Introduction to Hybrid Logic from Semantic Viewpoints

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JAIST Spring School @ Kanazawa, Ishikawa
7th March, 2012

Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder
2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics

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Modal Formalism

- p is true at w:
  - p holds at the world w.
  - p holds at the point of time w.
  - p holds at the coordinate w.
- $\Box p$ is true at w:
  - p is true at all possible worlds relative to w.
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Fathers/Mothers of Hybrid Logics

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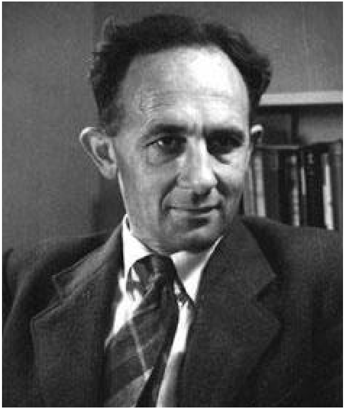

Prior

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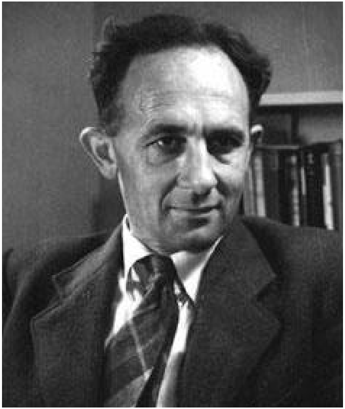
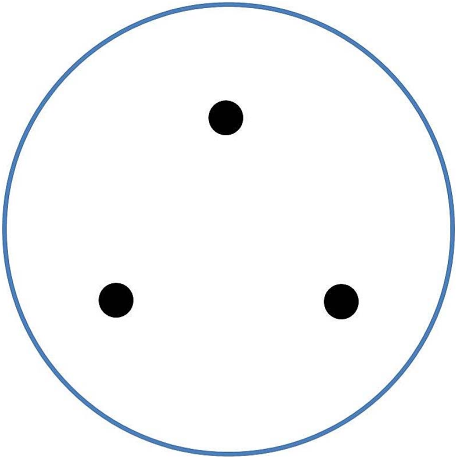


Prior

Gargov-Passy-Tinchev

Naming Points

Nominals and Satisfaction Operators



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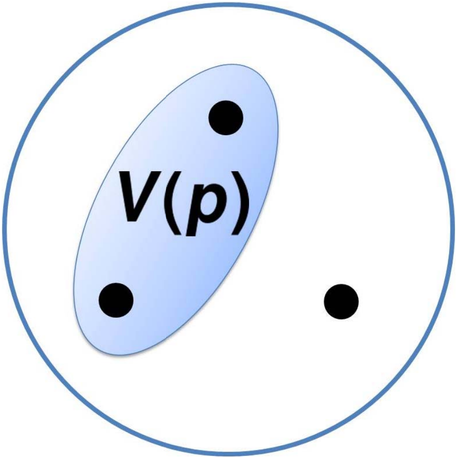

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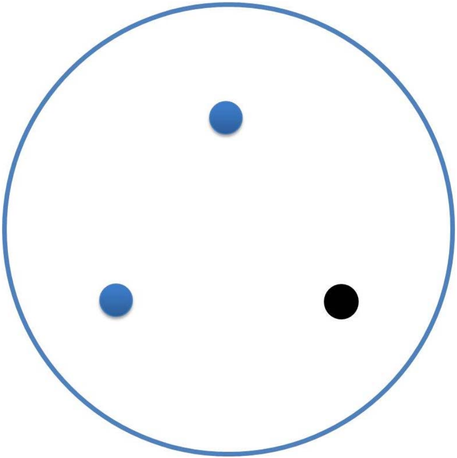

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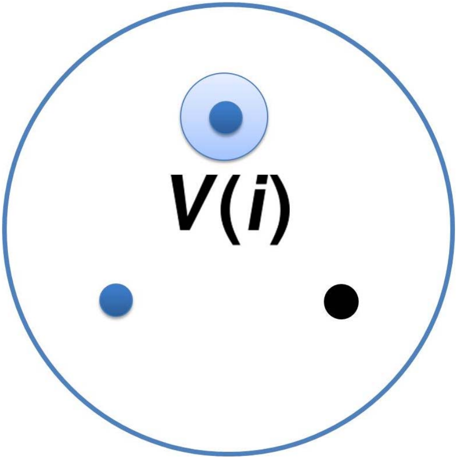

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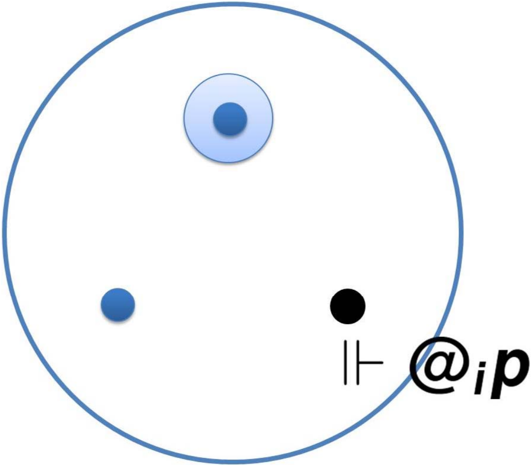

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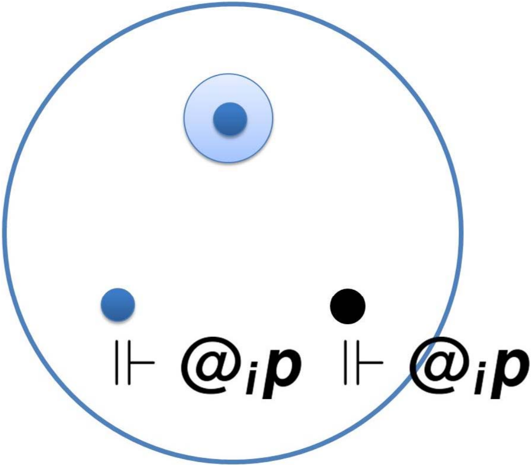

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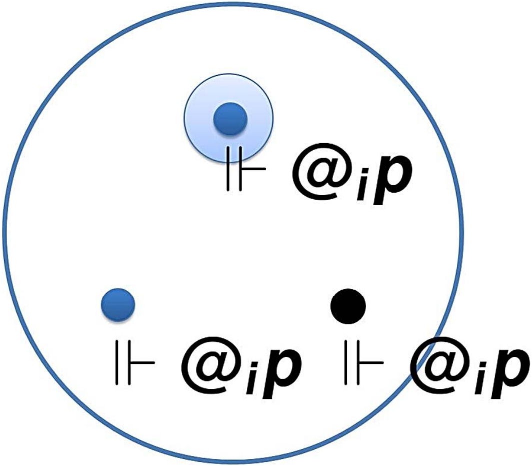

Hybrid Formalism by Examples

- Nominal i is true at w iff i is a name of w.
  - time: $17:00$, $07/03/2012$, 2012, etc.
  - space: Hirokasa Hall, Kanazawa, Japan, etc.
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  - $@_{18:20}$(Mary runs)
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First Merit of Hybrid Logic

- 08/03/2012 is future.
- He/she drinks much on 08/03/2012.
- Thus: He/she will drink much.

