frac{}{a < a, \Gamma \to \Delta}$$

that corresponds to *irreflexivity*, $\neg(a < a)$.
In some cases, the addition imposed by the closure condition looks like a contraction on atomic formulas.
Legitimate worry: Does the closure condition in practice re-introduce contraction for just atomic formulas in some cases?
Answer: No! If a rule arising from the closure condition is an instance of contraction, then it is admissible.

An example: nondegenerate linear order

The rules of *LO* are

$$\frac{a \leqslant b, \Gamma \rightarrow \Delta \quad b \leqslant a, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,Lin \qquad \frac{a \leqslant a, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,Ref$$

$$\frac{a \leqslant c, \Gamma \rightarrow \Delta}{a \leqslant b, b \leqslant c, \Gamma \rightarrow \Delta}\,Tr \qquad \frac{}{1 \leqslant 0, \Gamma \rightarrow \Delta}\,Ndeg$$

The first and last rules correspond to the axioms $a \leqslant b \vee b \leqslant a$ and $\neg\, 1 \leqslant 0$

Rule *Lin* introduces terms that are difficult to trace but we have:

Theorem. Word problem for linear order. *If $\Gamma \rightarrow \Delta$ has only atoms and is derivable in LO, terms in the derivation can be restricted to those in Γ, Δ.*

An application: Szpilrajn's theorem

Lemma. *If* $\Gamma \to P$ *is derivable in LO, it is derivable in PO (i.e., without Lin)*.

Definition. *An ordering* Σ *is* **inconsistent** *if* $\Gamma \to 1 \leqslant 0$ *is derivable for some finite subset* Γ *of* Σ*, otherwise it is* **consistent**.

An application: Szpilrajn's theorem

Szpilrajn's theorem. *Given a set Σ of atoms in a consistent PO, it can be extended to a consistent LO.*

Proof. Let a, b be any two elements in Σ not ordered in Σ. We claim that either $\Sigma, a \leqslant b$ or $\Sigma, b \leqslant a$ is consistent in *PO*. Let us assume the contrary, i.e., that there exists a finite subset Γ of Σ such that both $\Gamma, a \leqslant b \rightarrow 1 \leqslant 0$ and $\Gamma, b \leqslant a \rightarrow 1 \leqslant 0$ are derivable in *PO*. We then have the step

$$\frac{a \leqslant b, \Gamma \rightarrow 1 \leqslant 0 \quad b \leqslant a, \Gamma \rightarrow 1 \leqslant 0}{\Gamma \rightarrow 1 \leqslant 0}\,\mathit{Lin}$$

Now $\Gamma \rightarrow 1 \leqslant 0$ is derivable in *LO*, and by the conservativity lemma, $\Gamma \rightarrow 1 \leqslant 0$ is already derivable in *PO*, contrary to the consistency assumption. Iteration of the procedure gives the desired extension.

Classical set-theoretic extension results are reformulated as proof-theoretical conservativity results: in pointfree topology, similar shift in the proof of the **Hahn-Banach theorem.**

3. Geometric and co-geometric theories

A is a **geometric formula** *if it does not contain* $\supset$ *or* $\forall$.

Geometric implications *have the form, with* A, B *geometric formulas,*

$$\forall x \ldots \forall z (A \supset B)$$

A **geometric theory** *is a theory axiomatized by geometric implications.*

Typical example:

$$\forall xyz(P_1 \& \ldots \& P_m \supset \exists uvw(Q_1 \& \ldots \& Q_n))$$

"For all x, y, z, if so-and-so, then there are u, v, w such that so-and-so."

- none of the "so-and-so's" can be conditionals or universals

- especially, no *negations*

Canonical form for geometric implications

Conjunctions of

$$\forall \overline{x}(P_1 \& \ldots \& P_m \supset \exists \overline{y}_1 M_1 \vee \cdots \vee \exists \overline{y}_n M_n)$$

P_i atomic formula
M_j conjunction of atomic formulas
none of the variables in $\overline{y}_j$ are free in P_i.

Geometric rules

Left rules:

$$\frac{\overline{Q}_1(\overline{z}_1/\overline{y}_1), \overline{P}, \Gamma \rightarrow \Delta \quad \dots \quad \overline{Q}_n(\overline{z}_n/\overline{y}_n), \overline{P}, \Gamma \rightarrow \Delta}{\overline{P}, \Gamma \rightarrow \Delta}{\scriptstyle GRS}$$

- The eigenvariables z_i must not be free in $\overline{P}, \Gamma, \Delta$.

- Equivalent to geometric implications

- Basic results the same as for universal axioms

- Straightforward proof of **Barr's theorem**: *If a geometric implication is proved classically in a geometric theory, then it can be proved constructively* (N 2003).
Model-theoretic proofs of the same result require the vast apparatus of topos theory. Similar proof-theoretic conservativity results use the Gödel-Gentzen translation and their variants (Ishihara 2000, 2011, Palmgren 2001, Schwichtenberg and Senjak 2011), permutability of rules in sequent calculus (Orevkov 1968), root-first constraints in sequent calculus derivations (Nadathur 1999).

Examples of geometric theories

The formulation of the axioms and the choice of the basic concepts is crucial for obtaining an axiomatization that follows the pattern of geometric implications.

1. Robinson arithmetic.

$\neg\, a = 0 \supset \exists y\; a = s(y)$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a = s(y)$ is

2. Ordered fields

$\neg\, a = 0 \supset \exists y\; a \cdot y = 1$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a \cdot y = 1$ is.

3. Real-closed fields

$\neg\, a_{2n+1} = 0 \supset \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is not geometric, but the equivalent

$a_{2n+1} = 0 \vee \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is.

Examples of geometric theories (cont.)

3. Classical projective geometry with constructions.

Not a geometric theory!

The reason is the axiom of existence of three non-collinear points

$\exists x \exists y \exists z (\neg\, x = y \,\&\, \neg\, z \in ln(x, y))$

if the basic notions are replaced by the constructive notions of apartness between points and lines and "outsideness" of a point from a line, a geometric axiomatization is found. In particular the axiom above is replaced by

$\exists x \exists y \exists z (x \neq y \,\&\, z \notin ln(x, y))$

Co-geometric theories

- a formula is *co-geometric* if it does not contain $\supset$ or $\exists$.

- a *co-geometric implication* has the form, with A and B co-geometric formulas,

$$\forall x \ldots \forall z (A \supset B)$$

- canonical form: conjunctions of
$\forall \overline{x} (\forall y_1 M_1 \& \ldots \& \forall y_n M_n \supset P_1 \vee \cdots \vee P_m)$
with the M_i disjunctions of atoms

- classical projective and affine geometries with the axiom of non-collinearity are co-geometric: write non-collinearity as

$$\neg \forall x \forall y \forall z (x = y \vee z \in ln(x, y))$$

- the notion was found by Jan von Plato and myself on the basis of a proof-theoretical duality in rule systems

The left-right duality

$$\frac{}{a \neq a, \Gamma \longrightarrow \Delta}\,\mathit{Irref} \qquad \frac{a \neq c, \Gamma \longrightarrow \Delta \quad b \neq c, \Gamma \longrightarrow \Delta}{a \neq b, \Gamma \longrightarrow \Delta}\,\mathit{Split}$$

$$\frac{}{\Gamma \longrightarrow \Delta, a = a}\,\mathit{Ref} \qquad \frac{\Gamma \longrightarrow \Delta, a = c \quad \Gamma \longrightarrow \Delta, b = c}{\Gamma \longrightarrow \Delta, a = b}\,\mathit{ETr}$$

ETr stands for the “Euclidean” form of *Tr*

Duality of derivations

Symmetry of apartness is $\rightarrow a \neq b \supset b \neq a$, derived by

$$\dfrac{\dfrac{}{a \neq a \rightarrow b \neq a}\,\text{Irref} \qquad b \neq a \rightarrow b \neq a}{\dfrac{a \neq b \rightarrow b \neq a}{\rightarrow a \neq b \supset b \neq a}\,R\supset}\,\text{Split}$$

Symmetry of equality has a mirror-image derivation

$$\dfrac{\dfrac{}{b = a \rightarrow a = a}\,\text{Ref} \qquad b = a \rightarrow b = a}{\dfrac{b = a \rightarrow a = b}{\rightarrow b = a \supset a = b}\,R\supset}\,\text{ETr}$$

Co-geometric rules

Formulate **right** rules as mirror images of geometric rules:

$$\frac{\Gamma \to \Delta, P_{1_1}, \ldots, P_{1_k} \quad \ldots \quad \Gamma \to \Delta, P_{m_1}, \ldots, P_{m_l}}{\Gamma, \to \Delta, Q_1, \ldots, Q_n,} R$$

The P_i can contain eigenvariables

Basic proof-theoretical results go through as for geometric rules

The duality between geometric and co-geometric theories can be used for changing the primitive notions in the sequent formulation of a theory. Meta-theoretical results can be imported from one theory to its dual by exploiting the symmetry of their associated sequent calculi. Herbrand's theorem for geometric and co-geometric theories

Example from plane geometry

Non-collinearity as a co-geometric rule

$$\frac{\Gamma \rightarrow \Delta, x = y, z \in ln(x, y)}{\Gamma \rightarrow \Delta}\, \textit{Non-coll}$$

The eigenvariables x, y, z must not be free in the conclusion

The cases that any two points are equal and that any point is incident on $ln(x, y)$ are excluded by the rule

Result. If $\Gamma \rightarrow \Delta$ *has only atoms and is derivable by the rules of projective or affine geometry, rule Non-coll not needed*

Result. Subterm property. If $\Gamma \rightarrow \Delta$ *has only atoms and is derivable by the rules of projective or affine geometry, no new terms are needed*

Proof analysis in non-classical and philos. logics

- Aim: Formal investigation of non-classical and philosophical logics, as formulated within the language of modal logic in the way initiated by Hintikka and von Wright.
- Two main traditions in logic, two ways of answering the question "What is a correct logical argument": Syntactic and semantic way, proof-theoretic and model theoretic. Completeness theorems guarantee that they are equivalent.

