Gentzen's Consistency Proofs for Arithmetic

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Gentzen's four proofs

- The earliest proofs of the consistency of Peano arithmetic were presented by Gentzen, who worked out a total of four proofs between 1934 and 1939.
- The first consistency proof was withdrawn from publication due to criticism by Bernays for implicit use of the fan theorem, although this assessment was later retracted (Bernays, 1970). However, a galley proof of the article was preserved and excerpts were published posthumously in English translation (Szabo, 1969), as well as unabridged in the German original (Gentzen, 1974).

Gentzen's first proof as a game

- Gentzen's first version of his consistency proof can be formulated as a game between a verifier and a falsifier.
- Gentzen's proof consists in effectively constructing, from each derivable sequent $\Gamma \rightarrow A$ of PA, a reduction of the sequent.
- The verifier wins if for each choice of the falsifier, there is a reduction to endform.
- The endform is defined as that either:
  - the succedent is a true atomic formula or
  - there is a false atomic formula in the antecedent.
- Gentzen shows that if a sequent is derivable, then there is a winning strategy for the verifier.
- Because the contradictory sequent $\rightarrow 0 = 1$ cannot be reduced in this way, it is not derivable.

Bernays' critique and the fan theorem

- Neither Bernays nor Gödel were satisfied with Gentzen's first consitency proof, which is shown in correspondence from Gentzen to Bernays in the fall of 1935.[1]
- The critizism was that the proof made imlicit use of the fan theorem. However, it has been noted by Kreisel in 1987 that this principle is not sufficient for proving the consistency of Peano arithmetic.
- According to (Tait, 2005) the principle used implicitly in the proof is recursion on well-founded trees, which is essentially bar recursion.This is corroborated by von Plato's research of Gentzen's writings about the first proof.[2]

[1] Jan von Plato, From Hilbert's programme to Gentzen's Programme, p. 392
[2] Jan von Plato, From Hauptsatz to Hilfssatz

Gentzen's second proof

- Gentzen, who had already thought of the objections, reworked his proof.
- The result was the published second proof (Gentzen, 1936), which is appended with an ordinal assignment and relies on a constructive proof of the principle of transfinite induction up to the ordinal ϵ_0.

Gentzen's third proof

- Gentzen's third proof from 1938 uses a standard sequent calculus and provides a reduction procedure for derivations of contradictions, represented by the empty sequent.
- The proof shows that if there exists a derivation of the empty sequent, then there exists another less complex derivation, and another derivation, etc.
- The reduction procedure must terminate in a simple derivation of the empty sequent. But it can be shown that no simple derivation exists and therefore the assumption that there exists an arbitrary derivation of the empty sequent leads to a contradiction. Thus, the system must be consistent.

Gentzen's fourth proof

- Gentzen's fourth proof published in 1943 proves consistency through a non-derivability.
- In his proof he represented transfinite induction up to ϵ_0 as an arithmetical formula and showed that it is not provable in Peano arithmetic, but that any weaker induction principle is provable.

How consistency proofs are possible

- The combinatorial methods of Gentzen's reduction procedure described in the third proof can be represented in primitive recursive arithmetic (PRA). PRA is a weaker theory than Peano arithmetic and it is generally included in, and often identified with, finitistic logic, because unbounded quantification over the domain of natural numbers is not allowed. Due to this feature, the primitive recursive operation on derivations, described in Gentzen's proof, corresponds to a quantifier-free formula.
- Therefore, finitistic reasoning together with the principle of transfinite induction restricted to quantifier-free formulas gives the consistency result.

- It should be noted that the theory, in which the proof is formalizable, is incomparable to Peano arithmetic. The theory is not stronger than Peano arithmeic, since complete induction cannot be proved for all formulas. But on the other hand, neither is the theory weaker, since it proves the consistency of Peano arithmetic.

