Introduction to Hybrid Logic from Semantic Viewpoints

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JAIST Spring School @ Kanazawa, Ishikawa
7th March, 2012
Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder

2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics
Outline

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Modal Formalism

- $p$ is true at $w$:
  - $p$ holds at the world $w$.
  - $p$ holds at the point of time $w$.
  - $p$ holds at the coordinate $w$.

- $\Box p$ is true at $w$:
  - $p$ is true at all possible worlds relative to $w$.
  - $p$ is true at all points of time later than $w$.
  - $p$ is true at all coordinates within 2km from $w$. 
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Fathers/Mothers of Hybrid Logics
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<table>
<thead>
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<th>Prior</th>
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*Introduction to Hybrid Logic*
What is Hybrid Logic?
How Can We Combine Hybrid Logics?

Fathers/Mothers of Hybrid Logics

Prior

Gargov-Passy-Tinchev
Naming Points
Nominals and Satisfaction Operators
Nominals and Satisfaction Operators
Nominals and Satisfaction Operators
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\[ V(i) \]
Nominals and Satisfaction Operators
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\[ \models @i\neg \]

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Hybrid Formalism by Examples

- **Nominal** $i$ is true at $w$ iff $i$ is a name of $w$.
- @/ is true at $w$ iff $p$ is true at the world named by $i$.
  - @/Mary runs
  - @/London (Games of the XXX Olympiad are held)
Hybrid Formalism by Examples

- **Nominal** \( i \) is true at \( w \) iff \( i \) is a name of \( w \).
- time: 17:00, 07/03/2012, 2012, etc.
- space: Hirokasa Hall, Kanazawa, Japan, etc.
- \( @_i p \) is true at \( w \) iff \( p \) is true at the world named by \( i \).
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First Merit of Hybrid Logic

- 08/03/2012 is future.
- He/she drinks much on 08/03/2012.
- Thus: He/she will drink much.

Hybrid Logic enables us to formalize the inference containing both local & global information!
First Merit of Hybrid Logic

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Within hybrid logic, we can prove the following as a theorem:

\[ \langle \text{Future} \rangle i \land @i p \rightarrow \langle \text{Future} \rangle p, \]

where \( i = '08/03/2012' \) and \( p = '\text{He/she drinks much}' \).

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where $i = '08/03/2012'$ and $p = 'He/she drinks much'$.

- Hybrid Logic enables us to formalize the inference containing both local & global information!
Second Merit of HL: Past and Perfect Expressions

- I drink (Present Tense)
  - \( p \)

- I drank (Past)
  - \( \llangle \text{Past} \rrangle p \)

- I had drunk (Pluperfect)
  - \( \llangle \text{Past} \rrangle \llangle \text{Past} \rrangle p \)

- I have drunk (Perfect)
  - \( \llangle \text{Past} \rrangle p?q \) or \( p?q \)
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- I have drunk (Perfect)
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*Elements of Symbolic Logic* (1947)

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Elements of Symbolic Logic
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Reichenbachian Tense Analysis

Key ingredients are:

- Point of speech \((S)\)
- Point of event \((E)\)
- Point of reference \((R)\)
Reichenbachian Tense Analysis (Cont.)

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### Reichenbachian Tense Analysis (Cont.)

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<td>$\langle \text{Past} \rangle(p \land i)$</td>
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<td>$i \land \langle \text{Past} \rangle p$</td>
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Hybrid Definability over Kripke Frames

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Note: \(\downarrow_{ij}\) expresses ‘\(i = j\)’ and \(\downarrow_i \diamond j\) expresses ‘\(iRj\)’.

\(\varphi\) is pure if \(\varphi\) contains no ordinary proposition variables.
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- Note: $@_i j$ expresses ‘$i = j$’ and $@_i \Diamond j$ expresses ‘$iRj$’.
- $\varphi$ is pure if $\varphi$ contains no ordinary proposition variables.
Let $\mathbf{K}_H$ be the axiomatization of HL.

**Pure Completeness wrt Kripke Frames**

For any set $\Lambda$ of pure formulas, $\mathbf{K}_H + \Lambda$ (as new axioms) is strongly complete wrt the class of frames defined by $\Lambda$.

E.g.: $\varphi$ is valid on any SPOs iff

$\varphi$ is a theorem of $\mathbf{K}_H + \{-@_i \Diamond i, (@_i \Diamond j \land @_j \Diamond k) \rightarrow @_i \Diamond k\}$. 

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Rules with Side-condition in $\mathbf{K}_{\mathcal{H}}$

(\textbf{Name}) $\vdash i \rightarrow \varphi \Rightarrow \vdash \varphi$, where $i$ does not occur in $\varphi$.

(\textbf{BG}) $\vdash @i \Diamond j \rightarrow @j \varphi \Rightarrow \vdash @i \Box \varphi$, where $j$ does not appear in $@i \Box \varphi$. 
Rules with Side-condition in $\mathbf{K}_H$

\[(\text{Name}) \quad \vdash i \to \varphi \Rightarrow \vdash \varphi, \text{ where } i \text{ does not occur in } \varphi.\]

\[(\text{BG}) \quad \vdash @_i \Diamond j \to @_j \varphi \Rightarrow \vdash @_i \Box \varphi, \text{ where } j \text{ does not appear in } @_i \Box \varphi.\]

Recall:

\[
\frac{\vdash \varphi \to \psi}{\vdash \exists x. \varphi \to \psi},
\]

where $x$ does not occur free in $\exists x. \varphi \to \psi$. 
Rules with Side-condition in $K_H$

(Nam) $\vdash i \rightarrow \varphi \Rightarrow \vdash \varphi$, where $i$ does not occur in $\varphi$.