  For non-classical and philosophical logics, limitations in standard proof systems (cf. SN 2011) and dominance of model-theoretic methods (cf. recent handbooks).
- These two traditions are reconciled in our method: On the syntactic side we follow sequent calculus, on the semantic side Kripke, or relational, semantics. Cf. Hintikka 1955.
- Resulting systems are well suited both for the theory (make derivations, automatic proof search) and the metatheory (embeddings between various logics, decidability and completeness results, negative results, etc.)

Model-theoretic semantics

Establishes correspondence between syntactic expressions and elements of formal structures through interpretation functions

1. Algebraic structures (algebraic semantics)
2. Categories (categorial semantics)
3. Relational structures (Kripke semantics)

....

Conceptual order in model-theoretic semantics:

1. Truth
2. Consequence
3. Proof

Traditional approach in many logic textbooks, e.g.
Mendelson: Introduction to Mathematical Logic;
Goldblatt: Logics of Time and Computation.

Completeness theorem = match between syntax and semantics

valid (in every model) = provable (in a calculus)

Validity usually straightforward: both interpretations and proofs are inductively defined

Completeness obtained by suitable constructions that single out a structure of the semantics from the logical system:

1. Lindenbaum-Tarski algebra
2. Term model construction
3. Canonical model construction based on Henkin sets

....

Syntax and semantics in logical consequence

Notion of **logical consequence** obtained by **universal** quantification over all possible assignments of variables, interpretations, etc. vs. syntactic notion of consequence based on **existence** of a derivation in a logical calculus.

> *(logical) consequence is a universal notion, defined by means of universal quantification over functions (or sets), since one considers* all *models satistying a certain condition... This universality of consequence is a typical feature which is retained also for more complex languages... (Sundholm 2007)*

> *On the contrary, syntactic consequence holds in virtue of the* existence *of a suitable derivation. (Sundholm 2007)*

From model- to proof-theoretic semantics

Girard (On the meaning of the logical rules I: syntax vs. semantics, 1998): Traditional semantics (“Gesticulation”, “Broccoli semantics”, “Treason”, “Tarskism”) vs. *internal* semantics of proofs:

> *“The meaning of the logical rules is to be found in the* well-hidden geometrical structure of the rules themselves*. ... logical rules must be understood in term of their inner harmony.”*

Proof-theoretic semantics

Term “proof-theoretic semantics” introduced by Schroöder-Heister in 1987 but basic idea already in Gentzen (1934) (von Kutschera 1968 used the term ‘Gentzen semantics’):

> *“The introductions represent, as it were, the ‘definitions’ of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions (...) By making these ideas more precise it should be possible to display the E-inferences as unique functions of the I-inferences.”*

Meaning of the logical constants given by their rules

Martin-Löf (1984): On the meaning of the logical constants and the justification of the logical laws.

Introduction rules for intuitionistic connectives justified by the BHK (Brouwer-Heyting-Kolmogorov) interpretation of the logical constants.

Elimination rules Justified by introduction rules through an argument which is the semantical counterpart of the *detour* reduction of Prawitz.

Autonomous justification of the constants by their rules is a delicate thing: Explanation implicitly uses **normalization** for NJ (proved indirectly by Gentzen (1934), directly by Prawitz (1965) and by Gentzen (2008)).

Hacking What is logic? (1979): Which definitions are admissible?
Example: Prior's tonk, in "The runabout inference ticket" (1960)

$$\frac{A}{A\ tonk\ B}\ tonk\text{-}I \qquad \frac{A\ tonk\ B}{B}\ tonk\text{-}E$$

non-eliminable detour

$$\dfrac{\dfrac{A}{A\ tonk\ B}\ tonk\text{-}I}{B}\ tonk\text{-}E$$

Destructive extensions!

Detour eliminability not enough: take the modified tonk

$$\frac{A}{A\ tonk'\ B}\ tonk'\text{-}I \qquad\qquad \frac{A\ tonk'\ B \quad B}{A}\ tonk'\text{-}E$$

has detour conversion

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad\begin{array}{c}\vdots\\ B\end{array}}{A\ tonk'\ B}\ tonk'\text{-}I \qquad \begin{array}{c}\vdots\\ B\end{array}}{A}\ tonk'\text{-}E \qquad\rightsquigarrow\qquad \begin{array}{c}\vdots\\ A\end{array}$$

but no permutation conversion

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ (A\ tonk'\ B) tonk'\ C\end{array}\quad\begin{array}{c}\vdots\\ C\end{array}}{A\ tonk'\ B}\ tonk'\text{-}E \qquad \begin{array}{c}\vdots\\ B\end{array}}{A}\ tonk'\text{-}E \qquad\rightsquigarrow\qquad ???$$

Harmony in natural deduction

N and von Plato (2001) **Inversion principle:**

> *Whatever follows from the* direct grounds *for asserting a proposition must follow from that proposition.*

Natural deduction with general elimination rules (von Plato 2001)

$$\dfrac{A\&B \qquad \begin{matrix}\overset{1}{A}, \overset{1}{B}\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \&E,1} \qquad\qquad \dfrac{A \supset B \quad A \qquad \begin{matrix}\overset{1}{B}\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \supset E,1}$$

Derivations are converted to **normal form**: all major premisses of elimination rules are assumptions.

Note that the rules for tonk don't satisfy the inversion principle, lack of harmony causes failure of normalization; with elimination rule obtained from the inversion principle of NvP (2001)

$$\dfrac{A\ tonk\ B \quad \begin{array}{c} \overset{1}{A} \\ \vdots \\ C \end{array}}{C}\ tonk\text{-}E', 1$$

the problem disappears.

Labelled sequent calculi overcome the traditional duality between syntax and semantics, usually considered as complementary yet distinct ingredients of a logic.
Interaction of syntax and semantics in logical calculi

1. At design level: semantics as guiding tool

2. At investigation level: study of meta-theoretical properties (completeness, embeddings, decidability...)

3. Deeper interaction in labelled calculi, disciplined and fertile co-existence.

Semantics in logical calculi

- **Implicit**: *Sequent calculus for classical logic*, *display calculi* (Wansing), *nested sequents* (Kashima 1994), *tree-sequents* (Cerrato 1996), *deep sequents* (Brünnler 2006, Stouppa 2007), *tree-hypersequents* (Poggiolesi 2008), *hypersequents, non-deterministic matrices* (Avron, Zamansky, Ciabattoni, et al.).
- **Explicit**: *Labelled sequents* (Mints 1997, Viganó 2000, Kushida and Okada 2003, Castellini and Smaill 2002, Castellini 2005), *labelled tableaux* (Fitting 1983, Catach 1991, Nerode 1991, Goré 1998, Massacci 2000), *labelled natural deduction* (Fitch 1966, Simpson 1994, Basin, Matthews, Viganó 1998), *hybrid logic* (Blackburn 2000), *Labelled Deductive Systems* (Gabbay, Russo, et al. 1996).

A historical parenthesis

Who invented Kripke semantics? (politically correct terminology: *relational* semantics)

- Copeland (2001) “The genesis of possible world semantics”
- Goldblatt (2005) “Mathematical Modal Logic: a View of its Evolution”

  `www.mcs.vuw.ac.nz/ rob/papers/modalhist.pdf`.
- Ilpo Halonen course “Mahdollisten maailmojen semantiikan synty ja kehitys”

  `http://www.helsinki.fi/hum/fil/filosofia/`.

Rule systems with labels

For logics characterized by a Kripke-style semantics:
If the accessibility (Hintikka's alternativeness) relation in Kripke frames is made part of a sequent calculus, frame properties typically turn into rules that maintain cut elimination

- Explanation of modal operators through harmonious introduction and elimination pairs of rules.
- Properties of Kripke frames though rules for the accessibility relation.

How?

- Add possible worlds as labels for formulas $x : A$
- Add properties of the accessibility relation xRy as rules, following the method of extension for mathematical theories

Basic modal logic K: Add to propositional logic:

1. $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
2. From A to infer $\Box A$.

Rules for basic modal logic obtained from the inductive definition of validity in a Kripke frame.

From

$$x \Vdash \Box A \iff \textit{for all } y,\ xRy \textit{ implies } y \Vdash A$$

- $$\frac{xRy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\ R\Box$$

  variable condition: y not (free) in Γ, Δ

- $$\frac{y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\ L\Box$$

The systems G3K

Initial sequents: $x : P, \Gamma \rightarrow \Delta, x : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \rightarrow \Delta}{x : A\&B, \Gamma \rightarrow \Delta}\,L\& \qquad \frac{\Gamma \rightarrow \Delta, x : A \quad \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A\&B}\,R\&$$

$$\frac{x : A, \Gamma \rightarrow \Delta \quad x : B, \Gamma \rightarrow \Delta}{x : A \vee B, \Gamma \rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : B}{\Gamma \rightarrow \Delta, x : A \vee B}\,R\vee$$

$$\frac{\Gamma \rightarrow \Delta, x : A \quad x : B, \Gamma \rightarrow \Delta}{x : A \supset B, \Gamma \rightarrow \Delta}\,L\supset \qquad \frac{x : A, \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A \supset B}\,R\supset$$

$$\frac{}{x : \perp, \Gamma \rightarrow \Delta}\,L\perp$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \rightarrow \Delta}{x : \Box A, xRy, \Gamma \rightarrow \Delta}\,L\Box \qquad \frac{xRy, \Gamma \rightarrow \Delta, y : A}{\Gamma \rightarrow \Delta, x : \Box A}\,R\Box$$

Extensions of basic modal logic:

System	Axiom
T	$\Box A \supset A$
4	$\Box A \supset \Box\Box A$
E	$\Diamond A \supset \Box\Diamond A$
B	$A \supset \Box\Diamond A$
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$
D	$\Box A \supset \Diamond A$
2	$\Diamond\Box A \supset \Box\Diamond A$

Rules for extensions obtained by adding to **G3K** the mathematical rules that correspond to the frame properties

Examples of universal extensions

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\ xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\ xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness
GL	$\Box(\Box A \supset A) \supset \Box A$	trans., irref., and Noetherian
Grz	$\Box(\Box(A \supset \Box A) \supset A) \supset A$	trans., refl., Noetherian

	Frame property	Rule
T	$\forall x\ xRx$ reflexivity	$\dfrac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}$
4	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$
E	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz\Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
B	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}$
3	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness	$\dfrac{yRz, \Gamma \to \Delta \quad zRy, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
D	$\forall x \exists y\ xRy$ seriality	
2	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	
GL Grz	trans., irref., and Noetherian trans., refl., and Noetherian	

Examples of geometric extensions

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\ xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset xRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\ xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness
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	Frame property	Rule
T	$\forall x\ xRx$ reflexivity	$\dfrac{xRx, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$
4	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, \Gamma \rightarrow \Delta}{xRy, yRz, \Gamma \rightarrow \Delta}$
E	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}$
B	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}$
3	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$	$\dfrac{yRz, \Gamma \rightarrow \Delta \quad zRy, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}$
D	$\forall x \exists y\ xRy$ seriality	$\dfrac{xRy, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ y$
2	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	$\dfrac{yRw, zRw, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}\ w$
GL Grz	trans., irref., and Noetherian trans., refl., and Noetherian	

Results

Let **G3K*** be any extension of **G3K** by universal or geometric rules for the accessibility relation.