- Hybrid Logic enables us to formalize the inference containing both local & global information!

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Within hybrid logic, we can prove the following as a theorem:

$$\langle \text{Future} \rangle i \land @_i p \to \langle \text{Future} \rangle p,$$

where i = '08/03/2012' and p = 'He/she drinks much'.

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Second Merit of HL: Past and Pefect Expressions

- I drink (Present Tense)
  - p
- I drank (Past)
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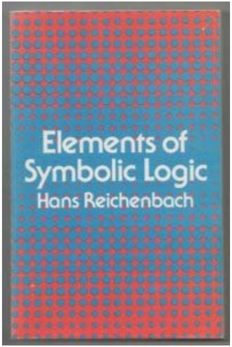

Elements of Symbolic Logic (1947)

Reichenbachian Tense Analysis

Key ingredients are:

- Point of speech (S)
- Point of event (E)
- Point of reference (R)

Reichenbachian Tense Analysis (Cont.)

Expression	Reichenbach	
I drank (Past)	$E = R < S$	
I have drunk (Pefect)	$E < R = S$	
I drink (Present)	$E = R = S$	

Reichenbachian Tense Analysis (Cont.)

Expression	Reichenbach	HL
I drank (Past)	$E = R < S$	$\langle \text{Past} \rangle (p \wedge i)$
I have drunk (Pefect)	$E < R = S$	$i \wedge \langle \text{Past} \rangle p$
I drink (Present)	$E = R = S$	$i \wedge p$

Hybrid Definability over Kripke Frames

Properties	ML	HL
Reflexivity	$\Box p \to p$	
Transitivity	$\Box p \to \Box\Box p$	
Irreflexivity	Undefinable	
Antisymmetry	Undefinable	
$\exists x, y.[xRy \& x \neq y]$	Undefinable	

- Note: $@_i j$ expresses '$i = j$' and $@_i \Diamond j$ expresses 'iRj'.
- φ is pure if φ contains no ordinary proposition variables.

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Transitivity	$\Box p \to \Box\Box p$	$(@_i \Diamond j \wedge @_j \Diamond k) \to @_i \Diamond k$
Irreflexivity	Undefinable	$\neg @_i \Diamond i$
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Pure Completeness wrt Kripke Semantics

Let $\mathbf{K}_{\mathcal{H}}$ be the axiomatization of HL.

Pure Completeness wrt Kripke Frames

For any set Λ of pure formulas, $\mathbf{K}_{\mathcal{H}} + \Lambda$ (as new axioms) is strongly complete wrt the class of frames defined by Λ.

E.g.: φ is valid on any SPOs iff
φ is a theorem of $\mathbf{K}_{\mathcal{H}} + \{ \neg @_i \Diamond i, (@_i \Diamond j \wedge @_j \Diamond k) \to @_i \Diamond k \}$.

Rules with Side-condition in $\mathbf{K}_{\mathcal{H}}$

(Name) $\vdash i \to \varphi \Rightarrow \vdash \varphi$, where i does not occur in φ.

(BG) $\vdash @_i\Diamond j \to @_j\varphi \Rightarrow \vdash @_i\Box\varphi$,
where j does not appear in $@_i\Box\varphi$.

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Recall:

$$\frac{\vdash \varphi \to \psi}{\vdash \exists x.\varphi \to \psi},$$

where x does not occur free in $\exists x.\varphi \to \psi$.

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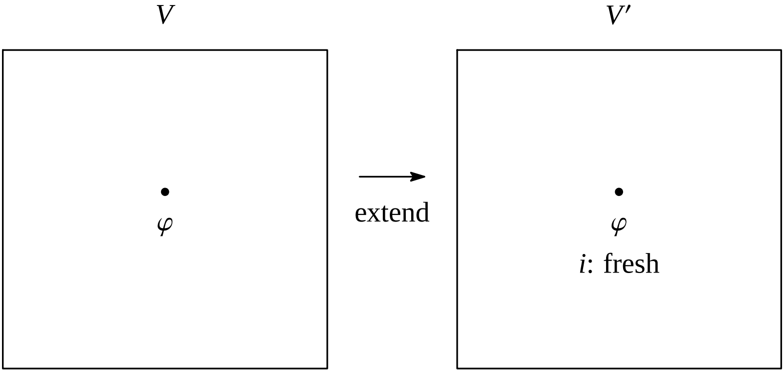
These are TABELEAU RULES!

Intuitive Meaning of (**Name**)

- (**Name**) If φ is consistent, then $i \wedge \varphi$ is consistent for some fresh i.



Intuitive Meaning of (**BG**)

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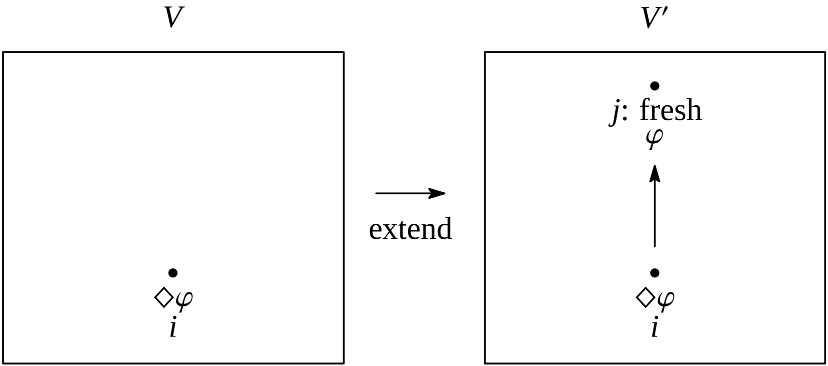

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Until-operator

- $w \models \mathcal{U}(\varphi, \psi)$ iff

  $\exists\, w'.[wRw'$ and $w' \models \varphi$ and

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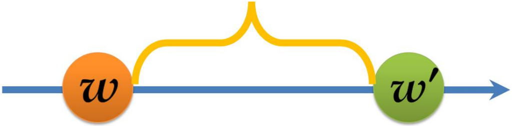