Gentzen's height-lines

- Gentzen's first and second proof are conducted in a natural deduction in sequent calculus style. The third proof, on the other hand, is conducted in pure sequent calculus.
- He uses a construction called the height-line argument to produce a derivation with a lower ordinal from a given derivation of the empty sequent. In the case that we can locate a suitable reducible cut on a compound formula in the derivation, our aim is to simpify this inference in the derivation. The simplification is however not a straightforward conversion of the cut into cuts on shorter formulas, due to the fact that we can have contractions on the cut formula. (Compare to a cut elimination theorem).

A Gentzen-style proof without height-lines

- What is required is a heightline construction, which means the introduction of additional cuts on compound formulas in the reduced derivation. As a consequence of these cuts the points in the derivation where the height of a sequent is dropped are permuted up in the derivation and the ordinal of the derivation is lowered, because of its dependance on the notion of height of a sequent.
- However a new result by the author gives reductions that directly turn suitable cuts on compound formulas into cuts on shorter formulas.
- The proof describes reductions for a consistency proof for an intuitionistic Heyting Arithmetic based on a normalization proof for Gödel's T in (Howard, 1970).

- The height-line construction is possible to avoid by a Howard-style vector assignment because the assignment is designed in such a way as to make it possible to duplicate the cut on a shorter formula if the original cut formula has been contracted. The reduction procedure places these new cuts on shorter formulas where each copy of the cut formula was introduced.
- The reductions resemble proofs of direct cut elimination without multicut. See (Buss, 1998), (Troelstra & Schwichtenberg, 1996) and (von Plato, 2001).

A vector assignment

- In contrast to Gentzen's proof, the procedure is appended with a vector assignment. The reduction reduces the first component of the vector and this component can be interpreted as an ordinal less than ϵ_0, thus ordering the derivations by complexity and proving termination of the process.
- The assignment uses vectors instead of a direct ordinal assignment, because the length of the vector is used as a parameter coding the complexity of the succedent formula of each sequent in the derivation.

Howard's vectors

- Howard gives a theory of expressions, which are then taken as the components of the vectors assigned to each sequent in a derivation.
- Whenever a copy of a formula, A, is first introduced in the antecedent of a sequent by a left rule (but not by weakening), then a variable corresponding to the formula, $\mathbf{x}^{\mathbf{A}}$ is introduced in the vector.
- He also defines two operations on the vectors, the box and the delta-operations.

- The box-operation is a form of addition and iteration of exponentiation and the delta-operation on a formula A gives a vector, denoted $\delta^A\mathbf{h}$, that does not contain any component of the variable vector $\mathbf{x^A}$.
- The main property proven for these operations is

$$((\delta^A\mathbf{h})\Box\mathbf{e})_i > (\mathbf{h}[\mathbf{e}/\mathbf{x^A}])_i$$

for all $i \leqslant \mathit{length}(\mathbf{h})$. This means that the vector $(\delta^A\mathbf{h})\Box\mathbf{e}$ is greater than a vector where each component of $\mathbf{e}$ has been substituted for the components of the variable vector $\mathbf{x^A}$ occurring in the vector $\mathbf{h}$.

The reduction procedure

- The reduction procedure for derivations of the empty sequent starts with substituting constants for all free variables.
- Now we search in the derivation for a suitable reduction.
- The purpose of this trace is to find a reduction, the rules below which preserve the inequality of the vectors. Because of this property we can never have a rule that performs the δ-operation on the reduced vector.

- We trace up from the end-sequent of the derivation.
  - If the last rule is an arithmetical rule, then by the assumption that the arithmetical rules are applied before all other rules, the derivation consists only of arithmetical rules and is simple, which is impossible.
  - If the last rule of the derivation is an induction, then this is a suitable reduction.
  - On the other hand, if the last rule is a cut. We trace up through the left premises of the lowermost cuts, until we reach a sequent which is not derived by a cut.
    The sequent is not an initial sequent because the antecedent is empty. The sequent can be derived by an induction, an arithmetical rule, right weakening or a logical right rule. This is the suitable reduction.