- All the structural rules–weakening, contraction, and cut–are admissible in the system **G3K***.
- The characteristic axioms are derivable.
- The necessitation rule is admissible.
- Indirect completeness, through equivalence with the corresponding axiomatic system.
- Direct completeness: proof search in the system either gives a derivation or a Kripke countemodel.
- Answers to questions of undefinability though conservativity theorems.
- Answers to decidability questions through algorithms of terminating proof search and countermodel constructions.

Some theorems

Lemma *Sequents of the form*

$$x : A, \Gamma \to \Delta, x : A$$

with A an arbitrary modal formula (not just atomic), are derivable in **G3K***.

Lemma *For arbitrary A and B, the sequent*

$$\to x : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

is derivable in **G3K***.

Proposition *The rules of left and right weakening are height-preserving admissible in* **G3K***.

Necessitation as an explicit rule would ruin the system. Instead it is an *admissible* rule. We exploit first-order features of the system...

Definition: Substitution.

$$
\begin{array}{rcl}
xRy(z/w) & \equiv & xRy \text{ if } w \neq x \text{ and } w \neq y \\
xRy(z/x) & \equiv & zRy \\
xRy(z/y) & \equiv & xRz \\
xRx(z/x) & \equiv & zRz \\
x : A(z/y) & \equiv & x : A \text{ if } y \neq x \\
x : A(z/x) & \equiv & z : A
\end{array}
$$

extend to multisets componentwise.

Lemma *If* $\Gamma \to \Delta$ *is derivable in* **G3K***, *then* $\Gamma(y/x) \to \Delta(y/x)$ *is also derivable, with the same derivation height.*

Proposition *The necessitation rule*

$$\frac{\to x : A}{\to x : \Box A}$$

is admissible in **G3K***.

Examples of derivations in G3K*

G3K $+$ *Ref* (**G3T**):

$$\cfrac{\cfrac{\cfrac{xRy, x : \Box A, x : A \rightarrow x : A}{xRx, x : \Box A \rightarrow x : A}\,L\Box}{x : \Box A \rightarrow x : A}\,Ref}{\rightarrow x : \Box A \supset A}\,R\supset$$

G3K $+$ *Trans* (**G3K4**):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{xRz, xRy, yRz, x : \Box A, z : A \rightarrow z : A}{xRz, xRy, yRz, x : \Box A \rightarrow z : A}\,L\Box}{xRy, yRz, x : \Box A \rightarrow z : A}\,Trans}{xRy, x : \Box A \rightarrow y : \Box A}\,R\Box}{x : \Box A \rightarrow x : \Box\Box A}\,R\Box}{\rightarrow x : \Box A \supset \Box\Box A}\,R\supset$$

G3K $+$ *Sym* (**G3KB**):

$$\cfrac{\cfrac{\cfrac{\cfrac{yRx, xRy, x : A \rightarrow x : A, y : \Diamond A}{yRx, xRy, x : A \rightarrow y : \Diamond A}\,R\Diamond}{xRy, x : A \rightarrow y : \Diamond A}\,Sym}{x : A \rightarrow x : \Box\Diamond A}\,R\Box}{\rightarrow x : A \supset \Box\Diamond A}\,R\supset$$

G3K $+$ *Ser*:

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xRy, x : \Box A \to x : \Diamond A, y : A}{xRy, x : \Box A \to x : \Diamond A, y : A}\, L\Box}{xRy, x : \Box A \to x : \Diamond A}\, R\Diamond}{x : \Box A \to x : \Diamond A}\, Ser}{\to x : \Box A \supset \Diamond A}\, R\supset$$

G3K $+$ *Dir*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRy, xRz, yRu, zRu, u : A, z : \Box A \to y : \Diamond A, u : A}{xRy, xRz, yRu, zRu, u : A, z : \Box A \to y : \Diamond A}\, R\Diamond}{xRy, xRz, yRu, zRu, z : \Box A \to y : \Diamond A}\, L\Box}{xRy, xRz, z : \Box A \to y : \Diamond A}\, Dir}{xRy, x : \Diamond\Box A \to y : \Diamond A}\, L\Diamond}{x : \Diamond\Box A \to x : \Box\Diamond A}\, R\Box}{\to x : \Diamond\Box A \supset \Box\Diamond A}\, R\supset$$

Structural properties of G3K*

- All the rules of **G3K*** are height-preserving invertible.

- The rules of weakening and contraction are height-preserving admissible in **G3K***.
- The rule of cut is admissible in **G3K***.

Undefinable properties

Alternative approach to proofs of negative results in correspondence theory.

Certain frame properties (irreflexivity, intransitivity, etc.) do not have any modal correspondent.

Usual proofs based on (complicated) model extension methods. Here an immediate consequence of a conservativity theorem.

Irreflexivity $\forall x \neg xRx$ corresponds to the rule

$$\frac{}{xRx, \Gamma \to \Delta}\; \mathit{Irref}$$

Theorem *The system* **G3K**+Irref *is conservative over* **G3K**.

Proof: Suppose that the sequent (not containing relational atoms) $\Gamma \to \Delta$ is derivable in **G3K+***Irref*. The atoms of the form xRy that appear on the left-hand side of sequents in the derivation originate from applications of rule $R\Box$. By the variable condition, $x \neq y$, so the derivation contains no atom of the form xRx, hence no application of *Irref*. Therefore the sequent is derivable in **G3K**. QED

Intransitivity

$\forall x \forall y \forall z (xRy \& yRz \supset \neg xRz)$ corresponds to

$$\frac{}{xRy, yRz, xRz\Gamma \to \Delta} \; \textit{Intrans}$$

Theorem *The system* **G3K**$+$Intrans *is conservative over* **G3K***.*

Result holds for a generalization of intransitivity:

Theorem *Let* $P_1, \ldots, P_n\Gamma \to \Delta$ *be a rule, called* G-Intrans*, that corresponds to the axiom* $\neg(P_1 \& \ldots \& P_n)$ *with* $P_i \equiv x_iRy_i$*, and assume that for some* i, j*,* $y_i = y_j$*. Then* **G3K**$+$G-Intrans *is conservative over* **G3K***.*

Similar result for $\exists x.xRx$ and $\forall x \exists y(xRy \;\&\; yRy)$

Analyticity

Cut elimination alone not enough to ensure terminating proof search.
Need subformula property.
Subformula property and *analytic* cut.
Subformula property not always sufficient for decidability:
- first order logic
- calculi with explicit structural rules

In our systems, a suitable version of the subformula property, adequate for proving syntactic decidability, is a consequence of the structural properties of the calculi.

Subformula: *For every propositional connective* $\circ$*, the subformulas of* $x : A \circ B$ *are* $x : A \circ B$*, and all the subformulas of* $x : A$ *and of* $x : B$*. The subformulas of* $x : \Box A$ *and* $x : \Diamond A$ *are* $x : \Box A$ *and* $x : \Diamond A$*, resp., and all the subformulas of* $y : A$ *for arbitrary* y*.*

Subformula property: *All formulas in a derivation are subformulas of (formulas in) the endsequent.*

Weak subformula property: *All formulas in a derivation are either subformulas of (formulas in) the endsequent or atomic formulas of the form* xRy*.*

Subterm property: *All terms (variables, worlds) in a derivation are either eigenvariables or terms (variables, worlds) in the conclusion.*

• Derivations in **G3K*** satisfy the weak subformula property

By h.p. substitution we can suppose that rules that remove labels (e.g. *Ref*) are applied, root first, only to labels in their conclusions, so that derivations in **G3K**, **G3T**, **G3K4**, **G3KB**, **G3S4**, **G3TB**, **G3S5**, **G3D**,**G3GL** have the subterm property.

Other source of potentially non-terminating proof search: the repetition of the principal formulas in the premisses of $L\Box$ and $R\Diamond$.

By permutation of rules and height preserving contraction, we prove that it is enough to apply $L\Box$ and $R\Diamond$ only once on any given pair of principal formulas $xRy, x : \Box A$ or $xRy, x : \Diamond A$.

Explicit bounds for proof search in **G3K**, **G3T**, **G3KB**, **G3TB**

In **G3S4** proof search may not terminate

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{zRw, xRz, xRy, yRz, x : \Box\neg\Box A \rightarrow y : B, z : A, w : A}\,R\Box}{xRz, xRy, yRz, x : \Box\neg\Box A \rightarrow y : B, z : A, z : \Box A}\,L\supset}{z : \neg\Box A, xRz, xRy, yRz, x : \Box\neg\Box A \rightarrow y : B, z : \Box A}\,L\Box}{xRz, xRy, yRz, x : \Box\neg\Box A \rightarrow y : B, z : \Box A}\,\mathit{Trans}}{xRy, yRz, x : \Box\neg\Box A \rightarrow y : B, z : A}\,R\Box}{xRy, x : \Box\neg\Box A \rightarrow y : B, y : \Box A}\,L\supset}{y : \neg\Box A, xRy, x : \Box\neg\Box A \rightarrow y : B}\,L\Box}{xRy, x : \Box\neg\Box A \rightarrow y : B}\,R\Box}{x : \Box\neg\Box A \rightarrow x : \Box B}\,R\supset}{\rightarrow x : \Box\neg\Box A \supset \Box B}
$$

Apply the substitution z/w and obtain

$$zRz, xRz, xRy, yRz, x : \Box\neg\Box A \to y : B, z : A, z : A$$

and continue using height-preserving contraction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ zRz, xRz, xRy, yRz, x : \Box\neg\Box A \to y : B, z : A\end{array}}{xRz, xRy, yRz, x : \Box\neg\Box A \to y : B, z : A}\,\mathit{Ref}}{xRy, yRz, x : \Box\neg\Box A \to y : B, z : A}\,\mathit{Trans}$$

the original derivation is shortened by two steps

By generalizing this argument we obtain a proof of:

Proposition *In a minimal derivation of a sequent in* **G3S4***, for each formula* $x : \Box A$ *in its positive part there are at most* $n(\Box)$ *applications of* $R\Box$ *iterated on a chain of accessible worlds* $xRx_1, x_1Rx_2, \ldots$*, with principal formula* $x_i : \Box A$*.*

Corollary *The systems* **G3S4** *and* **G3S5** *allow terminating proof search.*

Decidability can be proved by an application of the proof of Kripke completeness through a suitable definition of saturation of labelled sequents / truncation of countermodel construction (this method has been used for the logic of linear time in Boretti and Negri 2009).

