Reverse Mathematics and Non-Standard Methods

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Foundational Doctrines ⇦ Key Persons

1. Feasibility (Poly-time) ⇦ Ko
2. Finitism ⇦ Hilbert
3. Computability ⇦ Aberth, Pour-El
4. Constructivity ⇦ Bishop
5. Finitistic Reductionism ⇦ Hilbert
6. Predicativity ⇦ Weyl
7. Predic. Reductionism ⇦ Feferman
8. Semi-Finit. Consistency ⇦ Takeuti
10. Ramified Type Theory ⇦ Russell
11. Constructible Universe ⇦ Gödel
12. Large Cardinals ⇦ Gödel
Foundational Doctrines $\Rightarrow$ Formal Systems

1. Feasibility $\Rightarrow$ $S_2^1$, BTFA
2. Finitism $\Rightarrow$ PRA
3. Computability $\Rightarrow$ RCA$_0$
4. Constructivity $\Rightarrow$ RCA$_0$
5. Finitistic Reductionism $\Rightarrow$ WKL$_0$
6. Predicativity $\Rightarrow$ ACA$_0$
7. Predic. Reductionism $\Rightarrow$ ATR$_0$
8. Semi-Finit. Consistency $\Rightarrow$ $\Pi^1_1$-CA$_0$
9. Second Order Arith. $\Rightarrow$ Z$_2$
10. Ramified Type Theory $\Rightarrow$ PM
11. Constructible Universe $\Rightarrow$ V=L
12. Large Cardinals $\Rightarrow$ V$_\alpha$
From Foundationalism to Reverse Math

Foundationalism (基礎付け主義):
Which systems are needed to do mathematics?

Reverse Mathematics (逆数学):
Which axioms are needed to prove a theorem?

Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics. The method can briefly be described as “going backwards from the theorems to the axioms”. This contrasts with the ordinary mathematical practice of deriving theorems from axioms. Wikipedia
Reverse Mathematics Phenomenon

Which axioms are needed to prove a theorem?

0. Fix a weak base system $S$ (e.g. $\text{RCA}_0$).

1. Pick a theorem $\Phi$ and formalize it in $S$.

2. Find a weakest axiom $\alpha$ to prove $\Phi$ in $S$.

3. Very often, we can show (over $S$) that $\alpha$ and $\Phi$ are logically equivalent.
Second Order Arithmetic (Hilbert Arithmetic)

A first order theory of natural numbers and sets of them.
Standard Model: \((\omega \cup \wp(\omega); +, \cdot, 0, 1, <, \in)\)

Second order arithmetic \(\mathbb{Z}_2\)

= Basic axioms for \((+, \cdot, 0, 1, <)\)

+ Comprehension (CA): \(\exists X \forall x (x \in X \leftrightarrow \varphi(x))\)

+ Induction: \(\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x \varphi(x)\)
Classifying Formulas

✓ Bounded formulas ($\Sigma^0_0$), only with $\forall x < t$, $\exists x < t$

✓ Arithmetical formulas ($\Sigma^1_0$), with no set quantifiers

$$\Sigma^0_n : \exists \overrightarrow{x_1} \forall \overrightarrow{x_2} \cdots Q \overrightarrow{x_n} \varphi \text{ with } \varphi \text{ bounded.}$$

$$\Pi^0_n : \forall \overrightarrow{x_1} \exists \overrightarrow{x_2} \cdots Q \overrightarrow{x_n} \varphi \text{ with } \varphi \text{ bounded.}$$

✓ Analytical formulas:

$$\Sigma^1_n : \exists \overrightarrow{X_1} \forall \overrightarrow{X_2} \cdots Q \overrightarrow{X_n} \varphi \text{ with } \varphi \text{ arithmetic.}$$

$$\Pi^1_n : \forall \overrightarrow{X_1} \exists \overrightarrow{X_2} \cdots Q \overrightarrow{X_n} \varphi \text{ with } \varphi \text{ arithmetic.}$$
Examples.

\[ \Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1 \]  
Recursive, Computable, Decidable

\[ \Sigma^0_1 \]  
r.e. (recursively enumerable), c.e.,  
the set of theorems of a formal system

\[ \Pi^0_1 \]  
co-r.e., finitistic assertions,  
the Gödel sentence, consistency,  
the Goldbach conjecture

\[ \Pi^0_2 \]  
1-consistency, the twin prime conj.,  
Paris-Harrington, P \neq NP
Big five subsystems

\[ RCA_0 = \Delta^0_1-CA + \Sigma^0_1-\text{ind} \]

\[ WKL_0 = RCA_0 + \text{weak König's lemma} \]
\[ (\Sigma^0_1 - \text{Separation}) \]

\[ ACA_0 = RCA_0 + \Sigma^0_1-CA \]

\[ ATR_0 = RCA_0 + \text{transfinite iteration of } \Sigma^0_1-CA \]
\[ (\Sigma^1_1 - \text{Separation}) \]

\[ \Pi^1_1-CA_0 = RCA_0 + \Pi^1_1-CA \]
Weak König's Lemma for infinite binary trees
Some results of R. M.