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where $j$ does not appear in $@i \Box \varphi$.

Recall:

$$\frac{\vdash \varphi \rightarrow \psi}{\vdash \exists x. \varphi \rightarrow \psi},$$

where $x$ does not occur free in $\exists x. \varphi \rightarrow \psi$.

These are TABELEAU RULES!
(Name) If \( \varphi \) is consistent, then \( i \land \varphi \) is consistent for some fresh \( i \).

\[
\begin{align*}
V & \quad \text{extend} \quad V' \\
\bullet & \quad \varphi & \quad \bullet & \quad \varphi \\
& \quad i: \text{fresh}
\end{align*}
\]
Intuitive Meaning of \((BG)\)

\((BG)\) If \(\Diamond_i \varphi\) is consistent, then \(\Diamond_i j \land \Diamond_j \varphi\) is consistent for some fresh \(j\).

\[ V \quad \xrightarrow{\text{extend}} \quad V' \]

- \(i\): fresh
- \(\varphi\)

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Until-operator

- \( w |\models \mathcal{U}(\varphi, \psi) \) iff

  \[ \exists w'. [wRw' \text{ and } w' |\models \varphi \text{ and } \forall w''. ((wRw'' \text{ and } w'' Rw') \text{ implies } w'' |\models \varphi)] \]
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Downarrow Binder

$V(i)$
Downarrow Binder
Downarrow Binder

\[ \models_{V[i:=\bullet]} \varphi \]
Until in terms of $\downarrow$

$w \models \downarrow i.(-)$
Until in terms of $\downarrow$

$w \models \downarrow i.⟨\text{Future}⟩(−)$

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Until in terms of $\downarrow$

$$w \models \downarrow i.\langle \text{Future} \rangle(\downarrow j.(\varphi \land \neg))$$
Until in terms of ↓

\[ w \models \downarrow i.\langle \text{Future} \rangle (\downarrow j.(\varphi \land @ i\leftarrow)) \]
Until in terms of $\downarrow$

$w \models \downarrow i.⟨\text{Future}⟩(\downarrow j.(\varphi \land @i(⟨\text{Future}⟩j \rightarrow \psi)))$
Pure Completeness with Downarrow

- We can capture the behavior of \( \downarrow \) by the axiom:

\[
\@_i(\downarrow j. \varphi \leftrightarrow \varphi[i/j]).
\]

- Pure Completeness holds.
Comments on Decidability

- Adding $i$ and $\&_i$ to Basic ML preserves decidability.
- Satisfiability problem of Basic HL on the class of all frames is still PSPACE-complete (Areces, Blackburn & Marx 1999).
- HL with $\downarrow '=$' the generated-submodel-invariant fragment of FOL.
- Adding $\downarrow$ to Basic HL gives rise to undecidability result (Areces, Blackburn & Marx 1999).
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- Adding $i$ and $@i$ to Basic ML preserves decidability.
- Satisfiability problem of Basic HL on the class of all frames is still PSPACE-complete (Areces, Blackburn & Marx 1999).
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Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder

2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics
A Need for Two Dimensional Hybrid Logics

Suppose that you are in a downstairs room of H-Hall on 4th March. Let us consider the following scenario:

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"You received a reminder from your Google calendar: JAIST-SS starts on 5th March at Hirosaka Hall. 5th March is still future. H-Hall is an upstairs room of this place. So, JAIST-SS will start in the room overhead."

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How can we formalize this inference?

$\Diamond_i \Diamond_a p \land \langle \text{Future} \rangle i \land \langle \text{Upstairs} \rangle a \rightarrow \langle \text{Future} \rangle \langle \text{Upstairs} \rangle p,$

where $i = '05/03.'$, $a = 'H-Hall'$ & $p = 'JAIST-SS starts'$. 
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Formalism for Hybrid Product

- We need two kinds of Boxes:
  - $\Box_1$ (e.g. for time)
  - $\Box_2$ (e.g. for space)

- We also need two kinds of nominals:
  - $t$-nominals: $i, j, k, \ldots$
  - $s$-nominals: $a, b, c, \ldots$

- Each kind of nominals has its satisfaction operators:
  - $\@_i, \@_j, \@_k, \ldots$
  - $\@_a, \@_b, \@_c, \ldots$

- But, we have only one kind of proposition letters.
Formalism for Hybrid Product

- We need two kinds of Boxes:
  - $\square_1$ (e.g. for time)
  - $\square_2$ (e.g. for space)

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Formalism for Hybrid Product

- We need two kinds of Boxes:
  - □₁ (e.g. for time)
  - □₂ (e.g. for space)

- We also need two kinds of nominals:
  - t-nominals: i, j, k, ...
  - s-nominals: a, b, c, ...

- Each kind of nominals has its satisfaction operators:
  - @₁, @₂, @₃, ...
  - @₄, @₅, @₆, ...

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  - \(\varrho_a, \varrho_b, \varrho_c, \ldots\)

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An Example: Product of Kripke Frames

Time: \( \langle T, R \rangle \)

Space: \( \langle X, S \rangle \)
Product of Kripke Frames

\( \mathcal{F} = \langle T, R \rangle \), \( \mathcal{G} = \langle X, S \rangle \): Kriphe frames. Then, we define the product of Kripke frames \( \mathcal{F} \times \mathcal{G} = \langle T \times X, R_h, R_v \rangle \) by:

- \( \langle t, x \rangle R_h \langle t', x' \rangle \) iff \( tRt' \) and \( x = x' \).
- \( \langle t, x \rangle R_v \langle t', x' \rangle \) iff \( t = t' \) and \( xSx' \).