The system **G3I**

Initial sequents: $x \leqslant y, x : P, \Gamma \to \Delta, y : P$

Logical rules: As in **G3K** for &, $\vee$, $\perp$,

$$\frac{x \leqslant y, x : A \supset B, \Gamma \to y : A, \Delta, \quad x \leqslant y, x : A \supset B, y : B, \Gamma \to \Delta}{x \leqslant y, x : A \supset B, \Gamma \to \Delta} \, L{\supset}$$

$$\frac{x \leqslant y, y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B} \, R{\supset}$$

Order rules:

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta} \, \mathit{Ref} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta} \, \mathit{Trans}$$

Rule $R{\supset}$ has the condition that y must not be in Γ, Δ.

Intermediate logics (joint work with Roy Dyckhoff)

G3I can be extended with rules expressing additional properties of the pre-order $\leqslant$ exactly as done for modal logic.

For example, Gödel-Dummett logic has a *linear* accessibility relation
$\forall x \forall y (x \leqslant y \vee y \leqslant x)$. This becomes the rule

$$\frac{x \leqslant y, \Gamma \rightarrow \Delta \quad y \leqslant x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta} \; Lin$$

Add the rule to **G3I** and obtain a (labelled) sequent system for Gödel-Dummett logic.

Denote by **G3I*** any extension of **G3I** with rules following the geometric rule scheme. Below more examples of intermediate logics.

Structural properties of **G3I***

All sequents of the following form are derivable in **G3I***:

1. $x \leqslant y, x : A, \Gamma \rightarrow \Delta, y : A$
2. $x : A, \Gamma \rightarrow \Delta, x : A$

The substitution rule

$$\frac{\Gamma \rightarrow \Delta}{\Gamma(y/x) \rightarrow \Delta(y/x)}\ (y/x)$$

is hp-admissible in **G3I***.

The rules of *Weakening*

$$\frac{\Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\ LW \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, x : A}\ RW \qquad \frac{\Gamma \rightarrow \Delta}{x \leqslant y, \Gamma \rightarrow \Delta}\ LW_{\leqslant}$$

are hp-admissible in **G3I***.

Structural properties of **G3I*** (cont.)

All the rules of **G3I*** are hp-invertible.

The rules of *Contraction*

$$\frac{x : A, x : A, \Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\ \textit{L-Ctr} \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : A}{\Gamma \rightarrow \Delta, x : A}\ \textit{R-Ctr}$$

$$\frac{x \leqslant y, x \leqslant y, \Gamma \rightarrow \Delta}{x \leqslant y, \Gamma \rightarrow \Delta}\ \textit{L-Ctr} \leqslant$$

are hp-admissible in **G3I***.

The *Cut* rule

$$\frac{\Gamma \rightarrow \Delta, x : A \quad x : A, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}\ \textit{Cut}$$

is admissible in **G3I***.

The rule

$$\frac{\rightarrow x : A \supset B \quad \rightarrow x : A}{\rightarrow x : B}\ MP$$

is admissible in **G3I*.**

The axioms that correspond to the frame properties are derivable in **G3I*.**

Each system in **G3I*** is equivalent to the intermediate logic obtained by adding to **Int** the axiom(s) that correspond((s)) to the frame property(ies)

$$\vdash_{Int+Ax} A \text{ iff } \mathbf{G3I}^{*}_{\mathbf{Ax}} \vdash \rightarrow x : A$$

- **Int** *Intuitionistic Logic*.
- **Jan** *Jankov-De Morgan Logic*: The relation $\leqslant$ is *directed* or *convergent*, i.e.

$$\forall xyz((x \leqslant y \;\&\; x \leqslant z) \supset \exists w(y \leqslant w \;\&\; z \leqslant w)).$$

  Logic also known as **KC**, and as the "logic of weak excluded middle". Axiomatised by either $\neg A \vee \neg\neg A$ or $\neg(A \& B) \supset (\neg A \vee \neg B)$.
- **GD** *Gödel-Dummett Logic*: The accessibility relation is *linear*, i.e.

$$\forall xy(x \leqslant y \vee y \leqslant x).$$

  Axioms: $(A \supset B) \vee (B \supset A)$ or $((A \supset B) \supset C) \supset (((B \supset A) \supset C) \supset C)$.

- **Bd**$_2$: Bounded *depth at most* 2

$$\forall xyz((x \leqslant y \leqslant z) \supset (y \leqslant x \vee z \leqslant y)).$$

Axiomatised by $A \vee (A \supset (B \vee \neg B))$.

- **GSc**:Depth at most 2 and at most 2 final elements

$$\forall xyz \exists v((x \leqslant v \,\&\, y \leqslant v) \vee (y \leqslant v \,\&\, z \leqslant v) \vee (x \leqslant v \,\&\, z \leqslant v)).$$

This logic is axiomatised by, for example,
$(A \supset B) \vee (B \supset A) \vee ((A \supset \neg B) \& (\neg B \supset A))$ and
$A \vee (A \supset (B \vee \neg B))$.

- **Sm**: Smetanich logic, also known as $\mathbf{LC_2}$ or **HT**, the “logic of here and there”, or as Gödel’s 3-valued logic. The accessibility relation is linear and has depth at most 2

$$\forall xy(x \leqslant y \lor y \leqslant x).$$
$$\forall xyz((x \leqslant y \leqslant z) \supset (y \leqslant x \lor z \leqslant y))$$

  added to **GD** and $\mathbf{Bd_2}$. Axiomatised by the **GD** axiom plus the $\mathbf{Bd_2}$ axiom, or, equivalently, by $(\neg B \supset A) \supset (((A \supset B) \supset A) \supset A)$.
- **Cl** *Classical logic*: The accessibility relation is *symmetric*,

$$\forall xy(x \leqslant y \ \supset \ y \leqslant x).$$

  Axiomatised by $A \lor \neg A$ or by $\neg\neg A \supset A$.

Not only the interpolable ones can be treated by this method: Several variants of these logics are non-interpolable but still have geometric frame conditions:
Bd_n for $n > 2$ ("Bounded depth n") and btw_n for $n > 2$ (approximately, "bounded top-width" n). For $n = 3$, frame condition is

$$\forall x x_0 x_1 x_2 x_3 \left(\bigwedge_{i=1}^{n} xRx_i \supset \exists y \left(\bigvee_{i \neq j} x_i R y \mathbin{\&} x_j R y \right) \right)$$

a geometric implication

Kreisel-Putnam logic, axiomatised over **Int** by the schema

$$(\neg A \supset (B \vee C)) \supset ((\neg A \supset B) \vee (\neg A \supset C)),$$

is a (non-interpolable) intermediate logic with a characteristic frame condition that is not a geometric implication (p. 55, CZ).

Gödel translation of **Int** to **S4**

1. Gödel (1933): $\vdash_{Int} A \Rightarrow \vdash_{S4} A^*$ soundness
2. McKinsey & Tarski (1948): $\nvdash_{Int} A \Rightarrow \nvdash_{S4} A^*$ faithfulness
3. Dummett & Lemmon (1959): $\vdash_{Int+Ax} A$ iff $\vdash_{S4+Ax^*} A^*$

A modal logic **M** is a **modal companion** of a superintuitionistic logic **L** if $\vdash_{\mathbf{L}} A$ iff $\vdash_{\mathbf{M}} A^*$. So **S4** is a modal companion of **Int**, **S4+Ax*** is a modal companion of **Int+Ax**.

2. and 3. are proved semantically. We look closer at McKinsey & Tarski (1948):

Proof by McKinsey & Tarski (1948) uses:

- 1. Completeness of intuitionistic logic wrt Heyting algebras (Brouwerian algebras) and of S4 wrt topological Boolean algebras (closure algebras)
- 2. Representation of Heyting algebras as the opens of topological Boolean algebras.
- 3. The proof is **indirect** because of 1. and **non-constructive** because of 2. (Uses Stone representation of distributive lattices, in particular Zorn's lemma)

The result was generalized to intermediate logics by Dummett and Lemmon (1959).
No syntactic proof of faithfulness in the literature except the complex proof of the embedding of **Int** into **S4** in Troelstra & Schwichtenberg (1996).

4. Troestra & Schwichtenberg (1996) use a variant of the translation and give a complex syntactic proof of faithfulness.

$$\begin{array}{rcl} P^{\Box} & := & \Box P \\ \bot^{\Box} & := & \bot \\ (A \supset B)^{\Box} & := & \Box(A^{\Box} \supset B^{\Box}) \\ (A \& B)^{\Box} & := & A^{\Box} \& B^{\Box} \\ (A \vee B)^{\Box} & := & A^{\Box} \vee B^{\Box} \end{array}$$

The translation $\Gamma^{\Box}$ of a multiset $\Gamma \equiv A_1, \ldots, A_n$ is defined componentwise by

$$(A_1, \ldots, A_n)^{\Box} := A_1^{\Box}, \ldots, A_n^{\Box}$$

Given an extension **G3I*** of **G3I** with rules for $\leqslant$, we denote by **G3S4*** the corresponding extension of **G3S4**.