Over $RCA_0$

$WKL_0 \iff$ the maximum principle
  $\iff$ the Cauchy–Peano theorem
  $\iff$ Brouwer’s fixed point theorem

$ACA_0 \iff$ the Bolzano-Weierstrass theorem
  $\iff$ the Ascoli lemma

$ATR_0 \iff$ the Luzin separation theorem
  $\iff$ $\Sigma^0_1$-determinacy

$\Pi^1_1$-$CA_0 \iff$ the Cantor-Bendixson theorem
  $\iff$ $\Sigma^0_1 \land \Pi^0_1$-determinacy
## Mathematics in the Big Five

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<tr>
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<th>$RCA_0$</th>
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Defining the real number system $\mathbb{R}$

The following definitions are made in $RCA_0$

- Using the pairing function, we define $\mathbb{N}$ and $\mathbb{Q}$.
- The basic operations on $\mathbb{N}$ and $\mathbb{Q}$ are also naturally defined.
- A real number is an infinite sequence $\{q_n\}$ of rationals such that $|q_n - q_m| \leq 2^{-n}$ for all $m > n$.
- The operations on $\mathbb{R}$ are also defined so that the resulting structure is a real closed order field.
Some results

   \[ \text{RCA}_0 \vdash \forall \sigma (\text{RCOF} \vdash \sigma \Rightarrow \mathbb{R} \models \sigma) \]
with the help of the fundamental theorem of algebra
   \[
   (\text{strong FTA}) \quad \text{RCA}_0 \vdash \forall p(x) \in \mathbb{Q}[x] \exists \alpha \in \mathbb{C}^{\mathbb{N}} p(x) = \prod_i (x - \alpha_i)
   \]

   \[ WKL_0 \vdash \sigma \Rightarrow \text{RCA}_0 \vdash \sigma \]
   for \( \sigma \equiv \forall X \exists ! Y \varphi(X, Y) \) with \( \varphi \) arith.
Thus, it suffices to show strong FTA in \( WKL_0 \).

3. Strong FTA can be proved by a non-standard method in \( WKL_0 \).
Satisfaction on $\mathbb{R}$

- Simpson-T.-Yamazaki

$$\text{Sat}_{\mathbb{R}}([\varphi(\overrightarrow{x})], \overrightarrow{\xi})$$ can be defined as a $\Delta_2^0$ formula.

In $\text{RCA}_0$, $\text{Sat}_{\mathbb{R}}$ satisfies the Tarski clauses for the standard formulas.


In $\text{RCA}_0$, $\text{Sat}_{\mathbb{R}}$ satisfies the Tarski clauses for all the formulas. In particular,

$$\text{Sat}_{\mathbb{R}}([\exists \overrightarrow{x} \varphi(\overrightarrow{x}, \overrightarrow{y})], \overrightarrow{\beta}) \leftrightarrow \exists \overrightarrow{\alpha} \text{Sat}_{\mathbb{R}}([\varphi(\overrightarrow{x}, \overrightarrow{y})], \overrightarrow{\alpha}, \overrightarrow{\beta})$$

- The following fact (called strong FTA) is essential:

$$\text{RCA}_0 \vdash \forall p(x) \in \mathbb{Q}[x] \exists \overrightarrow{\alpha} \in \mathbb{C}^{<\mathbb{N}} p(x) = \prod_i (x-\alpha_i)$$
Applications of Sakamoto-T’s result

\[ \text{RCA}_0 \vdash \text{Hilbert’s Nullstellensatz} : \]

\[ p_1, \cdots, p_m \in \mathbb{C}[\overrightarrow{x}] \text{ have no common zeros} \]

\[ \Rightarrow \exists q_1 \cdots \exists q_m \in \mathbb{C}[\overrightarrow{x}] \ p_1 q_1 + \cdots + p_m q_m = 0 \]

\[ \text{RCA}_0 \vdash \text{strong FTA} \]
Conservation results

- **Shoenfield:**
  \[ ZF + V = L \vdash \sigma \Rightarrow ZF \vdash \sigma \text{ for } \sigma \in \Sigma^1_2 \cup \Pi^1_2 \]