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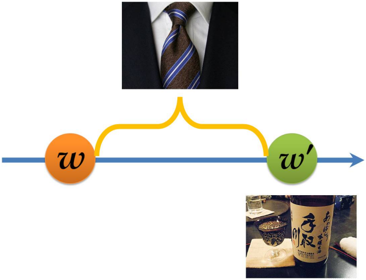
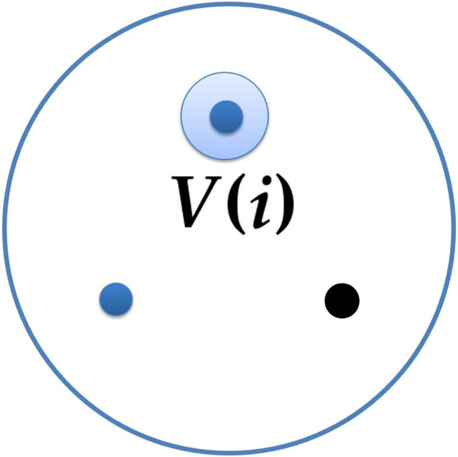


Downarrow Binder



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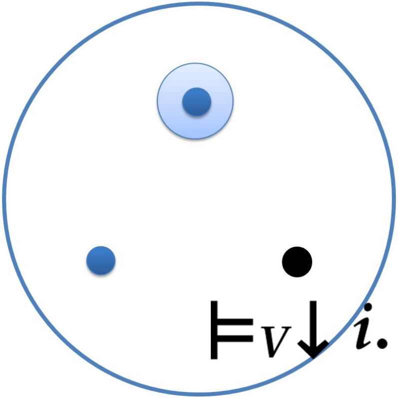
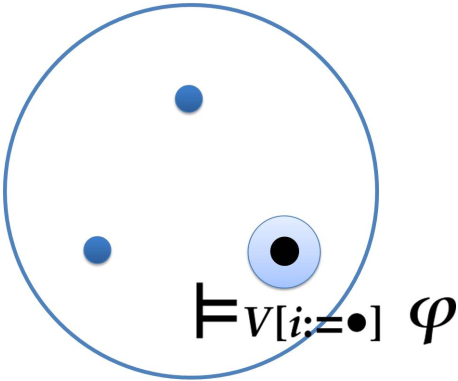


$\models_V \downarrow i.\, \varphi$

Downarrow Binder



Until in terms of ↓

$$w \models \downarrow i.(-)$$

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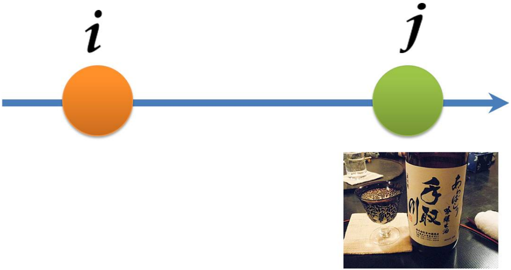

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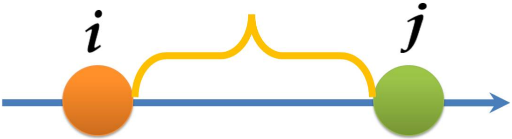

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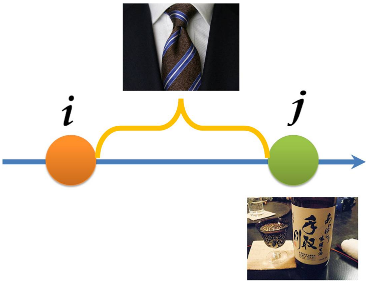
$$w \models \downarrow i.\langle \text{Future} \rangle(\downarrow j.(\varphi \wedge @_i(\langle \text{Future} \rangle j \to \psi)))$$



Pure Completeness with Downarrow

- We can capture the behavior of $\downarrow$ by the axiom:
$$@_i(\downarrow j.\varphi \leftrightarrow \varphi[i/j]).$$
- Pure Completeness holds.

Comments on Decidability

- Adding i and $@_i$ to Basic ML preserves decidability.
- Satisfiablity problem of Basic HL on the class of all frames is still PSPACE-complete (Areces, Blackburn & Marx 1999).
- HL with $\downarrow$ '=' the generated-submodel-invariant fragment of FOL.
- Adding $\downarrow$ to Basic HL gives rise to undecidability result (Areces, Blackburn & Marx 1999).

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Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder
2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics

A Need for Two Dimensional Hybrid Logics

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Formalism for Hybrid Product

- We need two kinds of Boxes:
  - $\Box_1$ (e.g. for time)
  - $\Box_2$ (e.g. for space)
- We also need two kinds of nominals:
  - t-nominals: $i, j, k, \ldots$
  - s-nominals: $a, b, c, \ldots$
- Each kind of nominals has its satisfaction operators:
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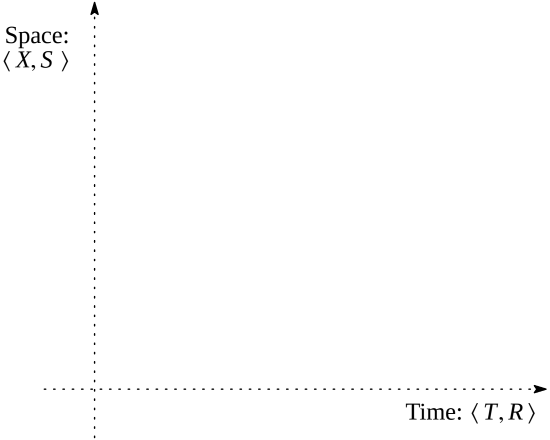
An Example: Product of Kripke Frames



Product of Kripke Frames

$\mathfrak{F} = \langle T, R \rangle$, $\mathfrak{G} = \langle X, S \rangle$: Kripke frames. Then, we define the product of Kripke frames $\mathfrak{F} \times \mathfrak{G} = \langle T \times X, R_h, R_v \rangle$ by:

- $\langle t, x \rangle R_h \langle t', x' \rangle$ iff tRt' and $x = x'$.
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What is a valuation V on $\mathfrak{F} \times \mathfrak{G}$?