Theorem. If **G3I*** $\vdash \Gamma \to \Delta$ then **G3S4*** $\vdash \Gamma^{\Box} \to \Delta^{\Box}$.

Proof: By induction on the structure of the derivation.

$x \leqslant y, \Gamma, x : P \to y : P, \Delta \quad \rightsquigarrow$

$$\dfrac{\dfrac{\dfrac{\overline{\ldots, \Gamma^{\Box}, x : \Box P, z : P \to z : P, \Delta^{\Box}}\ Ax}{x \leqslant y, y \leqslant z, x \leqslant z, \Gamma^{\Box}, x : \Box P \to z : P, \Delta^{\Box}}\ L\Box}{x \leqslant y, y \leqslant z, \Gamma^{\Box}, x : \Box P \to z : P, \Delta^{\Box}}\ Trans}{x \leqslant y, \Gamma^{\Box}, x : \Box P \to y : \Box P, \Delta^{\Box}}\ R\Box$$

$$\frac{x \leqslant y, \Gamma, y : A \to y : B, \Delta}{\Gamma \to x : A \supset B, \Delta}\ R\supset \quad \rightsquigarrow \quad \dfrac{\dfrac{x \leqslant y, \Gamma^{\Box}, y : A^{\Box} \to y : B^{\Box}, \Delta^{\Box}}{x \leqslant y, \Gamma^{\Box} \to y : A^{\Box} \supset B^{\Box}, \Delta^{\Box}}\ R\supset}{\Gamma^{\Box} \to x : \Box(A^{\Box} \supset B^{\Box}), \Delta^{\Box}}\ R\Box$$

$L\supset$ similar; conjunction, disjunction and absurdity routine.

Frame rules identical in the two systems, so nothing to prove for them.

Faithfulness. If **G3S4*** $\vdash \Gamma^{\Box} \rightarrow \Delta^{\Box}$ then **G3I*** $\vdash \Gamma \rightarrow \Delta$.

Follows as a special case from:

If Γ, Δ are multisets of labelled formulas (with relational atoms also possibly in Γ) and Γ', Δ' are multisets of labelled atomic formulas, and **G3S4*** $\vdash \Gamma^{\Box}, \Gamma' \rightarrow \Delta^{\Box}, \Delta'$, then **G3I*** $\vdash \Gamma, \Gamma' \rightarrow \Delta, \Delta'$.

Proof: By induction on the derivation. We show here only one case (the others are even easier)

$$\frac{x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), y : A^{\Box} \supset B^{\Box}, \Gamma''^{\Box}, \Gamma' \rightarrow \Delta^{\Box}, \Delta'}{x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), \Gamma''^{\Box}, \Gamma' \rightarrow \Delta^{\Box}, \Delta'} \; L\Box$$

Observe that $A^\Box \supset B^\Box$ is not a translated formula, nor an atomic one. By hp-invertibility of $L\supset$ in **G3S4*** we have

$$x \leqslant y, x : \Box(A^\Box \supset B^\Box), \Gamma''^\Box, \Gamma' \to \Delta^\Box, \Delta', y : A^\Box$$

and

$$x \leqslant y, x : \Box(A^\Box \supset B^\Box), y : B^\Box, \Gamma''^\Box, \Gamma' \to \Delta^\Box, \Delta'$$

Now the inductive hypothesis applies. We therefore have the derivation in **G3I***

$$\dfrac{\overset{\mathcal{I.H.}}{x \leqslant y, x : A \supset B, \Gamma'', \Gamma' \to \Delta, \Delta', y : A} \quad \overset{\mathcal{I.H.}}{x \leqslant y, x : A \supset B, y : B, \Gamma'', \Gamma' \to \Delta, \Delta'}}{x \leqslant y, x : A \supset B, \Gamma'', \Gamma' \to \Delta, \Delta'} L\supset$$

Compare the above proof of faithfulness of the embedding with a (standard) proof of faithfulness of the embedding of **Int** into **S4** for an unlabelled sequent calculus (Troelstra & Schwichtenberg 1996)

Faithfulness of the embedding into its (smallest) modal companion maintained for each of the intermediate logics characterised by frames satisfying geometric implications. Well-known modal companions are **S4** for **Int**, **S4.2** for **Jan**, **S4.3** for **GD**, **S5** for Cl.

Core of the above proof, erasure of all $\Box$, reminiscent of an analogous reduction in the model-theoretic proof of faithfulness of the embedding of **Int** into **S4**: Countermodel for an unprovable sequent in **Int** turned into countermodel for the translation of that sequent in **S4**; in particular, "it can be treated as a modal frame isomorphic to its skeleton" (see theorem 3.83 in [CZ]).

Gödel-Löb provability logic

Solovay (1976): GL characterized as the logic of arithmetic provability; characterization of Kripke models.

Cut elimination for GL

semantic proofs:

1. Sambin & Valentini (1982); 2. Avron (1984)

syntactic proofs:

3. Leivant (1981); 4. Valentini (1983); 5. Borga (1983)
6. Moen (2003)

4 gives a counterexample to 3
6 raises some doubts on 4 but uses a different calculus with explicit contraction.
In 1–5 calculi with *contexts-as-sets*; contraction seems to be the problematic issue.

Lack of harmony: Only one rule (both left and right) for $\Box$

$$\frac{\Box\Gamma, \Gamma, \Box A \to A}{\Box\Gamma, \Gamma' \to \Delta, \Box A}$$

Here: Calculus with admissible contraction for sequents labelled by possible worlds, with left and right rules for $\Box$, allows for a transparent proof of cut elimination.

Kripke frames for provability logic

Accessibility relation R is

irreflexive

transitive

Noetherian (every R-chain eventually becomes stationary)

equivalently: *transitive and all R-chains are finite*

Characterizing frame condition is not first order, so cannot apply the method of universal / geometric extensions.

But can nevertheless be internalized as follows:

Lemma In irreflexive, transitive, and Noetherian Kripke frames

$$x \Vdash \Box A \;\Leftrightarrow\; \textit{for all } y,\; \textit{from } xRy \textit{ and } y \Vdash \Box A \textit{ follows } y \Vdash A$$

"$\Rightarrow$" gives the right rule for $\Box$

$$\frac{xRy, y : \Box A, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\, R\Box\text{-}L$$

(variable condition: y not in the conclusion)
"$\Leftarrow$" the left rule

$$\frac{x : \Box A, xRy, \Gamma \to \Delta, y : \Box A \quad y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\, L\Box\text{-}L$$

The systems G3GL

Initial sequents:

$x : P, \Gamma \rightarrow \Delta, x : P \qquad x : \Box A, \Gamma \rightarrow \Delta, x : \Box A$

Logical rules:

As in **G3K** for &, $\vee$, $\supset$, $\bot$; $L\Box$-L, $R\Box$-L

Mathematical rules: *Ref, Trans*

Preliminary results

- All the rules are sound wrt Kripke semantics.
- The axioms of the axiomatic system are derivable.
- Substitution is height-preserving admissible.
- Weakening is height-preserving admissible.
- The necessitation rule is admissible.
- All the rules are height-preserving invertible.
- The rules of contraction are admissible. Elimination of contraction does not introduce new worlds in the derivation (i.e. contraction is *range*-preserving admissible).

Cut elimination for G3GL

Typical procedure for G3-like systems: Consider topmost cuts and perform reductions that either decrease the *height* (permutations - cut formula not principal in at least one of the premisses)
or
size (detours - cut formula principal in both premisses)
until cuts reach initial sequents and disappear

With **G3GL** does not work in the case of detour cuts on $x : \Box A$

Principal cut on $x : \Box A$

$$\dfrac{\dfrac{xRy, y{:}\Box A, \Gamma \rightarrow \Delta, y{:}A}{\Gamma \rightarrow \Delta, x{:}\Box A}\, R\Box\text{-}L \qquad \dfrac{xRz, x{:}\Box A, \Gamma' \rightarrow \Delta', z{:}\Box A \quad z{:}A, xRz, x{:}\Box A, \Gamma' \rightarrow \Delta'}{xRz, x{:}\Box A, \Gamma' \rightarrow \Delta'}\, L\Box\text{-}L}{xRz, \Gamma', \Gamma \rightarrow \Delta, \Delta'}\, Cut$$

The derivation is transformed into one containing four cuts:

1. First, we construct the derivation

$$\dfrac{\Gamma \rightarrow \Delta, x : \Box A \quad xRz, x : \Box A, \Gamma', \rightarrow \Delta', z : \Box A}{xRz, \Gamma, \Gamma' \rightarrow \Delta, \Delta', z : \Box A}\, Cut$$

using a cut of smaller weight, on the same labelled formula $x : \Box A$ (and thus the same range) but with lower sum of heights of derivations.

2. Second, we construct the derivation

$$\dfrac{\Gamma \rightarrow \Delta, x : \Box A \quad xRz, x : \Box A, z : A, \Gamma' \rightarrow \Delta'}{xRz, z : A, \Gamma', \Gamma \rightarrow \Delta, \Delta'}\, Cut$$

reduced in weight in the same way.

3. Third, we use derivation 1. and height-preserving substitution (z/y) on the premiss of $R\Box$-L to obtain

$$\dfrac{\begin{array}{c}\vdots\\ xRz, \Gamma', \Gamma \to \Delta, \Delta', z : \Box A\end{array} \quad xRz, z : \Box A, \Gamma \to \Delta, z : A}{xRz, xRz, \Gamma, \Gamma', \Gamma \to \Delta, \Delta', \Delta, z : A}\ \mathit{Cut}$$

using a cut on the labelled formula $z : \Box A$. This cut, however, is not reduced according to the usual size/height measure.