What is a valuation \( V \) on \( \mathcal{F} \times \mathcal{G} \)?
Product of Kripke Frames

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A Valuation of Product of Kripke Frames

For any $p$, it suffices to define $V(p) \subseteq T \times X$, i.e. a subset of ‘2D-plane’. How about nominals?
A Valuation of Product of Kripke Frames

For any $p$, it suffices to define $V(p) \subseteq T \times X$, i.e. a subset of ‘2D-plane’. How about nominals?
Naming Lines
Semantic Idea behind Two Nominals

To \( t \)-nominal \( i \), we assign a vertical line \( \{ i^V \} \times X \).
To s-nominal $a$, we assign a horizontal line $T \times \{ a^V \}$. 

Semantic Idea behind Two Nominals (Cont.)
Truth Condition for Satisfaction Operators

- $@_i@_ap$ is true at $⟨x, y⟩$
- iff $@_ap$ is true at $⟨i^v, y⟩$
- iff $p$ is true at $⟨i^v, a^v⟩$
Truth Condition for Satisfaction Operators

- $\langle i, a \rangle p$ is true at $\langle x, y \rangle$
- iff $\langle i, a \rangle p$ is true at $\langle i^V, y \rangle$
- iff $p$ is true at $\langle i^V, a^V \rangle$
Truth Condition for Satisfaction Operators

- $\@ i \@ a p$ is true at $\langle x, y \rangle$
- iff $\@ a p$ is true at $\langle i^V, y \rangle$
- iff $p$ is true at $\langle i^V, a^V \rangle$
Semantic Understanding of Our Example (1)

@_i@_a p ∧ ⟨Future⟩i ∧ ⟨Upstairs⟩a → ⟨Future⟩⟨Upstairs⟩p.

@_i@_a p is true at ⟨t, x⟩ iff:

- Space: ⟨X, S⟩
- Time: ⟨T, R⟩

The denotation of p
Semantic Understanding of Our Example (2)

\( \langle \text{Future} \rangle i \) is true at \( \langle t, x \rangle \) iff:

- **Time:** \( \langle T, R \rangle \)
- **Space:** \( \langle X, S \rangle \)

\[ \langle t, x \rangle \]

\[ i^V \]

\[ \text{Time: } \langle T, R \rangle \]
Semantic Understanding of Our Example (3)

⟨ Upstairs ⟩a is true at ⟨ t, x ⟩ iff:

Space:
⟨ X, S ⟩

Time: ⟨ T, R ⟩

⟨ t, x ⟩ V

Space:
⟨ X, S ⟩

Time: ⟨ T, R ⟩

⟨ t, x ⟩

a V

4
Semantic Understanding of Our Example (4)

Thus, $\langle \text{Future} \rangle \langle \text{Upstairs} \rangle p$ is true at $\langle t, x \rangle$. 

Space: $\langle X, S \rangle$

Time: $\langle T, R \rangle$

the denotation of $p$
Roughly, we need the two kinds of axioms and rules: $\mathbf{K}_H$ for $\Box_1$ and $@_i \& \mathbf{K}_H$ for $\Box_2$ and $@_a$.

Furthermore, we also need the five ‘interaction’ axioms:

- $@_a @_i p \iff @_i @_a p$.
- $@_{H-Hall} @_{05/03} p \iff @_{05/03} @_{H-Hall} p$.
- $\Diamond_1 @_a p \iff @_a \Diamond_1 p$.
- $\langle \text{Future} \rangle @_{H-Hall} p \iff @_{H-Hall} \langle \text{Future} \rangle p$.
- $\Diamond_2 @_i p \iff @_i \Diamond_2 p$.
- $\langle \text{Upstairs} \rangle @_{05/03} p \iff @_{05/03} \langle \text{Upstairs} \rangle p$.
- $@_i a \iff a$.
- $@_a i \iff i$. 
Hilbert-style Axiomatization of Hybrid Products

Roughly, we need the two kinds of axioms and rules: $K_H$ for $\Box_1$ and $@i \& K_H$ for $\Box_2$ and $@a$.
Furthermore, we also need the five ‘interaction’ axioms:

1. $@a @ip \leftrightarrow @i @a p$.
2. $@H-Hall @05/03 p \leftrightarrow @05/03 @H-Hall p$.
3. $\Diamond 1 @ap \leftrightarrow @a \Diamond 1 p$.
4. $\langle Future \rangle @H-Hall p \leftrightarrow @H-Hall \langle Future \rangle p$.
5. $\Diamond 2 @ip \leftrightarrow @i \Diamond 2 p$.
6. $\langle Upstairs \rangle @05/03 p \leftrightarrow @05/03 \langle Upstairs \rangle p$.
7. $@ja \leftrightarrow a$.
8. $@ai \leftrightarrow i$. 
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- $\langle \text{Future} \rangle @_{H-Hall} p \leftrightarrow @_{H-Hall} \langle \text{Future} \rangle p$.
- $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$.
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- $\langle a \rangle \langle i \rangle p \leftrightarrow \langle i \rangle \langle a \rangle p$.
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- $\langle 1 \rangle \langle a \rangle p \leftrightarrow \langle a \rangle \langle 1 \rangle p$.
- $\langle Future \rangle\langle H-Hall \rangle p \leftrightarrow \langle H-Hall \rangle \langle Future \rangle p$.
- $\langle 2 \rangle \langle i \rangle p \leftrightarrow \langle i \rangle \langle 2 \rangle p$.
- $\langle Upstairs \rangle @_05/03 p \leftrightarrow @_05/03 \langle Upstairs \rangle p$.
- $\langle i \rangle a \leftrightarrow a$.
- $\langle a \rangle i \leftrightarrow i$. 