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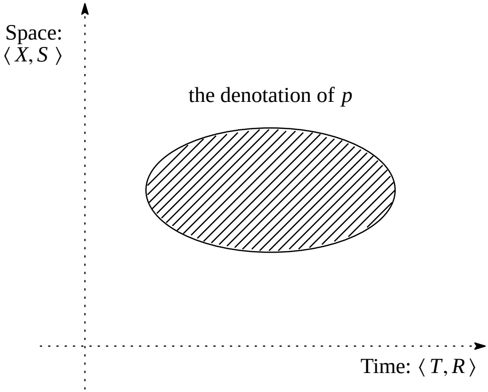
A Valuation of Product of Kripke Frames



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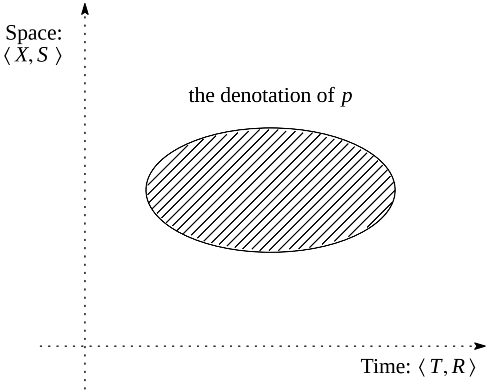

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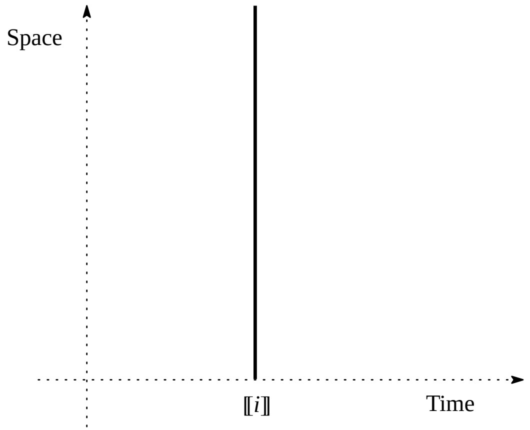
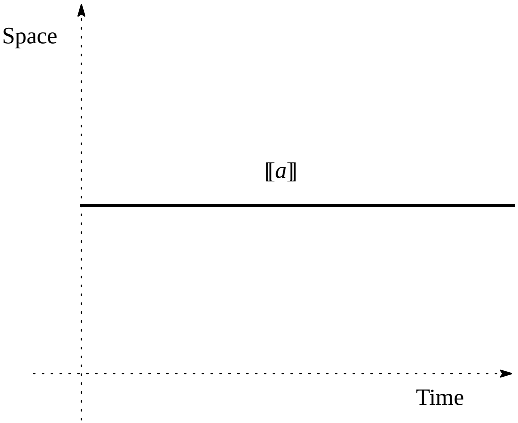
Naming Lines

Semantic Idea behind Two Nominals



To *t*-nominal *i*, we assign a vertical line $\{ i^V \} \times X$.

Semantic Idea behind Two Nominals (Cont.)



To s-nominal a, we assign a horizontal line $T \times \{ a^V \}$.

Truth Condition for Satisfaction Operators

- $@_i @_a p$ is true at $\langle x, y \rangle$
- iff $@_a p$ is true at $\langle i^V, y \rangle$
- iff p is true at $\langle i^V, a^V \rangle$

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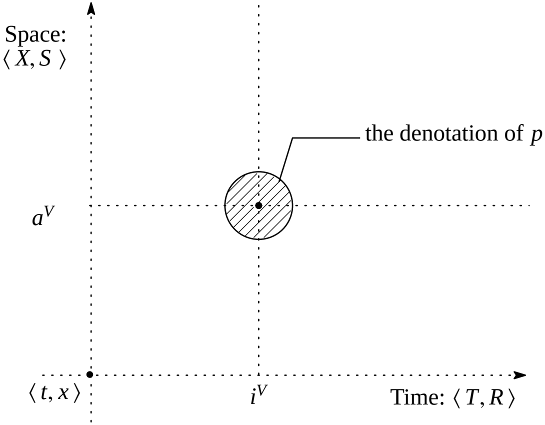
Semantic Understanding of Our Example (1)

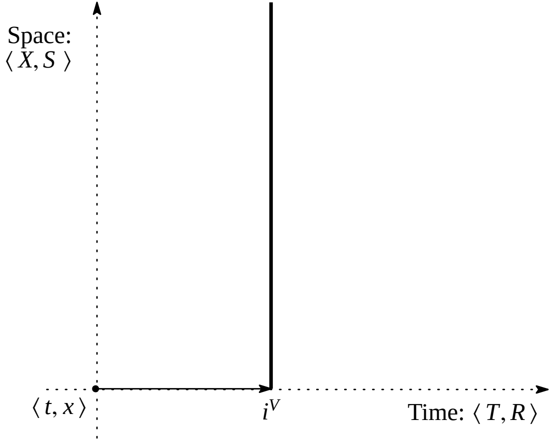
$@_i @_a p \wedge \langle \text{Future} \rangle i \wedge \langle \text{Upstairs} \rangle a \rightarrow \langle \text{Future} \rangle \langle \text{Upstairs} \rangle p$.

$@_i @_a p$ is true at $\langle t, x \rangle$ iff:



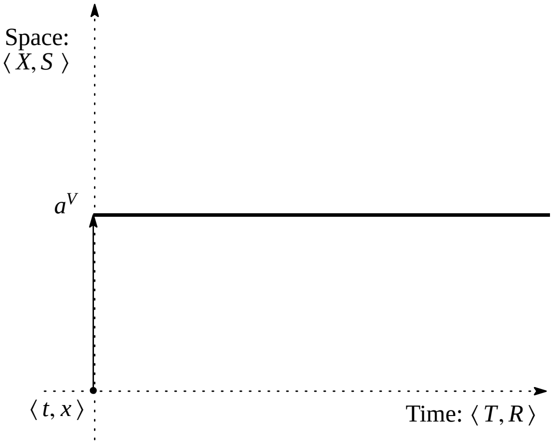
Semantic Understanding of Our Example (2)

$\langle \text{Future} \rangle i$ is true at $\langle t, x \rangle$ iff:



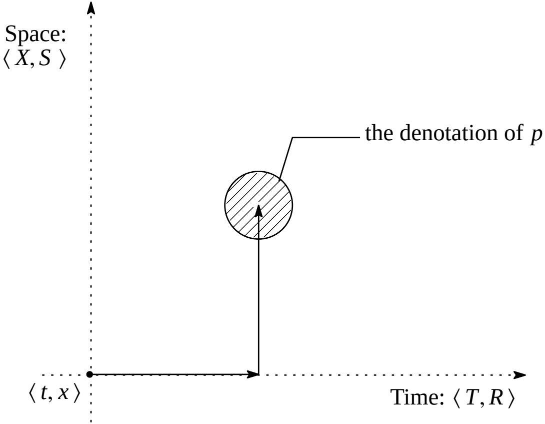
Semantic Understanding of Our Example (3)

$\langle$ Upstairs $\rangle a$ is true at $\langle t, x \rangle$ iff:



Semantic Understanding of Our Example (4)

Thus, $\langle$ Future $\rangle\langle$ Upstairs $\rangle p$ is true at $\langle\, t, x\,\rangle$.