4. Fourth, we combine 3. and 2. by a cut on the labelled formula $z : A$ of smaller size, followed by several contractions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ xRz, xRz, \Gamma', \Gamma, \Gamma \to \Delta, \Delta, \Delta', z : A\end{array} \quad \begin{array}{c}\vdots\\ xRz, z : A, \Gamma', \Gamma \to \Delta, \Delta'\end{array}}{xRz, xRz, xRz, \Gamma', \Gamma', \Gamma, \Gamma, \Gamma \to \Delta, \Delta, \Delta, \Delta', \Delta'}\ \mathit{Cut}}{xRz, \Gamma', \Gamma \to \Delta, \Delta'}\ \mathit{Ctr}^*$$

Way out: introduce a third parameter
$range(x)$ is the set of labels y such that xRy is in the transitive closure of relational atoms occurring in antecedents of sequents in the derivation.
The range satisfies:

- $x \notin range(x)$
- If $y \in range(x)$ then $range(y) \subset range(x)$
- If $y, z \in range(x)$ and y is an eigenvariable, then $range(y) \cup range(z) \subset range(x)$

provided we assume (wlog):

- There are no cuts with xRx or $xRx_1, \ldots, x_nRx$ in the antecedents of their conclusions (if there, they are eliminated using *Irref* and *Trans*).
- Eigenvariables are *pure*, i.e., appear only in the subtree above the step introducing them.

Therefore cut is reduced wrt the inductive parameter (size, range, height)

Lemma All sequents of the form $xRy, x : \Box A, \Gamma \to \Delta, y : \Box A$ are derivable in **G3GL**.

Corollary The standard $L\Box$ rule

$$\frac{y : A, xRy, x : \Box A, \Gamma \to \Delta}{xRy, x : \Box A, \Gamma \to \Delta}\ {\scriptstyle L\Box}$$

is derivable in **G3GL.**

Remark: The two left $\Box$ rules are interderivable, but use of $L\Box$-L seems essential in the proof of cut elimination. If the standard $L\Box$ were used instead, a cut with a (derived) sequent of the form $xRy, x : \Box A, \Gamma \to \Delta, y : \Box A$ would be needed. However, its derivation introduces new worlds, thus breaking the property of range admissibility of all cut reductions.

Corollary The Löb axiom is derivable in **G3GL.**

$$\cfrac{\cfrac{\cfrac{y : \Box A \supset A, xRy, x : \Box(\Box A \supset A), y : \Box A \to y : A}{xRy, x : \Box(\Box A \supset A), y : \Box A \to y : A}\ {\scriptstyle L\Box}}{x : \Box(\Box A \supset A) \to x : \Box A}\ {\scriptstyle R\Box\text{-}L}}{\to x : \Box(\Box A \supset A) \supset \Box A}\ {\scriptstyle R\supset}$$

As an application of the cut-free calculus we get an immediate proof of the **second incompleteness theorem**:

The sequent $\rightarrow x : \neg\Box\bot$ is not derivable in **G3GL.**

Proof: Proceeding root first, if a derivation exists, it ends with

$$\frac{x : \Box\bot \rightarrow x : \bot}{\rightarrow x : \Box\bot \supset \bot} R\supset$$

but no rule of **G3GL** is applicable to the premiss. QED

Displayable logics

Properly displayable logics are captured by the extension with rules for geometric implications.
By Kracht's results, displayable extensions of basic modal logic are characterized by *primitive* frame conditions, of the form

$$(\forall^R)(\exists^R)A$$

$$\forall^R \ldots\ldots \forall y(xRy \supset Ay)$$

$$\exists^R \ldots\ldots \exists y(xRy \,\&\, Ay)$$

A built from conjunctions and disjunctions of $x = y, xRy, xR^{-1}y$ where x and y not both in the scope of an $\exists$

Displayable logics (cont.)

Through standard conversions of first order logic, primitive frame conditions convert to the form of a geometric implication:

$$\forall x_1(At_1(x_1) \supset (\forall x_2 At_2(x_1, x_2) \supset \ldots \exists y_1(Bt_1(y_1)\&(\exists y_2 Bt_2(y_2)\& \ldots)))))$$

$$\rightsquigarrow$$

$$\forall x_1 \forall x_2 \ldots \forall x_n(At_1(x_1)\&At_2(x_1, x_2) \supset \exists y_1 \exists y_2 Bt_1(y_1)\&Bt_2(y_1, y_2)....))))$$

Not every geometric implication satisfies the additional conditions on variables, but those that are needed in our context do: the existential label licences additional steps if related to a universal label. If both labels in an atom were bound by the existential quantifier they would be both fresh in the geometric rule scheme and thus useless.

First-order modal logic

Quantificational model (Kripke 1963): To every world w is associated a domain of interpretation of individual variables $D(w)$.
$D \equiv \bigcup_{w \in K} D(w)$
Valuation of atomic predicates under an assignment extended to arbitrary formulas by the standard inductive clauses for propositional connectives. For the quantifiers

$w \Vdash \forall x A(x)$ whenever for all a in $D(w)$, $w \Vdash A(a/x)$.

$w \Vdash \exists x A(x)$ whenever for some a in $D(w)$, $w \Vdash A(a/x)$.

Different possible assumptions about the domains $D(w)$ give rise to different notions of quantificational models.

We add to our calculus expressions of the form $a \in D(w)$.

Rules for the quantifiers

$$\frac{a \in D(w), \Gamma \rightarrow \Delta, w : A(a/x)}{\Gamma \rightarrow \Delta, w : \forall x A}\, R\forall$$

$$\frac{w : A(a/x), w : \forall x A, a \in D(w), \Gamma \rightarrow \Delta}{w : \forall x A, a \in D(w), \Gamma \rightarrow \Delta}\, L\forall$$

$$\frac{a \in D(w), \Gamma \rightarrow \Delta, w : \exists x A, w : A(a/x)}{a \in D(w), \Gamma \rightarrow \Delta, w : \exists x A}\, R\exists$$

$$\frac{a \in D(w), w : A(a/x), \Gamma \rightarrow \Delta}{w : \exists x A, \Gamma \rightarrow \Delta}\, L\exists$$

Rules $R\forall$, $L\exists$ have the condition $a \notin \Gamma, \Delta$.

Barcan and the like

In axiomatic approaches to modal logic (e.g. Hughes & Cresswell) first-order modal logic obtained as an extension of first-order classical logic;

Mismatch between axiomatization and semantics: universal instantiation, in the form $\forall x A(x) \supset A(a)$, in general is not valid.

Seen through a failed proof search by our rules:

$$\dfrac{\begin{array}{c}\vdots\\ w : \forall x A(x) \rightarrow w : A(a)\end{array}}{\rightarrow w : \forall x A(x) \supset A(a)}\; R\supset$$

After this single step, no rule is applicable; The only way to continue would be a step of $L\forall$ but this would require the additional assumption $a \in D(w)$.

Similarly to **G3K**, we may add to **G3Kq** properties of the accessibility relation and obtain, for example, a system for **S5** with quantifiers by adding rules *Ref*, *Trans*, and *Sym*.
Also properties of the domain function can be required. For instance, it can be postulated that for every world, the corresponding domain of interpretation be **non-empty**:

$$\forall w \exists a\, a \in D(w)$$

Another condition is that domains are **increasing**:

$$\forall wo \forall a (wRo \;\&\; a \in D(w) \supset a \in D(o))$$

They can also be **decreasing**:

$$\forall wo \forall a (wRo \;\&\; a \in D(o) \supset a \in D(w))$$

All the above properties follow the geometric rule scheme and their rule form is a follows, with the variable condition $a \notin \Gamma, \Delta$ in the first:

$$\frac{a \in D(w), \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\, \textit{Nonempty}$$

$$\frac{a \in D(o), wRo, a \in D(w), \Gamma \rightarrow \Delta}{wRo, a \in D(w), \Gamma \rightarrow \Delta}\, \textit{Incr}$$

$$\frac{a \in D(w), wRo, a \in D(o), \Gamma \rightarrow \Delta}{wRo, a \in D(o), \Gamma \rightarrow \Delta}\, \textit{Decr}$$

Property of **nonemptiness** usually part of the ontology of the intended semantics for quantified systems of logic and implicit in the rule of elimination of the universal quantifier. We gain a more flexible approach by not having it inbuilt in the rules. Formally similar to the property of seriality added to **G3K** to obtain deontic logic. This latter characterized by the axiom $\Box A \supset \Diamond A$.

Nonemptiness corresponds to the axiom $\forall x A \supset \exists x A$

$$\dfrac{\dfrac{\dfrac{\dfrac{w : A(a/x), a \in D(w), w : \forall x A \rightarrow w : \exists x A, w : A(a/x)}{w : A(a/x), a \in D(w), w : \forall x A \rightarrow w : \exists x A}\, R\exists}{a \in D(w), w : \forall x A \rightarrow w : \exists x A}\, L\forall}{w : \forall x A \rightarrow w : \exists x A}\, \textit{Nonempty}}{\rightarrow w : \forall x A \supset \exists x A}\, R\supset$$

Properties of permutability of the necessity modality and the universal quantifier have been the object of a long philosophical discussion.

Barcan formula $\forall x \Box A \supset \Box \forall x A$

Converse Barcan formula $\Box \forall x A \supset \forall \Box x A$.

The Barcan formula is derivable in **G3Kq**+*Decr*

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{o : A(a/x), w : \Box A(a/x), a \in D(w), a \in D(o), wRo, w : \forall x \Box A \to o : A(a/x)}{w : \Box A(a/x), a \in D(w), a \in D(o), wRo, w : \forall x \Box A \to o : A(a/x)}\ L\Box}{a \in D(w), a \in D(o), wRo, w : \forall x \Box A \to o : A(a/x)}\ L\forall}{a \in D(o), wRo, w : \forall x \Box A \to o : A(a/x)}\ Decr}{wRo, w : \forall x \Box A \to o : \forall x A}\ R\forall}{w : \forall x \Box A \to w : \Box \forall x A}\ R\Box}{\to w : \forall x \Box A \supset \Box \forall x A}\ R\supset
$$

Structural properties of **G3Kq***

All the structural properties proved for **G3Kq*** easily extend to **G3Kq***.

In addition to substitution on labels we have substitution on domain elements:

$$a \in D(w)(b/a) \equiv b \in D(w)$$
$$a \in D(w)(o/w) \equiv a \in D(o)$$

In addition to the weakening and contraction rules of **G3K**, we have to consider also weakening and contraction rules that operate on domain atoms, such as

$$\frac{\Gamma \rightarrow \Delta}{x \in D(w), \Gamma \rightarrow \Delta} LW_D \qquad \frac{a \in D(w), a \in D(w), \Gamma \rightarrow \Delta}{a \in D(w), \Gamma \rightarrow \Delta} L\text{-}Ctr_D$$

- If $\Gamma \to \Delta$ is derivable in **G3Kq***, then also $\Gamma(o/w) \to \Delta(o/w)$ and $\Gamma(b/a) \to \Delta(b/a)$ are derivable, with the same derivation height (substitution of world labels and of domain elements is heigh-preserving admissible).
- All the structural rules (weakening and contraction, both on labelled formulas and on relational and domain atoms) are height-preserving admissible in **G3Kq***.
- The rule of necessitation is admissible in **G3Kq***.
- The rule of cut is admissible in **G3Kq***.
- Direct and uniform completeness proof.