Katsuhiko Sano
Introduction to Hybrid Logic
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- $\lozenge_1 @_a p \leftrightarrow @_a \lozenge_1 p$.
- $\langle \text{Future} \rangle @_{\text{H-Hall}} p \leftrightarrow @_{\text{H-Hall}} \langle \text{Future} \rangle p$.
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- $\langle \text{Future} \rangle @_{H-Hall} p \leftrightarrow @_{H-Hall} \langle \text{Future} \rangle p$.
- $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$.
- $\langle \text{Upstairs} \rangle @_{05/03} p \leftrightarrow @_{05/03} \langle \text{Upstairs} \rangle p$.
- $@_i a \leftrightarrow a$.
- $@_a i \leftrightarrow i$. 
Interaction Axioms for Two kinds of Nominals

We explain $a \rightarrow @j a$ alone. Assume that $a$ is true at $\langle t, x \rangle$. 

Space: $\langle X, S \rangle$

Time: $\langle T, R \rangle$

$\langle t, x \rangle$
Interaction Axioms for Two kinds of Nominals (Cont.)

Then, $a$ is true also at $\langle i^V, x \rangle$, i.e. $@_ia$ is true at $\langle t, x \rangle$. 
Main Result: Pure Completeness of Hybrid Products

Let \([K_H, K_H]\) be our axiomatization of hybrid products.

Pure Completeness wrt Product Frames (S.2010)
For any set \(\Lambda\) of pure formulas, \([K_H, K_H] + \Lambda\) (as new axioms) is strongly complete wrt the class of product Kripke frames defined by \(\Lambda\).

K. Sano
Axiomatizing Hybrid Products.
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K. Sano
Axiomatizing Hybrid Products.
Heart of Our Proof

Henkin
Product of Modal Logics

When we want to take the product of modal logics, e.g. $\mathbf{K}$ and $\mathbf{K}$, we need the following two axioms:

1. **(com)** $\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p$
2. **(chr)** $\Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p$
Product of Modal Logics

When we want to take the product of modal logics, e.g. $K$ and $K$, we need the following two axioms:

- (com) $\Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p$
- (chr) $\Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$

Note that our axiomatization does not contain these guys! But, our completeness results assure us that these are theorems of $[K_H, K_H]$. 
Outline

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Nominals and Satisfaction Operators

Structures (relational, topological) on the domain are irrelevant to a hybridization.
Nominals and Satisfaction Operators

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Overview of Hybrid Product Methods
## Overview of Hybrid Product Methods

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<thead>
<tr>
<th>Time \ Space</th>
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4. $@_ia \leftrightarrow a$
5. $@_ai \leftrightarrow i$
What is Topological Space?

~l & ~2 & ~3 \( \because \)
\( \therefore \)
\( \neg (\neg l \leftrightarrow \neg p) \),
\( \neg (\neg l \land \neg 2 \land \neg 3 \land \neg p) \),
\( \neg (\neg l \land \neg 2 \land \neg 3 \land \neg \neg p) \),
------
\( S \)
What is Topological Space?
What is Topological Space?

~\textit{-worlds} where \( p \) is false. The opposite counterfactual \( \neg \neg \),...,\( \neg p \) is false: there are \( \neg \neg \)-permitting spheres, and both of them contain \( \neg \neg \)-worlds where the new consequent \( \neg p \) is false.

Case (C) is the other way around. There are \( \neg \neg \)-permitting spheres, but none in which \( p \) holds at every \( \neg \neg \)-world, so none throughout which \( \neg \neg \neg > p \) holds. Therefore \( \neg \neg \neg p \) is false. In the inner one of the two \( \neg \neg \)-permitting spheres, \( \neg p \) holds at every \( \neg \neg \)-world; so the opposite counterfactual \( \neg \neg \neg p \) is true.

In case (D), finally, there are \( \neg \neg \)-permitting spheres, and both of them contain a mixture of \( \neg \neg \)-worlds where \( p \) holds and \( \neg \neg \)-worlds where \( \neg p \) holds. Therefore \( \neg \neg \neg p \) and its opposite \( \neg \neg \neg \neg p \) both are false.

Let us reconsider the sequences of true counterfactuals and their true negated opposites that drove us to give up the theory that the counterfactual is a constantly strict conditional based on similarity:

\begin{align*}
\neg l \& \sim 1 \& \sim 2 \& \sim 3 \& p \\
\neg (\neg l \& \sim 1), \\
\neg (\neg l \& \sim 2), \\
\neg (\neg l \& \sim 2 \& \sim 3 \& p), \\
\end{align*}

\textbf{FIGURE 2}

\textbf{An Analysis of Counterfactuals}
What is Topological Space?

\[ \tau(W) \subseteq \mathcal{P}(W) \]
Let $\langle W_l, \tau_l \rangle$ ($l = 1, 2$) be a topological space. Define the horizontal topological space $\tau_h$ by:

$$P \in \tau_h(x, y) \quad \text{iff} \quad ??$$
Product of Topologies (Van Benthem, et. al. 2006, S.2011)

- Let \( \langle W_l, \tau_l \rangle (l = 1, 2) \) be a topological space.
- Define the horizontal topological space \( \tau_h \) by:

\[
P \in \tau_h(x, y) \quad \text{iff} \quad ???
\]
Product of Topologies (Van Benthem, et. al. 2006, S.2011)