Hilbert-style Axiomatization of Hybrid Products

Roughly, we need the two kinds of axioms and rules: $\mathbf{K}_{\mathcal{H}}$ for $\Box_1$ and $@_i$ & $\mathbf{K}_{\mathcal{H}}$ for $\Box_2$ and $@_a$.
Furthermore, we also need the five 'interaction' axioms:

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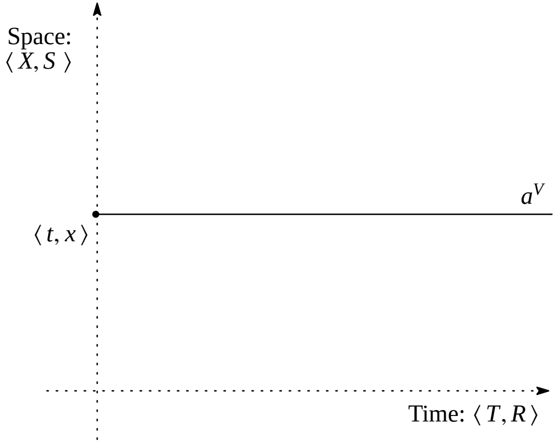
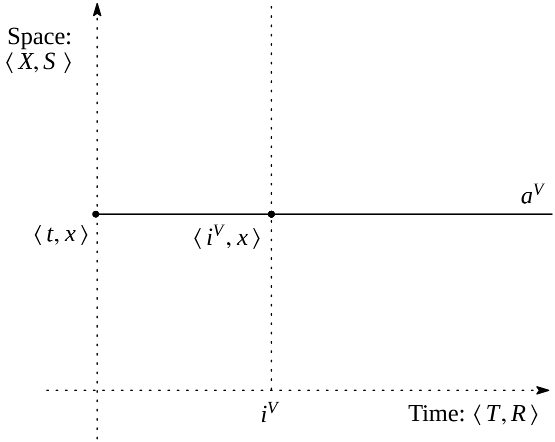
Interaction Axioms for Two kinds of Nominals

We explain $a \to @_i a$ alone. Assume that a is true at $\langle t, x \rangle$.



Interaction Axioms for Two kinds of Nominals (Cont.)

Then, a is true also at $\langle i^V, x \rangle$, i.e. $@_i a$ is true at $\langle t, x \rangle$.



Main Result: Pure Completeness of Hybrid Products

Let $[\mathbf{K}_{\mathcal{H}}, \mathbf{K}_{\mathcal{H}}]$ be our axiomatization of hybrid products.

Pure Completeness wrt Product Frames (S.2010)

For any set Λ of pure formulas, $[\mathbf{K}_{\mathcal{H}}, \mathbf{K}_{\mathcal{H}}] + \Lambda$ (as new axioms) is strongly complete wrt the class of product Kripke frames defined by Λ.

K. Sano
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Heart of Our Proof

Henkin

Product of Modal Logics

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- (com) $\Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p$
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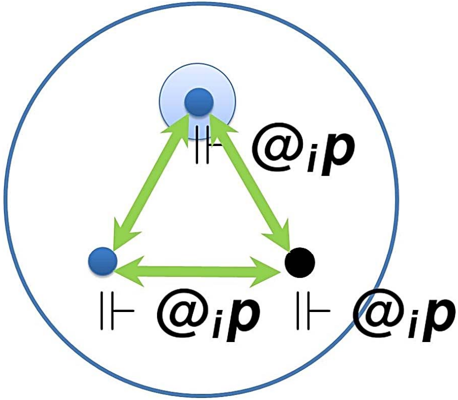
Note that our axiomatization does not contain these guys!

But, our completeness results assure us that these are theorems of $[\mathbf{K}_{\mathcal{H}}, \mathbf{K}_{\mathcal{H}}]$.

Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder
2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics

Nominals and Satisfaction Operators



Structures (relational, topological) on the domain are irrelevant to a hybridization.

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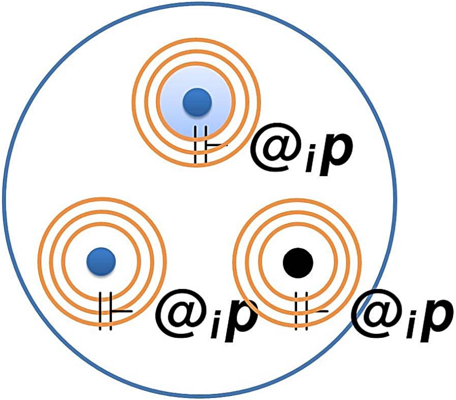

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Overview of Hybrid Product Methods

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Time \ Space	Rel. $\langle X, S \rangle$	Top. $\langle X, \sigma \rangle$
Rel. $\langle T, R \rangle$	(Sano 2010)	??
Top. $\langle T, \tau \rangle$	??	??

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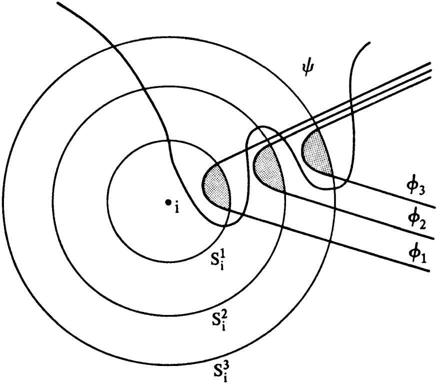

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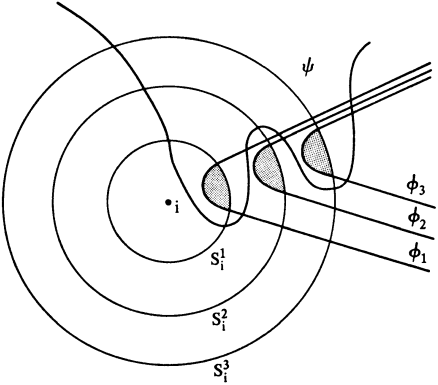