Other approaches: Fitting (1998) and Fitting and Mendelsohn (1998): tableaux systems; Arlo-Costa and Pacuit (2001): neighborhoods semantics ; Corsi (2002): unified completeness theorem; Garson (2005): calculus with existence predicate; Goldblatt and Mares (2006): general frames.

Systems with transitive closure of accessibility relations

Linear time

Temporal operator: **T**, 'tomorrow', and **G**, 'it will always be the case that',
are necessities w.r.t. the accessibility relation of immediate predecessor $x \prec y$ and its (reflexive) and transitive closure $x \leq y$.
G has dual **F** (possibility in the future)
T self-dual under the condition of uniqueness of immediate successor

$x \Vdash \mathbf{T}A$ *iff for all* $y,\ x \prec y$ *implies* $y \Vdash A$
$x \Vdash \mathbf{G}A$ *iff for all* $y,\ x \leq y$ *implies* $y \Vdash A$
$x \Vdash \mathbf{F}A$ *iff there exists* y *such that* $x \leq y$ *and* $y \Vdash A$

Rules for G

$$\frac{y : A, x : \mathbf{G}A, x \leq y, \Gamma \to \Delta}{x : \mathbf{G}A, x \leq y, \Gamma \to \Delta}\, L\mathbf{G} \qquad \frac{x \leq y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathbf{G}A}\, R\mathbf{G}$$

Rules for F

$$\frac{x \leq y, y : A, \Gamma \to \Delta}{x : \mathbf{F}A, \Gamma \to \Delta}\, L\mathbf{F} \qquad \frac{x \leq y, \Gamma \to \Delta, x : \mathbf{F}A, y : A}{x \leq y, \Gamma \to \Delta, x : \mathbf{F}A}\, R\mathbf{F}$$

Rules for T

$$\frac{y : A, x : \mathbf{T}A, x \prec y, \Gamma \to \Delta}{x : \mathbf{T}A, x \prec y, \Gamma \to \Delta}\, L\mathbf{T} \qquad \frac{x \prec y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathbf{T}A}\, R\mathbf{T}$$

(Rules $R\mathbf{G}$, $L\mathbf{F}$ and $R\mathbf{T}$ have the condition that y is not in Γ, Δ)

Infinitary Rule

$$\frac{\{x \prec^n y, x \leq y, \Gamma \to \Delta\}_{n \in \mathbb{N}}}{x \leq y, \Gamma \to \Delta}\; T^\omega$$

where
$x \prec^0 y \;\equiv\; x = y$
$x \prec^1 y \;\equiv\; x \prec y$
$x \prec^{n+1} y \;\equiv\; \exists z(x \prec^n z \,\&\, z \prec y)$, for $n > 0$
Two forms of finitization are available (Boretti and Negri 2010):

- Non-standard system: replace rule with a rule that corresponds to Robinson's induction. Conservative for the fragment without **G** in the positive part and **F** in the negative part.
- Finite bound to T^ω on the basis of the **T**-complexity of the endsequent. Conservative for the fragment without **G** in the negative part and **F** in the positive part.

Completeness

- Kripke (1959) Completeness for 1st-order S5 with equality. A formula is derivable in S5 if and only if it is valid. Uses an adaptation to modal logic of Beth's method of semantical tableaux. If $A_1 \& \ldots \& A_n \supset B$ is not valid, the tableau construction cannot be closed; If the tableau construction is not closed, then a countermodel is found.
- Kripke (1963) Extension of results of Kripke (1959) to T, B, S4, and S5. Properties of the accessibility relation not part of the tableau syntax.
  Validity: If the construction for A is closed then A is valid.
  Completeness: By a systematic search of a countermodel. Uses König's lemma; Applications to decidability.
  Modal disjunction property (in S4, if $\Box A \vee \Box B$ derivable, then $\Box A$ is derivable, or $\Box B$ is derivable, McKinsey-Tarski 1948, Lemmon 1960) here with "glueing of Kripke models".

History: Criticisms and their effect

- **Arnould Bayart** (1966), review of Kripke (1959): Criticizes lack of determinism in the tableau construction for the quantifier rules.
- **David Kaplan** (JSL, 1966), review to Kripke (1963): Criticizes lack of rigor and use of informal arguments in the tableau construction. Suggests a completeness proof a' la Henkin, as foreseen already by Kanger, as more rigorous.
- **Henkin-style completeness**
  Henkin (1949) for 1st-order logic
  Bayart (1958,1959) Sequent calculus, possible worlds semantics, and Henkin style completeness for 1st- and 2nd-order S5
  Lemmon, Scott (Kaplan 1966)
  Makinson (1966), Cresswell (1967)

Henkin-style completeness proof for modal logic:

Soundness proved by induction of the derivation in $\mathcal{L}$. If $\mathcal{L}$ has additional axioms then it is proved that they are valid in the class of frames considered.

Completeness proved by the canonical model construction. From $\mathcal{L}$ a special model is built in which validity and derivability coincide. The canonical model is a Kripke model in which the nodes are maximal consistent sets of formulas, the accessibility relation is such that two nodes Γ, Δ are related if all the necessary truths in the former are in the latter and a formula is forced at a node if it belongs to that node.

Henkin-style completeness proof

Henkin (1949) completeness for 1st-order logic

Basic idea:
A set of formulas Δ is a *maximal set* if

- Δ is consistent
- for every A either A or $\neg A$ is in Δ

equivalently, there is no consistent extension of Δ

The *deductive closure* of Γ is $\overline{\Gamma} \equiv \{A \mid \Gamma \vdash A\}$
If Δ is a *maximal set* then $\Delta = \overline{\Delta}$
Fact: $\overline{\Gamma} \equiv \bigcap\{\Delta \text{ maximal set} \mid \Gamma \subset \Delta\} \equiv \Gamma^*$
In the *canonical model* $V_\Delta(P) = \textit{ true } \equiv P \in \Delta$. By induction one sees that this is extended to arbitrary formulas (**truth lemma**), so $\Gamma \models A$ means $A^* \subset \Gamma^*$
Completeness: If $\Gamma \models A$ then $\Gamma \vdash A$

Henkin-style completeness proof

Adaptation to modal logic:

Canonical Kripke frame:

$K \equiv \{s \mid s \text{ maximal set }\}$

$sRt \equiv$ for all $A. \Box A \in s$ implies $A \in t$

$s \Vdash P \equiv P \in s$

Truth Lemma: $s \Vdash A$ if and only if $A \in s$

The rest is identical to the proof for first-order logic.

Uniform completeness

- The use of a labelled system allows a direct proof of completeness via Schütte's method of reduction trees.
- The proof is not constructive (König's lemma), however ...
- The countermodels of unprovable sequents are obtained directly from a failed proof search.
- No need for the somewhat artificial constructions of Henkin sets of formulas and of "bulldozing" methods for imposing irreflexivity (in systems with irreflexive relations).
- The proof can be adapted to all the systems of modal/temporal/epistemic logic considered.
- Gives a heuristics for finding frame conditions that correspond to modal formulas.

Uniform completeness: soundness

Consider a derivation in **G3K***:

K frame with accessibility relation $\mathcal{R}$ that satisfies the properties $*$.

W the set of world labels used in derivations in **G3K***.

Interpretation of W in $K \equiv [\![\cdot]\!] : W \rightarrow K$ such that,

If wRo in the derivation, then $[\![w]\!]\mathcal{R}[\![o]\!]$ in K.

Valuation of atomic formulas $\mathcal{V} : AtFrm \rightarrow \mathcal{P}(K)$

$w \in \mathcal{V}(P)$ iff $[\![w]\!] \Vdash P$.

Uniform completeness: soundness (cont.)

Valuations extended to arbitrary formulas:

For all w, it is not the case that $[\![w]\!] \Vdash \bot$ (abbr. $[\![w]\!] \nVdash \bot$);

$[\![w]\!] \Vdash A \& B$ if $[\![w]\!] \Vdash A$ and $[\![w]\!] \Vdash B$;

$[\![w]\!] \Vdash A \vee B$ if $[\![w]\!] \Vdash A$ or $[\![w]\!] \Vdash B$;

$[\![w]\!] \Vdash A \supset B$ if $[\![w]\!] \Vdash A$ implies $[\![w]\!] \Vdash B$;

$[\![w]\!] \Vdash \Box A$ if for all o, $[\![w]\!]\mathcal{R}[\![o]\!]$ implies $[\![o]\!] \Vdash A$;

$[\![w]\!] \Vdash \Diamond A$ if there exists o such that $[\![w]\!]\mathcal{R}[\![o]\!]$ and $[\![o]\!] \Vdash A$.

Uniform completeness: soundness (cont.)

$\Gamma \rightarrow \Delta$ *valid* for a given interpretation of labels and valuation of propositional variables in a frame, if for all labelled formulas $w : A$ and relational atoms oRr in Γ, if $[\![w]\!] \Vdash A$ and $[\![o]\!]\mathcal{R}[\![r]\!]$ in K, then for some $l : B$ in Δ, $[\![l]\!] \Vdash B$. A sequent is valid if it it valid for every interpretation and every valuation of propositional variables in the frame.

Validity: If sequent $\Gamma \rightarrow \Delta$ is derivable in $\mathbf{G3K}^*$, then it is valid.

Uniform completeness: completeness

Completeness: Let $\Gamma \to \Delta$ be a sequent in the language of $\mathbf{G3K}^*$. Then either the sequent is derivable in $\mathbf{G3K}^*$ or it has a Kripke countermodel.

Proof: We define for an arbitrary sequent $\Gamma \to \Delta$ in the language of $\mathbf{G3K}^*$ a reduction tree by applying root first the rules of $\mathbf{G3K}^*$ in all possible ways. If the construction terminates we obtain a proof, else we obtain an infinite tree. By König's lemma an infinite tree has an infinite branch, which is used to define a countermodel to the endsequent.