- **Barwise-Schlipf:**
  \[ \Sigma^1_1-AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma \text{ for } \sigma \in \Pi^1_2 \]

- **Harrington:**
  \[ WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \text{ for } \sigma \in \Pi^1_1 \]

- **Simpson-T.-Yamazaki (2002):**
  \[ WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \text{ for } \sigma \equiv \forall X \exists! Y \varphi(X, Y) \text{ with } \varphi \text{ arith.} \]
Application of Simpson-T.-Yamazaki's result

The fundamental theorem of algebra (FTA):

*Any complex polynomial of a positive degree has a unique factorization into linear terms.*

By the STY result, we have

\[ \text{WKL}_0 \vdash (\text{strong}) \ FTA \Rightarrow \text{RCA}_0 \vdash (\text{strong}) \ FTA. \]

By the usual mathematical argument, we have

\[ \text{WKL}_0 \vdash \ FTA \ (\text{for any particular standard polynomial}). \]

Thus, we have

\[ \text{RCA}_0 \vdash \ FTA \ (\text{standard}), \]

which is not enough for our purpose.
Non-Standard Models

Theorem (H. Friedman, Kirby-Paris)

Suppose $M \models PRA$, countable.

Suppose $b \ll_M c$ (i.e., $f(b) <_M c$ for all prim. rec. $f$).

Then $\exists I \subseteq_e M$ s.t. $b \in I, c \notin I$ and $I \models \text{I} \Sigma_1$

Moreover, if $C(M) = \{X \subseteq M : \exists a \in M \text{ codes } X\}$,

$(I, C(M) \upharpoonright I) \models WKL_0$.

Theorem (T.) A converse to the above holds.

Suppose $(M, S) \models WKL_0$, countable, $M \neq \omega$.

Then $\exists^* M \supseteq_e M$ s.t. $^*M \models \text{I} \Sigma_1$ and $S = C(^*M) \upharpoonright M$. 
Self-Embedding Theorems

Thm. (self-embedding for $WKL_0$, T. 1997)

Suppose $(M, S) \models WKL_0$, countable, $M \neq \omega$.

Then $\exists I \subseteq M$ s.t. $(M, S) \simeq (I, S \cap I)$.

• History of self embedding results.
  
  H. Friedman (1970's) for PA.
  
  Ressayre, Dimitracopoulos and Paris (1980's) for $I\Sigma_1$.

(Proof) By a back-and-forth argument.

Cor. Suppose $(M, S) \models WKL_0$, countable, $M \neq \omega$.

Then $\exists^* M \supseteq M, \exists^* S$ s.t. $(*M, *S) \models WKL_0$

and $S = *S \upharpoonright M$. 
Application (the maximum principle)

\[ WKL_0 \vdash \text{Any cont. function } f : [0, 1] \to [0, 1] \text{ has a max.} \]

(Proof)

\[
\begin{align*}
V &= (M, S) \\
*V &= (*M, *S) \\
\end{align*}
\]

\[
\begin{align*}
&f : [0, 1] \cap \mathbb{Q} \to [0, 1] \\
&\{q_i\}_{i \in M} \quad 2^M \\
\Rightarrow &\quad *f : \{q_i\}_{i < a} \to 2^b \\
&(a, b \in *M - M, f = *f \cap M) \\
\Downarrow &\quad *m \cap M \text{ is sup } f \\
\iff &\quad *m = \max\{*f(q_i)\}_{i < a}
\end{align*}
\]
Application

\[ WKL_0 \vdash \text{Strong FTA.} \]

**Proof**

\[ V = (M, S) \]  \[ \iff \]  \[ *V = (*M, *S) \]

\[ f : \mathbb{Q}[x] \to (\mathbb{C} \cap \mathbb{Q}^2)^{<\mathbb{N}} \]

\[ \{ p_i \}_{i \in M} \text{ with infinite repetition} \]

s.t. \( f(p_i) \) is a list of rational approximations of the roots of \( p_i \) with error \( < 2^{-i} \).

\[ *f : \{ p_i \}_{i < a} \to (*\mathbb{C} \cap *\mathbb{Q}^2)^{<b} \]

\[ (a, b \in *M - M, f = *f \cap M) \]

\[ \iff \]

\[ *f(p_{j_i}) \cap M \text{ is the list of roots of } p_i. \]

\[ \{ p_i \}_{i \in M} = \{ p_{j_i} \}_{j_i \notin M, i \in M} \]
Other applications

\[ WKL_0 \vdash \text{The Cauchy–Peano theorem} \ (Tanaka, \ 1997) \]

\[ WKL_0 \vdash \text{The existence of Haar measure for a compact group} \ (Tanaka-Yamazaki, \ 2000) \]

\[ WKL_0 \vdash \text{The Jordan curve theorem} \ (Sakamoto-Yokoyama, \ 2007) \]
**Application (Sakamoto, Yokoyama)**