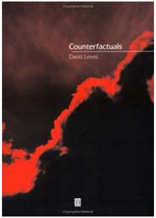

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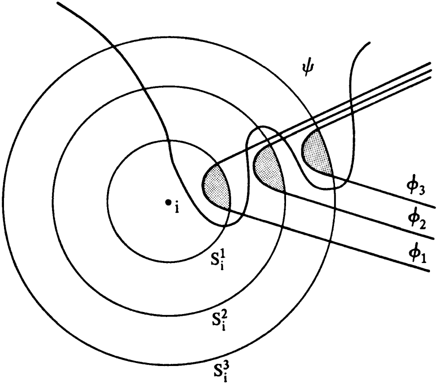
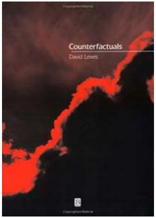


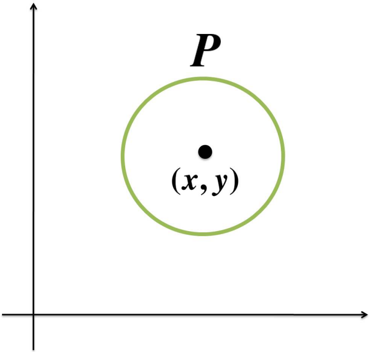


$$\tau(w) \subseteq \mathcal{P}(W)$$

Product of Topologies (Van Benthem, et. al. 2006, S.2011)

- Let $\langle W_l, \tau_l \rangle$ $(l = 1, 2)$ be a topological space.
- Define the horizontal topological space τ_h by:

$$P \in \tau_h(x, y) \quad \text{iff} \quad ???$$



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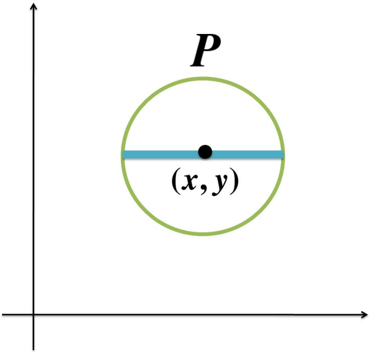

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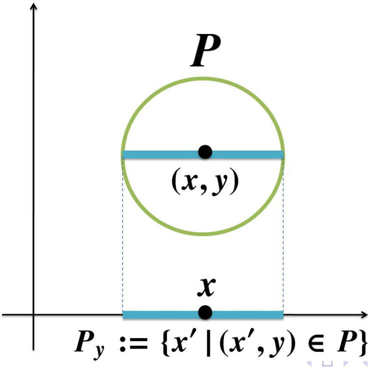

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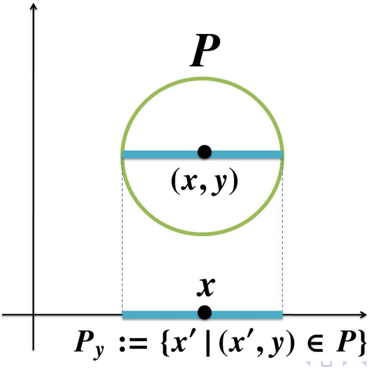

Semantic Clauses on Product of Topologies

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where $[\![\varphi]\!] := \{\, (x, y) \mid (x, y) \models \varphi \,\}$

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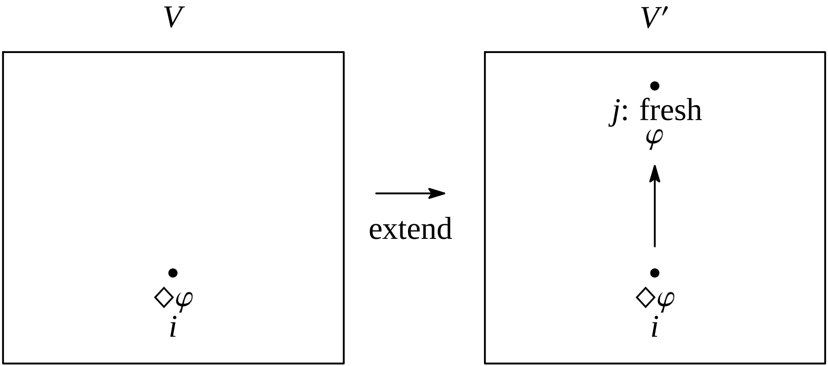
- Our five interaction axioms are all valid on any product of topologies.
- There is a problem on **BG**.

Intuitive Meaning of **BG** on Kripke Frame

- **BG**: If $@_i\Diamond\varphi$ is consistent, then $@_i\Diamond j \wedge @_j\varphi$ is consistent for some fresh j.



‘BG = Kripke Frames’ in Topological Setting

- **S4**-frame = $\langle W, \tau \rangle$ s.t. $\tau(w)$ has the smallest element.
- On $\mathbb{R}^2$, we can consider the smaller and smaller neighbhorhood around w.

Ten Cate & Litak (2007)

(**BG**) characterizes the notion of **S4**-frames (Alexandrov spaces) within the class of topological spaces.

So, we should drop two kinds of **BG** from our axiomatization.

Pure completeness wrt Products of Topologies

Let $\mathbf{S4}^{-}_{\mathcal{H}} = (\mathbf{K}_{\mathcal{H}} - \mathbf{BG}) + \{\Box p \to p, \Box p \to \Box\Box p\}$.

Pure Completeness for Product of Topologies

For any set Λ of pure formulas, $[\mathbf{S4}^{-}_{\mathcal{H}}, \mathbf{S4}^{-}_{\mathcal{H}}] + \Lambda$ is strongly complete wrt the class of product of topologies defined by Λ.

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If we put $\Lambda = \emptyset$, we obtain:

Cor.

$[\mathbf{S4}^-_{\mathcal{H}}, \mathbf{S4}^-_{\mathcal{H}}]$ is strongly complete wrt the class of all products of topologies.

Topological Definability in Hybrid Logic

Here 'definability' means definability by a single formula.

Properties	ML	HL
T_0	Undef.	$@_i\Diamond j \wedge @_j\Diamond i \to @_i j$
T_1	Undef.	$\Diamond i \to i$
T_2	Undef.	Undef. by Sustretov (2005)
density-in-itself	Undef.	$\neg\Box i$
compactness	Undef.	Undef.
discreteness	$\Diamond p \to p$	$\Diamond p \to p$

- Undef. in ML is due to the result of McKinsey-Tarski.
- T_1 says that $\{ x \}$ (e.g. $[x, x]$) is closed.
- Density-in-itself says $\{ x \} \notin \tau(x)$.