- Let $\langle W_l, \tau_l \rangle$ ($l = 1, 2$) be a topological space.
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$$P \in \tau_h(x, y) \text{ iff } ???$$

$$P_y := \{x' \mid (x', y) \in P\}$$
Let $\langle W_l, \tau_l \rangle$ ($l = 1, 2$) be a topological space. Define the horizontal topological space $\tau_h$ by:

$P \in \tau_h(x, y)$ if and only if $P_y \in \tau_1(x)$
Semantic Clauses on Product of Topologies

\[ w \models \Box \varphi \iff [\varphi] \in \tau(w). \]
Semantic Clauses on Product of Topologies

\[ w \models \Box \varphi \iff \llbracket \varphi \rrbracket \in \tau(w). \]

\[(x, y) \models \Box_1 \varphi \iff \llbracket \varphi \rrbracket \in \tau_h(x, y) \]

where \( \llbracket \varphi \rrbracket := \{ (x, y) \mid (x, y) \models \varphi \} \)
Semantic Clauses on Product of Topologies

\[ w \models \Box \varphi \iff \llbracket \varphi \rrbracket \in \tau(w). \]

\[ (x, y) \models \Box_1 \varphi \iff \llbracket \varphi \rrbracket \in \tau_h(x, y) \]
\[ \iff \llbracket \varphi \rrbracket_y \in \tau_1(x), \]

where \( \llbracket \varphi \rrbracket := \{ (x, y) | (x, y) \models \varphi \} \)
Our five interaction axioms are all valid on any product of topologies.

There is a problem on BG.
Intuitive Meaning of $\textbf{BG}$ on Kripke Frame

- $\textbf{BG}$: If $@_i \Diamond \varphi$ is consistent, then $@_i \Diamond j \land @_j \varphi$ is consistent for some fresh $j$.

\[ V \]

\[ V' \]

$V$: fresh extend $/\Diamond \varphi$

$i$  \hspace{1cm} \hspace{1cm}  \\

extend

\[ j: \text{fresh} \]

$\varphi$

\[ i \\

$V'$: fresh extend $/\Diamond \varphi$

$i$
'BG = Kripke Frames' in Topological Setting

- **S4-frame** = \( \langle W, \tau \rangle \) s.t. \( \tau(w) \) has the smallest element.
- On \( \mathbb{R}^2 \), we can consider the smaller and smaller neighborhood around \( w \).

---

Ten Cate & Litak (2007)

(\( BG \)) characterizes the notion of **S4**-frames (Alexandrov spaces) within the class of topological spaces.

So, we should drop two kinds of **BG** from our axiomatization.
Pure completeness wrt Products of Topologies

Let $\mathbf{S4}_H^- = (\mathbf{K}_H - \mathbf{BG}) + \{ \Box p \rightarrow p, \Box p \rightarrow \Box \Box p \}$. 

Pure Completeness for Product of Topologies

For any set $\Lambda$ of pure formulas, $[\mathbf{S4}_H^-, \mathbf{S4}_H^-] + \Lambda$ is strongly complete wrt the class of product of topologies defined by $\Lambda$. 

Katsuhiko Sano  
Introduction to Hybrid Logic
Let $\mathbf{S}^\land_4\mathcal{H} = (\mathbf{K}_\mathcal{H} - \text{BG}) + \{ \Box p \to p, \Box p \to \Box \Box p \}$.

**Pure Completeness for Product of Topologies**

For any set $\Lambda$ of pure formulas, $[\mathbf{S}^\land_4\mathcal{H}, \mathbf{S}^\land_4\mathcal{H}] + \Lambda$ is strongly complete wrt the class of product of topologies defined by $\Lambda$.

If we put $\Lambda = \emptyset$, we obtain:
Pure completeness wrt Products of Topologies

Let $S_{4_H} = (K_H - BG) + \{ \Box p \rightarrow p, \Box p \rightarrow \Box \Box p \}$.

Pure Completeness for Product of Topologies

For any set $\Lambda$ of pure formulas, $[S_{4_H}, S_{4_H}] + \Lambda$ is strongly complete wrt the class of product of topologies defined by $\Lambda$.

If we put $\Lambda = \emptyset$, we obtain:

Cor.

$[S_{4_H}, S_{4_H}]$ is strongly complete wrt the class of all products of topologies.
Topological Definability in Hybrid Logic

Here ‘definability’ means definability by a single formula.

<table>
<thead>
<tr>
<th>Properties</th>
<th>ML</th>
<th>HL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>Undef.</td>
<td>$\Diamond i \land \Diamond j \rightarrow \Diamond i j$</td>
</tr>
<tr>
<td>$T_1$</td>
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<td>$\Diamond i \rightarrow i$</td>
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<tr>
<td>$T_2$</td>
<td>Undef.</td>
<td>Undef. by Sustretov (2005)</td>
</tr>
<tr>
<td>density-in-itself</td>
<td>Undef.</td>
<td>$\neg \Box i$</td>
</tr>
<tr>
<td>compactness</td>
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<td>Undef.</td>
</tr>
<tr>
<td>discreteness</td>
<td>$\Diamond p \rightarrow p$</td>
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- Undef. in ML is due to the result of McKinsey-Tarski.
- $T_1$ says that $\{x\}$ (e.g. $[x, x]$) is closed.
- Density-in-itself says $\{x\} \notin \tau(x)$. 