Uniform completeness: completeness (cont.)

Construction of the countermodel:
Let $\Gamma_0 \to \Delta_0 \equiv \Gamma \to \Delta, \Gamma_1 \to \Delta_1 \ldots, \Gamma_i \to \Delta_i, \ldots$ be the infinite branch branch. Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i$$

$$\mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$, and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Uniform completeness: completeness (cont.)

Frame $K \equiv$ labels appearing in the relational atoms in $\mathbf{\Gamma}$

Relation $R \equiv$ all the wRo's in $\mathbf{\Gamma}$.

Construction of the reduction tree imposes the frame properties of the countermodel: For instance, in the system **G3S4** the constructed frame is reflexive and transitive.

Valuation: For all the atomic formulas $w : P$ in $\mathbf{\Gamma}$, set $w \Vdash P$, and for all atomic formulas $o : Q$ in $\mathbf{\Delta}$, set $o \nVdash Q$.

Finally show inductively on the weight of formulas that A is forced in the model at node w if $w : A$ is in $\mathbf{\Gamma}$ and A is not forced at node w if $w : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \rightarrow \Delta$.

Completeness and decidability

The proof of uniform completeness can be turned into a decision procedure for vast classes of modal logics, with methods popular in the tableaux literature:

Consider root-first proof search in the labelled calculus for the given logic

Show that proof search does not need to go on forever. Saturation of sequents together with loop-detection allow to find a bound.

Construction of countermodels on saturated sequents.

Advantage here: contrary to tableaux, sequent systems are *completely local*.

Decidability for the logic of linear time (Boretti and Negri 2009).

Multi-modal systems for logics of social interaction

- Theories of collective intentionality and social choice theory study aggregation of individual attitudes (preferences, judgements) into collective attitudes.
    - Summative aggregation: shared beliefs, mutual beliefs, distributed knowledge, common knowledge (Fagin et al. 1995).
    - Non-summative aggregation: group beliefs attributed to the collectivity, not reducible to individuals beliefs; distinction between belief and acceptance (Gaudou et al., Lorini et al.).
- Raul Hakli and SN (2008, 2011): Proof theory for the logic of shared and distributed knowledge and of the logic of acceptance; formal analysis of voting procedures and location of sources of inconsistences.
- Systems with actions that modify the models: Dynamic Epistemic Logic, in particular Public Announcement Logic (P. Maffezioli and SN 2010).
- *Knowability logic*: Proof analysis of Fitch's paradox (Maffezioli, Naibo and SN 2012).

Knowability logic

Here knowability logic is treated only as a motivating problem for a methodological extension of the method of proof analysis. For the specific results, see the paper Maffezioli, Naibo, Negri (2012). Epistemic conceptions of truth justify the *knowability principle*:

If A is true, then it is possible to know that A $\quad A \supset \Diamond \mathcal{K} A$ (**KP**)

Fitch's paradox: formal derivation that poses minimal assumptions on the alethic and epistemic operators, and that starts from the knowability principle to conclude *(collective) omniscience*:

All truths are actually known $\quad A \supset \mathcal{K} A$ (**OP**)

The paradox was presented by Fitch (1963) but found by Joe Salerno and Julien Murzi to have actually been suggested by Church in a series of referee's reports that date back to 1945 (Salerno 2009).

Knowability logic

The paradox has given rise to a flourishing and ever expanding literature (can be found even in social networks). The main goal has been to show that the paradox does not affect an *intuitionistic* conception of truth.
The derivation of the paradox is indeed done in classical logic. Intuitionistic logic proves its negative version, but to prove intuitionistic *underivability* of the positive version, a careful proof analysis is needed.
So the goal has been to develop a proof theory for *knowability logic*: a cut-free sequent system for bimodal logic extended by the knowability principle.

The knowability principle does not reduce to atomic instances, so it cannot be translated into rules through the methodology recalled above, and the more expressive language of labelled sequent calculi comes to use.

A labelled calculus for knowability logic

The rules for intuitionistic implication and for the modalities from the forcing clauses of Kripke semantics:

- $x \Vdash A \supset B$ iff for all y, from $x \leqslant y$ and $y \Vdash A$ follows $y \Vdash B$
- $x \Vdash \mathcal{K}A$ iff for all y, $xR_{\mathcal{K}}y$ implies $y \Vdash A$
- $x \Vdash \Diamond A$ iff for some y, $xR_{\Diamond}y$ and $y \Vdash A$

The clauses are converted into rules:

$$\frac{x \leqslant y, y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B}\, R\supset \qquad \frac{x \leqslant y, x : A \supset B, \Gamma \to \Delta, y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma \to \Delta}{x \leqslant y, x : A \supset B, \Gamma \to \Delta}\, L\supset$$

$$\frac{y : A, x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \to \Delta}{x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \to \Delta}\, L\mathcal{K} \qquad \frac{xR_{\mathcal{K}}y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathcal{K}A}\, R\mathcal{K}$$

$$\frac{xR_{\Diamond}y, y : A, \Gamma \to \Delta}{x : \Diamond A, \Gamma \to \Delta}\, L\Diamond \qquad \frac{xR_{\Diamond}y, \Gamma \to \Delta, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma \to \Delta, x : \Diamond A}\, R\Diamond$$

In $R\supset$, $R\mathcal{K}$, $L\Diamond$, y does not appear in Γ and Δ

From (modal) axioms to (frame) rules

Various extensions obtained by adding the frame properties that correspond to the added axioms, for example

Logic	Axiom	Frame property	Rule
T	$\Box A \supset A$	$\forall x\, xRx$ reflexivity	$\dfrac{xRx, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, xRy, yRz, \Gamma \rightarrow \Delta}{xRy, yRz, \Gamma \rightarrow \Delta}$
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, xRy, xRz, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}$
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, xRy, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}$
D	$\Box A \supset \Diamond A$	$\forall x \exists y\, xRy$ seriality	$\dfrac{xRy, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\, y$
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	$\dfrac{yRw, zRw, xRy, xRz, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}\, w$
W	$\Box(\Box A \supset A) \supset \Box A$	trans., irref., and no infinite R-chains	modified $L\Box$ and $R\Box$

but *knowability* is different from all such cases...

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\frac{}{\Rightarrow x : A \supset \Diamond \mathcal{K} A}$$

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Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant y, y \leqslant w, yR_\Diamond z, zR_\mathcal{K} w, y : A \to y : \Diamond\mathcal{K}A, w : A}{x \leqslant y, yR_\Diamond z, , zR_\mathcal{K} w, y : A \to y : \Diamond\mathcal{K}A, w : A}\;{\scriptstyle \Diamond\mathcal{K}\text{-}Tr}}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathcal{K}A, z : KA}\;{\scriptstyle R\mathcal{K}}}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathcal{K}A}\;{\scriptstyle R\Diamond}}{x \leqslant y, y : A \to y : \Diamond\mathcal{K}A}\;{\scriptstyle Ser_\Diamond}}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\;{\scriptstyle R\supset}
$$

the uppermost sequent is derivable by *monotonicity*.

Finding the right rules for knowability logic (cont.)

The two extra-logical rules used are:

$$\frac{xR_{\Diamond}y, \Gamma \to \Delta}{\Gamma \to \Delta}\ \mathit{Ser}_{\Diamond} \qquad \frac{x \leqslant z, xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma \to \Delta}{xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma \to \Delta}\ \Diamond\mathcal{K}\text{-}\mathit{Tr}$$

$\mathit{Ser}_{\Diamond}$ has the condition $y \notin \Gamma, \Delta$. The rules correspond to the frame properties

$$\forall x \exists y . xR_{\Diamond}y \qquad \mathbf{Ser}_{\Diamond}$$

$$\forall x \forall y \forall z (xR_{\Diamond}y \,\&\, yR_{\mathcal{K}}z \supset x \leqslant z) \qquad \Diamond\mathcal{K}\text{-}\mathbf{Tr}$$

The universal frame property $\Diamond\mathcal{K}$**-Tr** is, however, **too strong**: The instance of rule $\Diamond\mathcal{K}$-*Tr* used in the derivation of **KP** is not applied (root first) to an arbitrary sequent, but to one in which the middle term is the eigenvariable introduced by $\mathit{Ser}_{\Diamond}$. So we have the requirements:

- *$\Diamond\mathcal{K}$-Tr has to be applied above $\mathit{Ser}_{\Diamond}$*
- The middle term of $\Diamond\mathcal{K}$-*Tr* is the eigenvariable of $\mathit{Ser}_{\Diamond}$.

Finding the right rules for knowability logic (cont.)

The move to consider the two rules not independently of each other, but as a *system of rules*, coupled together by the side condition on the eigenvariable.
With this proviso, the system of rules is equivalent to the frame property

$$\forall x \exists y (x R_{\Diamond} y \,\&\, \forall z (y R_{\mathcal{K}} z \supset x \leqslant z)) \quad \textbf{KP-Fr}$$

KP-Fr is derivable in a G3-sequent system for intuitionistic first-order logic extended by the two rules $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-Tr with the side condition.
Conversely, any derivation that uses rules $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-Tr *in compliance with the side condition*, can be transformed into a derivation that uses cuts with **KP-Fr**.

The system with rules $\Diamond\mathcal{K}$-*Tr* and $Ser_\Diamond$ that respect the side condition is a cut-free equivalent of the system that employs ***KP-Fr*** as an axiomatic sequent in addition to the structural rules.

The rules that correspond to **KP-Fr do not** follow the geometric rule scheme. However, all the structural rules are still admissible in the presence of such rules. In particular, cut elimination holds and the proof follows the usual pattern; **a genuine extension of the method of conversion of axioms into rules**.

The system obtained by the addition of suitable combinations of these two rules provides a complete contraction- and cut-free system for the *knowability logic* **G3KP**, that is, intuitionistic bimodal logic extended with **KP**.

Intuitionistic solution to Fitch's paradox through an exhaustive proof analysis in **KP**: **OP** is **not** derivable in **G3KP**.

Warning: Be careful with the side condition in the system of rules!

More about the Church-Fitch paradox in the article.

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