Overview of Hybrid Product Methods

Time \ Space	Rel. $\langle X, S \rangle$	Top. $\langle X, \sigma \rangle$
Rel. $\langle T, R \rangle$	(Sano 2010)	Delete **BG** for $\Box_2$
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3. $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$: Space str. is independent of Time
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How to Capture Dependence of Space on Time (Rel)

- The accessible space-area may vary with the time: $y S(t) y'$ rather than $y S y'$.

- $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$ is invalid, $@_i \Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$ is valid.
- A pure completeness result still holds.

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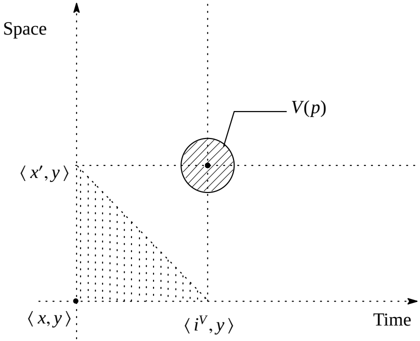

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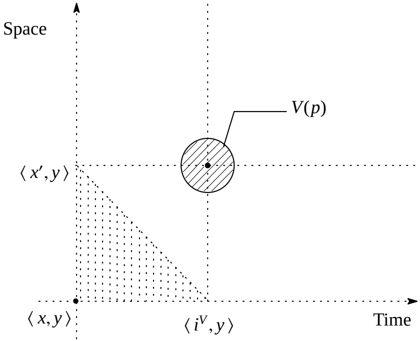

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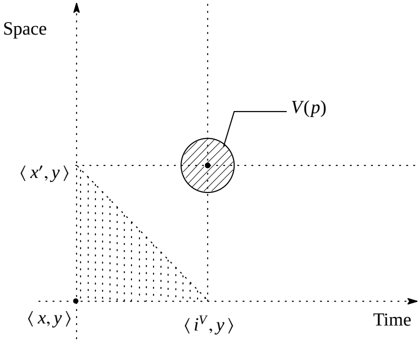

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How to Capture Dependence of Space on Time (Top)

- Spatial topology might depend on Time.
- Then, our spatial topology and semantics should be defined as:

$$(\sigma_t : X \to \mathcal{PP}(X))_{t \in T}.$$

$$\langle t, x \rangle \models \Box_2 \varphi \text{ iff } [\![\varphi]\!]_t \in \sigma_t(x).$$

- Syntactically, this change corresponds to:
  - $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$: Invalid
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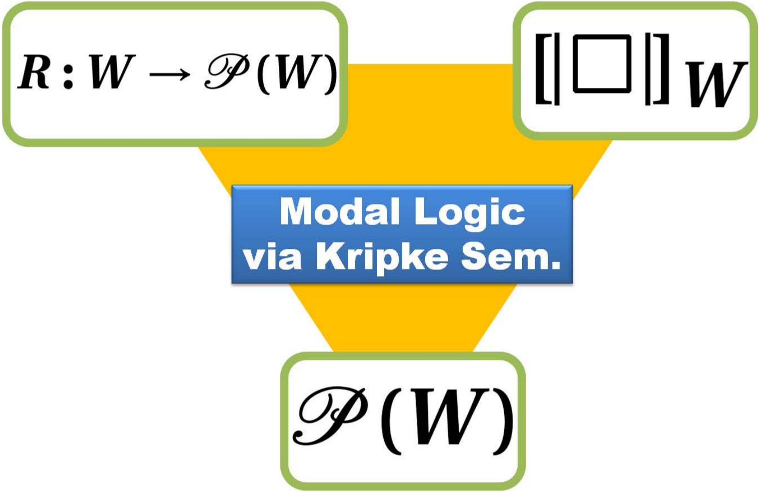
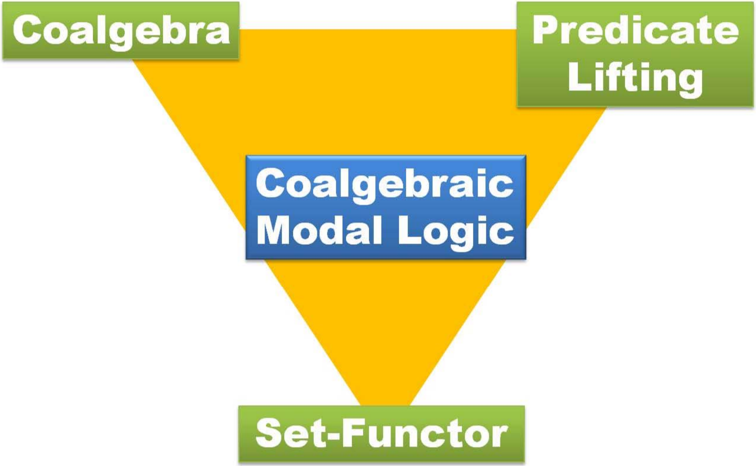
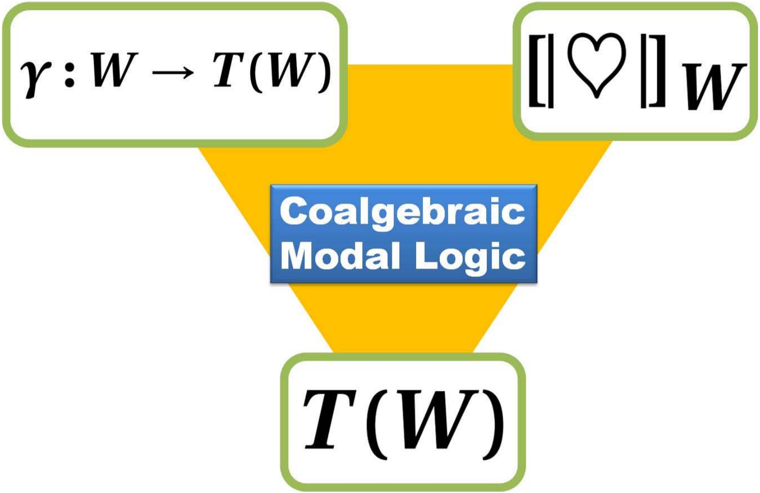
Coalgebraic View on Kripke Semantics for ML

$$\begin{aligned} w \models \Box p \quad & \text{iff} \quad R(w) \subseteq [\![p]\!], \\ & \text{iff} \quad R(w) \in \{ X \subseteq W \mid X \subseteq [\![p]\!] \}, \\ & \text{iff} \quad R(w) \in [\![\Box]\!]_W([\![p]\!]), \end{aligned}$$

where $[\![\Box]\!]_W(A) := \{ X \subseteq W \mid X \subseteq A \}$.