Katsuhiko Sano

Introduction to Hybrid Logic
Overview of Hybrid Product Methods

<table>
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<tr>
<th>Time \ Space</th>
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<th>Top. $\langle X, \sigma \rangle$</th>
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1. $\@a \@i p \leftrightarrow \@i \@a p$
2. $\Diamond_1 \@a p \leftrightarrow \@ a \Diamond_1 p$
3. $\Diamond_2 \@i p \leftrightarrow \@i \Diamond_2 p$
4. $\@ i a \leftrightarrow a$
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Overview of Hybrid Product Methods

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1. $\downarrow a \downarrow i p \leftrightarrow \downarrow i \downarrow a p$
2. $\lozenge_1 \downarrow a p \leftrightarrow \downarrow a \lozenge_1 p$: Time str. is independent of Space
3. $\lozenge_2 \downarrow i p \leftrightarrow \downarrow i \lozenge_2 p$: Space str. is independent of Time
4. $\downarrow i a \leftrightarrow a$
5. $\downarrow a i \leftrightarrow i$
How to Capture Dependence of Space on Time (Rel)

- The accessible space-area may vary with the time: \( yS(t)y' \) rather than \( ySy' \).

\[ \diamond _2 @ip \leftrightarrow @i \diamond _2 p \text{ is invalid, } @i \diamond _2 @ip \leftrightarrow @i \diamond _2 p \text{ is valid.} \]

- A pure completeness result still holds.
How to Capture Dependence of Space on Time (Rel)

- The accessible space-area may vary with the time: $yS(t)y'$ rather than $ySy'$.

$\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$ is invalid, $@_i \Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$ is valid.

A pure completeness result still holds.
How to Capture Dependence of Space on Time (Rel)

- The accessible space-area may vary with the time: \( yS(t)y' \) rather than \( ySy' \).

\[ \langle x', y \rangle \quad \langle i^V, y \rangle \quad \langle x', y \rangle \]

\[ V(p) \]

- \( \Diamond_2 @ip \leftrightarrow @i \Diamond_2 p \) is invalid, \( @i \Diamond_2 @ip \leftrightarrow @i \Diamond_2 p \) is valid.

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Katsuhiko Sano Introduction to Hybrid Logic
How to Capture Dependence of Space on Time (Rel)

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- A pure completeness result still holds.
Spatial topology might depend on Time.

Then, our spatial topology and semantics should be defined as:

\[ (\sigma_t : X \rightarrow \mathcal{P}(X))_{t \in T}. \]

\[ \langle t, x \rangle \models \Box_2 \varphi \text{ iff } \llbracket \varphi \rrbracket_t \in \sigma_t(x). \]

Syntactically, this change corresponds to:

- $\psi \Box_2 \varphi \rightarrow @_2 \Box \varphi$: Invalid
- $@_2 \Box \varphi \rightarrow \Box_2 \Box \varphi$: Valid

We can still establish a pure completeness result.
Spatial topology might depend on time. Then, our spatial topology and semantics should be defined as:

\[(\sigma_t : X \to \mathcal{P}(X))_{t \in T}.\]

\[\langle t, x \rangle \models \Box_2 \varphi \text{ iff } \llbracket \varphi \rrbracket_t \in \sigma_t(x).\]

Syntactically, this change corresponds to:

- \(\Box_2 \circ \! @ \ p \leftrightarrow @_1 \Box_2 \ p: \text{ Invalid}\)
- \(\circ_2 \Box @_1 \ p \leftrightarrow @_1 \circ_2 \ p: \text{ Valid}\)

We can still establish a pure completeness result.
Spatial topology might depend on Time.

Then, our spatial topology and semantics should be defined as:

\[(\sigma_t : X \rightarrow \mathcal{P}(X))_{t \in T}.\]

\[\langle t, x \rangle \models \Box_2 \phi \iff [\phi]_t \in \sigma_t(x).\]

Syntactically, this change corresponds to:

- \(\Diamond_2 \iota p \leftrightarrow @i \Diamond_2 p: Invalid\)
- \(@i \Diamond_2 @i p \leftrightarrow @i \Diamond_2 p: Valid\)

We can still establish a pure completeness result.
How to Capture Dependence of Space on Time (Top)

- Spatial topology might depend on Time.
- Then, our spatial topology and semantics should be defined as:
  \[(\sigma_t : X \to \mathcal{P}(X))_{t \in T}\].
  \[\langle t, x \rangle \models □_2 \varphi \iff [\varphi]_t \in \sigma_t(x)\].

- Syntactically, this change corresponds to:
  - \(\Diamond_2 @ ip \leftrightarrow @i \Diamond_2 p\): Invalid
  - \(@i \Diamond_2 @ ip \leftrightarrow @i \Diamond_2 p\): Valid

- We can still establish a pure completeness result.
Spatial topology might depend on Time.

Then, our spatial topology and semantics should be defined as:

$$(\sigma_t : X \rightarrow \mathcal{P}(X))_{t \in T}.$$

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- $@_i \Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$: Valid

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We can still establish a pure completeness result.
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1. $@_a @_i p \leftrightarrow @_i @_a p$
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5. $@_a i \leftrightarrow i$
Outline

1. What is Hybrid Logic?
   - Basic Hybrid Logic
   - Downarrow Binder

2. How Can We Combine Hybrid Logics?
   - Kripke Semantics
   - Topological Semantics
   - Coalgebraic Semantics
Coalgebraic View on Kripke Semantics for ML

\[ w \models \Box p \text{ iff } R(w) \subseteq \llbracket p \rrbracket, \]
\[ \text{iff } R(w) \in \{ X \subseteq W \mid X \subseteq \llbracket p \rrbracket \}, \]
\[ \text{iff } R(w) \in \llbracket \Box \rrbracket_w(\llbracket p \rrbracket), \]

where \( \llbracket \Box \rrbracket_w(A) := \{ X \subseteq W \mid X \subseteq A \} \).
What is Hybrid Logic?
How Can We Combine Hybrid Logics?
Kripke Semantics
Topological Semantics
Coalgebraic Semantics

\[ \gamma : W \rightarrow T(W) \]

[\text{Coalgebraic Modal Logic}]

\[ T(W) \]
Recall: Semantic Clauses on Product of Nbhd Frames

\[(x, y) \models □_1 ϕ \iff [[ϕ]] \in \tau_h(x, y) \]

\[\iff [[ϕ]]_y \in \tau_1(x).\]
Recall: Semantic Clauses on Product of Nbhd Frames

\[(x, y) \models \Box_1 \varphi \iff \sem{\varphi} \in \tau_h(x, y) \]
\[\iff \sem{\varphi}_y \in \tau_1(x).\]

An essence for the semantic clause

You can use the original transition map.
What is Hybrid Logic?
How Can We Combine Hybrid Logics?