Assume that the suitable reduction is a logical right rule. There are different cases depending on the outermost connective of the formula.
Assume that the cut formula of the suitable reduction is a conjunction $A\&B$. Now the derivation has the following form:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdots \\ \xrightarrow{\alpha} A\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdots\\ \xrightarrow{\beta} B\end{matrix}}{\xrightarrow{\mu_1} A\&B}\,R\& \qquad \begin{matrix}\Pi_3\\ \vdots\\ A\&B \xrightarrow{\mu_2} C\end{matrix}}{\begin{matrix}\to C\\ \vdots\\ \to\end{matrix}}\,Cut$$

The vector assigned to the conclusion of the cut is $\delta^{A\&B}\mu_2\Box\mu_1 = \delta^{A\&B}\mu_2\Box(\alpha+\beta)$.

- We now do a trace in the derivation of the right cut premise to find the rules in the derivation Π_3 that introduced the formula $A \& B$ in the derivation.
- The only way we may introduce a conjunction in the antecedent of a sequent is by weakening or a logical left rule (not initial sequents because they are restricted to atomic formulas).

Trace. Trace up from the right cut premise the occurrence of the cut formula in question. If at some stage the formula is principal in contraction, trace up from both occurrences of the contraction formula in the premise. In this way, a number of first occurrences of the formula are located.
(i) If a first occurrence of $A\&B$ is obtained by left weakening, remove the rule and the weakening formula. Modify the derivation below by removing the formula in each context and if a step is found where a contraction on $A\&B$ was done in the derivation, then remove the contraction rule. This modification does not alter the vectors of the sequents in the derivation.

(ii) If a logical left rule is reached, then the premise of the logical rule has either A or B as active formula. Assume that the active formula is A. Then the rule is

$$\frac{A, \Gamma \xrightarrow{\gamma} D}{A \& B, \Gamma \xrightarrow{\lambda} D} \; L\&$$

The vector assigned to the conclusion of this rule is $\lambda = \mathbf{x}^{\mathbf{A\&B}} \Box \delta^{A} \gamma$.

Now replace this rule by a cut with the premise of the right rule as a left cut premise.

$$\cfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \xrightarrow{\alpha} A\end{array} \quad A, \Gamma \xrightarrow{\gamma} D}{\Gamma \xrightarrow{\lambda'} D}\ \mathit{Cut}$$

Modify the derivation below the new cut as in (i) of the trace. The vector assigned to the conclusion of the cut is $\lambda' = \alpha \Box \delta^A \gamma = \lambda[\alpha/\mathbf{x}^{\mathbf{A\&B}}]$.

The vector assigned to the sequent $\rightarrow C$ corresponding to the conclusion of the cut in the reducibe derivation is

$$\mu_2[\nu/\mathbf{x}^{\mathbf{A}\&\mathbf{B}}],$$

where the vector ν that is substituted is in some cases α and in some cases β. For both you have $\alpha \leqslant \alpha + \beta$ and $\beta \leqslant \alpha + \beta$ respectively. One can see that the vector of the reduced derivation is smaller than

$$\delta^{A\&B}\mu_2\Box(\alpha+\beta)$$

by the main result for Howard's operations.
This completes the reduction for a suitable reduction in the case of conjunction. The other cases are similar (except for disjunction, which we take to be a defined concept in the calculus).

The consistency theorem

Theorem (The consistency of Heyting arithmetic)

The empty sequent $\rightarrow$ is not derivable in HA, that is, HA is consistent.

Assume that there is a derivation of the empty sequent. By performing the reduction procedure on this derivation, a reduced derivation of the empty sequent with a lower ordinal is obtained. Because it is possible to continue the reductions the procedure would not terminate and we would have an infinite succession of decreasing ordinals all less than ϵ_0. But this is impossible due to the well-ordering of the ordinals. Thus, our initial assumption of the existence of a derivation of the empty sequent must be wrong and the system of Heyting arithmetic, HA, is consistent.

References

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(4) J. VON PLATO: From Hilbert's programme to Gentzen's Programme, in *Logic's lost genius: The life of Gerhard Gentzen*, E. Menzler-Trott, American mathematical society, 2007.

(5) A. SIDERS: Gentzen's Consistency Proof Without Heightlines. (Submitted).