$R : W \to \mathscr{P}(W)$
$[\![\Box]\!]_W$
Modal Logic
via Kripke Sem.
$\mathscr{P}(W)$


Coalgebra
Predicate Lifting
Coalgebraic Modal Logic
Set-Functor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Coalgebraic Modal Logic
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Recall: Semantic Clauses on Product of Nbhd Frames

$$
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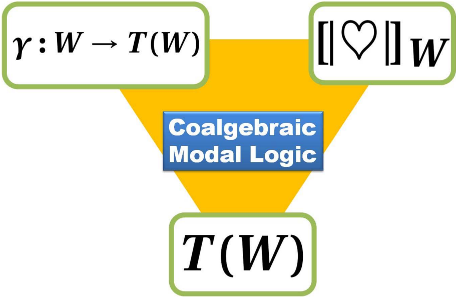
An essence for the semantic clause

You can use the original transition map.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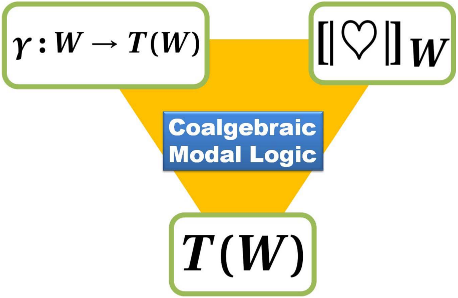
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Coalgebraic Semantics on Product of Coalgebras

- $\langle W_1, \gamma \rangle$: a T_1-coalgebra & $\langle W_2, \delta \rangle$: a T_2-coalgebra.
- We should define:

$$(x, y) \models \heartsuit_1 \varphi \text{ iff } \gamma_h(x, y) \in [\![\heartsuit_1]\!]_{W_1 \times W_2}([\![\varphi]\!]),$$

  where $[\![\varphi]\!] = \{ (x', y') \mid (x', y') \models \varphi \} \subseteq W_1 \times W_2$.
- It suffices to have:

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$$(x, y) \models \heartsuit_1 \varphi \text{ iff } \gamma_h(x, y) \in [\![\heartsuit_1]\!]_{W_1 \times W_2}([\![\varphi]\!]),$$

  where $[\![\varphi]\!] = \{ (x', y') \mid (x', y') \models \varphi \} \subseteq W_1 \times W_2$.
- It suffices to have:

$$(x, y) \models \heartsuit_1 \varphi \text{ iff } \gamma(x) \in [\![\heartsuit_1]\!]_{W_1}([\![\varphi]\!]_y).$$

- We can also define γ_h (Tensorial Strength (Kock 1972), suggested by Dirk Pattinson and Fredrik Dahlqvist).

Semantic Summary

- $\langle W_1, \gamma \rangle$: a T_1-coalgebra & $\langle W_2, \delta \rangle$: a T_2-coalgebra.
- Given any valuation:

$$
\begin{array}{lll}
(x, y) \models \heartsuit_1 \varphi & \text{iff} & \gamma(x) \in [\![\heartsuit_1]\!]_{W_1}([\![\varphi]\!]_y) \\
(x, y) \models \heartsuit_2 \varphi & \text{iff} & \delta(y) \in [\![\heartsuit_2]\!]_{W_2}([\![\varphi]\!]_x)
\end{array}
$$

Five Interaction Axioms

1. $@_a @_i p \leftrightarrow @_i @_a p$
2. $\heartsuit_1 @_a p \leftrightarrow @_a \heartsuit_1 p$
3. $\heartsuit_2 @_i p \leftrightarrow @_i \heartsuit_2 p$
4. $@_i a \leftrightarrow a$
5. $@_a i \leftrightarrow i$

are valid on all products of T_1-coalgebra and T_2-coalgebra.

Validity of Five Interaction Axioms

Let us focus on $\heartsuit_1 @_a p \leftrightarrow @_a \heartsuit_1 p$.

- $(x, y) \models \heartsuit_1 @_a p$
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  - $\because [\![p]\!]_{a^V} = [\![@_a p]\!]_y$.
  - $\because x \in [\![p]\!]_{a^V}$ iff $x \in [\![@_a p]\!]_y$.
  - $\because (x, a^V) \models p$ iff $(x, y) \models @_a p$.
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If each R_i is strongly one-step complete, the corresponding product of hybrid logics is strongly complete. (S.2011)

Schröder, L. and Pattinson, D.
'Named models in Coalgebraic Hybrid Logic',
Proceedings of STACS 2010, 2010, pp.645-656.

Summary

- I have shown you how to combine two hybrid logics. A key idea is: Naming Lines.
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- My method is also robust for completeness results. ANY way of 'combining' hybrid logics always enjoys a general completeness result (pure completeness). It can be generalize to a coalgebraic level (S.2011).

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Further Directions

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- Construct Gentzen-style sequent calculus and establish cut elimination theorem possibly extended with geometric theory.
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Take-home Message

Naming Lines provides a modular and robust way of combining two hybrid logics

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Thank You

Hilbert-style Axiomatization of HL

In addition to (**BG**) + (**Name**), **K** plus

(K@) $@_i(p \to q) \to (@_i p \to @_i q)$.

(Self-Dual) $\neg @_i p \leftrightarrow @_i \neg p$.

(Ref) $@_i i$.

(Intro) $i \wedge p \to @_i p$.

(Back) $@_i p \to \Box @_i p$.

(Agree) $@_i @_j p \to @_j p$.

(Nec@) From φ, we may infer $@_i \varphi$.

(Hsub) From φ, we may infer $\sigma(\varphi)$, where σ is a *H*-uniform substitution, eg.:

$$\frac{(i \wedge p) \to (q \to p)}{(j \wedge \varphi) \to (\psi \to \varphi))}$$

Tensorial Strength on **Sets**

- Define $\iota_y : W_1 \to W_1 \times W_2$ by $\iota_y(x) := (x, y)$ $(y \in W_2)$.
- Tensorial strength $\mathrm{st}_{W_1,W_2} \colon T_1(W_1) \times W_2 \to T_1(W_1 \times W_2)$ is defined by

$$\mathrm{st}_{W_1,W_2}(t, y) := T_1(\iota_y)(t),$$

  where $T_1(\iota_y) \colon T_1(W_1) \to T_1(W_1 \times W_2)$.
- Define $\gamma_h :=$

$$W_1 \times W_2 \xrightarrow{\gamma \times \mathrm{id}} T_1(W_1) \times W_2 \xrightarrow{\mathrm{st}_{W_1,W_2}} T_1(W_1 \times W_2)$$

- This gives us the same definitions as R_h and τ_h.