Kripke Semantics
Topological Semantics
Coalgebraic Semantics

\[ \gamma : W \to T(W) \]
\[ [\Diamond \Diamond] W \]
\[ T(W) \]

Coalgebraic Modal Logic

\[ \gamma : W \to T(W) \]
\[ [\Diamond \Diamond] W \]
\[ T(W) \]

Coalgebraic Modal Logic
Coalgebraic Semantics on Product of Coalgebras

- $\langle W_1, \gamma \rangle$: a $T_1$-coalgebra & $\langle W_2, \delta \rangle$: a $T_2$-coalgebra.

- We should define:

$$
(x, y) \models \Box_1 \varphi \text{ iff } \gamma_h(x, y) \in \llbracket \Box_1 \rrbracket_{W_1 \times W_2}([\varphi]),
$$

where $[\varphi] = \{(x', y') | (x', y') \models \varphi \} \subseteq W_1 \times W_2$.

- It suffices to have:

$$
(x, y) \models \Box_1 \varphi \text{ iff } \gamma(x) \in \llbracket \Box_1 \rrbracket_{W_1}.
$$

- We can also define $\gamma_h$ (Tensorial Strength (Kock 1972), suggested by Dirk Pattinson and Fredrik Dahlqvist).
Coalgebraic Semantics on Product of Coalgebras

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  where $\llbracket \varphi \rrbracket = \{(x', y') | (x', y') \models \varphi \} \subseteq W_1 \times W_2$.
- It suffices to have:

  $$(x, y) \models \Diamond_1 \varphi \text{ iff } \gamma(x) \in \llbracket \Diamond_1 \rrbracket_{W_1}(???) .$$

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  \]

  where \( \llbracket \varphi \rrbracket = \{(x', y') \mid (x', y') \models \varphi \} \subseteq W_1 \times W_2. \)

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  \]

- We can also define \( \gamma_h \) (Tensorial Strength (Kock 1972), suggested by Dirk Pattinson and Fredrik Dahlqvist).
Coalgebraic Semantics on Product of Coalgebras

- \( \langle W_1, \gamma \rangle \): a \( T_1 \)-coalgebra & \( \langle W_2, \delta \rangle \): a \( T_2 \)-coalgebra.
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  where \( \llbracket \varphi \rrbracket = \{(x', y') \mid (x', y') \models \varphi \} \subseteq W_1 \times W_2. \)
- It suffices to have:
  \[(x, y) \models \Diamond_1 \varphi \iff \gamma(x) \in \llbracket \Diamond_1 \rrbracket_{W_1}(\llbracket \varphi \rrbracket_y).\]
- We can also define \( \gamma_h \) (Tensorial Strength (Kock 1972), suggested by Dirk Pattinson and Fredrik Dahlqvist).
Semantic Summary

- $\langle W_1, \gamma \rangle$: a $T_1$-coalgebra & $\langle W_2, \delta \rangle$: a $T_2$-coalgebra.
- Given any valuation:

\[(x, y) \models \Diamond_1 \varphi \iff \gamma(x) \in \downarrow_1 \downarrow W_1(\downarrow \varphi)_y\]

\[(x, y) \models \Diamond_2 \varphi \iff \delta(y) \in \downarrow_2 \downarrow W_2(\downarrow \varphi)_x\]
Five Interaction Axioms

1. $@a @i p \iff @i @a p$
2. $\Diamond_1 @a p \iff @a \Diamond_1 p$
3. $\Diamond_2 @i p \iff @i \Diamond_2 p$
4. $@i a \iff a$
5. $@a i \iff i$

are valid on all products of $T_1$-coalgebra and $T_2$-coalgebra.
Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @_a p$
  - iff $\gamma(x) \in [\Diamond_1]W_1([@_a p]y)$
  - iff $\gamma(x) \in [\Diamond_1]W_1([p]a^y)$
    - iff $(p^{W_1} = [p]^{W_1})$
    - iff $(x, y) \models @_a p$
    - iff $(x, a^y) \models p$ if $(x, y) \models @_a p$
    - iff $(x, a^y) \models \Diamond_1 p$
  - iff $(x, y) \models @_a \Diamond_1 p$
Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @_a p$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1 (\llbracket @_a p \rrbracket_y)$
  - iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1 (\llbracket p \rrbracket_{av})$
    - iff $\llbracket p \rrbracket_{av} = \llbracket @_a p \rrbracket_y$
    - iff $x \in \llbracket p \rrbracket_{av}$ if $x \in \llbracket @_a p \rrbracket_y$
    - iff $(x, a^v) \models p$ if $(x, y) \models @_a p$