$WKL_0 \vdash \text{The Jordan Curve Theorem}$

(Proof) \hspace{1cm} \begin{align*}
V &= (M, S) \\
*V &= (*M, *S)
\end{align*}

\begin{align*}
U_0 &\hspace{1cm} U_1 \\
*U_0 &\hspace{1cm} *U_1
\end{align*}
Outer model method for \(ACA_0\)

Suppose \((M, S) \models ACA_0\), countable, \(M \neq \omega\).
Then \(\exists^* M \supseteq_e M \exists^* S\)

s.t. \((^*M, ^*S) \models ACA_0\), \(S = ^*S \upharpoonright M\)

and \(\exists^* : S \to ^*S \forall \varphi(x, X) \in \Sigma^1 \cup \Pi^1\)

\((M, S) \models \varphi(m, A) \leftrightarrow (^*M, ^*S) \models \varphi(m, ^*A)\)

This easily follows from

Theorem (Gaifman): Every model \(M\) of \(PA\) has a
 conservative extension \(K\), i.e., (the sets definable
 in \(K\)) \(\upharpoonright M = \) the sets definable in \(M\).
Applications

\( \text{ACA}_0 \vdash \text{Any Cauchy sequence converges.} \)

(Proof) \[ V = (M, S) \quad \Rightarrow \quad *V = (*M, *S) \]

\( \{a_i\}_{i \in M} \text{ a Cauchy seq.} \quad \Rightarrow \quad *\{a_i\}_{i \in M} = \{(*a)_i\}_{i \in *M} \cdot \]

\[ \forall n \exists m \forall k > m |a_k - b| < 2^{-n} \quad \iff \quad |(*a)_k - (*a)_j| < 2^{-n}. \]

\[ *b \approx (*a)_j. \]

\( \text{ACA}_0 \vdash \text{The Riemann mapping theorem.} \)

(Yokoyama)
Outer model method II for $ACA_0$

Suppose $(M, S) \models \Sigma^1_1$-$AC_0$, countable.

Then $\exists^* M \supseteq e M \exists^* S$

s.t. $(^* M, ^* S) \models \Sigma^1_1$-$AC_0$, $S = ^* S \upharpoonright M$

and $\exists^* : S \rightarrow ^* S \ \forall \varphi(x, X) \in \Sigma^1_2 \cup \Pi^1_2$

$(M, S) \models \varphi(m, A) \leftrightarrow (^* M, ^* S) \models \varphi(m, ^* A)$

This can be used with

$\Sigma^1_1$-$AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma \ for \ \sigma \in \Pi^1_2$
Some general results (due to Schmerl)

Suppose \((M, S) \models \Sigma^1_n\text{-AC}_0\), countable.

Then \(\exists^* M \supseteq M \exists^* S \text{ s.t. } S = *S \upharpoonright M\)

\[\text{and } \exists^*: S \to *S \forall \varphi(x, X) \in \Sigma^1_{n+1} \cup \Pi^1_{n+1} \]

\((M, S) \models \varphi(m, A) \iff (*M, *S) \models \varphi(m, *A)\)

Suppose \((M, S) \models \Pi^1_n\text{-CA}_0 + \Sigma^1_n\text{-AC}_0\), countable.

Then \(\exists^* M \supseteq M \exists^* S \text{ s.t. } S = *S \upharpoonright M\)

\[(*M, *S) \models \Pi^1_n\text{-CA}_0 + \Sigma^1_n\text{-AC}_0\]

\[\text{and } \exists^*: S \to *S \text{ as above.}\]

Suppose \((M, S) \models \Pi^1_n\text{-CA}_0 + \Sigma^1_n\text{-AC}_0\)

Then \(\exists^* M \supseteq M \exists^* S \text{ s.t. } (*M, *S) \models \Sigma^1_{n+1}\text{-AC}_0\)

\[\text{and } \exists^*: S \to *S \text{ as above.}\]
Other nonstandard methods

- Comparing nonstandard arithmetic with second-order arithmetic (Keisler, et al.)
- Nonstandardizing second-order arithmetic (Yokoyama)
- Analyzing the strength of transfer principles over very weak arithmetic (Impens, Sanders)
- Relating the existence of cuts or end-extensions of a nonstandard model of arithmetic to second order principles (Kaye, Wong)
THANK YOU