- iff $(x, a^v) \models \Diamond_1 p$
- iff $(x, y) \models @_a \Diamond_1 p$
Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @ a p \leftrightarrow @ a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @ a p$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket @ a p \rrbracket y)$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket p \rrbracket a^\vee)$
  - $\therefore \llbracket p \rrbracket a^\vee = \llbracket @ a p \rrbracket y$.
  - $\therefore x \in \llbracket p \rrbracket a^\vee$ iff $x \in \llbracket @ a p \rrbracket y$.
  - $\therefore (x, a^\vee) \models p$ iff $(x, y) \models @ a p$.
- iff $(x, a^\vee) \models \Diamond_1 p$
- iff $(x, y) \models @ a \Diamond_1 p$
Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @_a p \iff @_a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @_a p$
- iff $\gamma(x) \in [\Diamond_1]_{W_1}([@_a p]_y)$
- iff $\gamma(x) \in [\Diamond_1]_{W_1}([p]_{a^\vee})$
  - $\therefore [p]_{a^\vee} = [@_a p]_y$.
  - $\therefore x \in [p]_{a^\vee}$ iff $x \in [@_a p]_y$.
  - $\therefore (x, a^\vee) \models p$ iff $(x, y) \models @_a p$.

- iff $(x, a^\vee) \models \Diamond_1 p$
- iff $(x, y) \models @_a \Diamond_1 p$
Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @ a p \leftrightarrow @ a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @ a p$
- $\iff \gamma(x) \in \sem{\Diamond_1} W_1(\sem{\Diamond_1} @ a p)_y$
- $\iff \gamma(x) \in \sem{\Diamond_1} W_1(\sem{p} a V)$
  - $\therefore \sem{p} a V = \sem{\Diamond_1} @ a p)_y$.
  - $\therefore x \in \sem{p} a V \iff x \in \sem{\Diamond_1} @ a p)_y$.
  - $\therefore (x, a V) \models p$ $\iff (x, y) \models @ a p$.
- $\iff (x, a V) \models \Diamond_1 p$
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Validity of Five Interaction Axioms

Let us focus on $\Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$.

- $(x, y) \models \Diamond_1 @_a p$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket @_a p \rrbracket_y)$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket p \rrbracket_a V)$
  - $\therefore \llbracket p \rrbracket_a V = \llbracket @_a p \rrbracket_y$.
  - $\therefore x \in \llbracket p \rrbracket_a V$ iff $x \in \llbracket @_a p \rrbracket_y$.
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- $(x, y) \models \Diamond_1 \Diamond a p$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket \Diamond a p \rrbracket y)$
- iff $\gamma(x) \in \llbracket \Diamond_1 \rrbracket W_1(\llbracket p \rrbracket a^V)$
  - $\therefore \llbracket p \rrbracket a^V = \llbracket \Diamond a p \rrbracket y$.
  - $\therefore x \in \llbracket p \rrbracket a^V$ iff $x \in \llbracket \Diamond a p \rrbracket y$.
  - $\therefore (x, a^V) \models p$ iff $(x, y) \models \Diamond a p$.
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  - $\therefore [[p]]_{a^V} = [[@_a p]]_y$.
  - $\therefore x \in [[p]]_{a^V}$ iff $x \in [[@_a p]]_y$.
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If each \( R_i \) is strongly one-step complete, the corresponding product of hybrid logics is strongly complete. (S.2011)

Schröder, L. and Pattinson, D.
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Summary

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Further Directions

- **Decidability** is still open, even for hybrid product of Kripke frames.
- Construct Gentzen-style sequent calculus and establish cut elimination theorem possibly extended with geometric theory.
- Build a corresponding first-order language for product of modal/hybrid logic over Kripke frames and investigate modal model theory, e.g. Van Benthen Characterization Theorem.
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<table>
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Naming Lines provides a modular and robust way of combining two hybrid logics.

Thank You
Take-home Message

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Thank You
In addition to (BG) + (Name), K plus

\((K@)\) \(\@_i(p \rightarrow q) \rightarrow (\@_i p \rightarrow \@_i q)\).

\((\text{Self-Dual})\) \(\neg\@_i p \leftrightarrow \@_i \neg p\).

\((\text{Ref})\) \(\@_i i\).

\((\text{Intro})\) \(i \land p \rightarrow \@_i p\).

\((\text{Back})\) \(\@_i p \rightarrow \Box\@_i p\).

\((\text{Agree})\) \(\@_i \@_j p \rightarrow \@_j p\).

\((\text{Nec}@)\) From \(\varphi\), we may infer \(\@_i \varphi\).

\((\text{Hsub})\) From \(\varphi\), we may infer \(\sigma(\varphi)\), where \(\sigma\) is a \(H\)-uniform substitution, eg.:

\[
\frac{(i \land p) \rightarrow (q \rightarrow p)}{(j \land \varphi) \rightarrow (\psi \rightarrow \varphi)}
\]
Tensorial Strength on **Sets**

- Define \( \iota_y : W_1 \to W_1 \times W_2 \) by \( \iota_y(x) := (x, y) \) (\( y \in W_2 \)).

- **Tensorial strength** \( \text{st}_{W_1, W_2} : T_1(W_1) \times W_2 \to T_1(W_1 \times W_2) \) is defined by

\[
\text{st}_{W_1, W_2}(t, y) := T_1(\iota_y)(t),
\]

where \( T_1(\iota_y) : T_1(W_1) \to T_1(W_1 \times W_2) \).

- Define \( \gamma_h := \)

\[
W_1 \times W_2 \xrightarrow{\gamma \times \text{id}} T_1(W_1) \times W_2 \xrightarrow{\text{st}_{W_1, W_2}} T_1(W_1 \times W_2)
\]

- This gives us the same definitions as \( R_h \) and \( \tau_h \).