Reverse Mathematics and Non-Standard Methods

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Logic & Foundations

England | **Continent**

17-18c — Newton vs. Leibniz

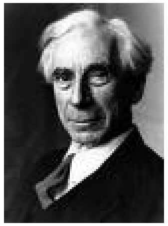
19c — Peacock, Herschel, Babbage (Symbolic algebra) | Gauss (Theory of numbers)

Boole, de Morgan | Cantor, Dedekind

Frege, Peano, Peirce

20c

Paradox



Russell — Logicism

Intuitionism — Poincare, Brouwer

Hilbert — Formalism

Turing | **Gödel**

Foundational Doctrines ⇨ Key Persons

1. Feasibility (Poly-time)	⇨	Ko
2. Finitism	⇨	Hilbert
3. Computability	⇨	Aberth, Pour-El
4. Constructivity	⇨	Bishop
5. Finitistic Reductionism	⇨	Hilbert
6. Predicativity	⇨	Weyl
7. Predic. Reductionism	⇨	Feferman
8. Semi-Finit. Consistency	⇨	Takeuti
9. Second Order Arith.	⇨	Hilbert
10. Ramified Type Theory	⇨	Russell
11. Constructible Universe	⇨	Gödel
12. Large Cardinals	⇨	Gödel

Foundational Doctrines ⇨ Formal Systems

- 1. Feasibility ⇨ S^1_2, BTFA
- 2. Finitism ⇨ PRA
- 3. Computability ⇨ RCA_0
- 4. Constructivity ⇨ RCA_0
- 5. Finitistic Reductionism ⇨ WKL_0
- 6. Predicativity ⇨ ACA_0
- 7. Predic. Reductionism ⇨ ATR_0
- 8. Semi-Finit. Consistency ⇨ Π^1_1-CA_0
- 9. Second Order Arith. ⇨ Z_2
- 10. Ramified Type Theory ⇨ PM
- 11. Constructible Universe ⇨ V=L
- 12. Large Cardinals ⇨ V_α

From Foundationalism to Reverse Math

Foundationalism (基礎付け主義):

Which systems are needed to do mathematics?

Reverse Mathematics (逆数学):

Which axioms are needed to prove a theorem?

Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics. The method can briefly be described as “going backwards from the theorems to the axioms”. This contrasts with the ordinary mathematical practice of deriving theorems from axioms. Wikipedia

Reverse Mathematics Phenomenon

Which axioms are needed to prove a theorem?

0. Fix a weak base system S (e.g. RCAo).

1. Pick a theorem Φ and formalize it in S.

2. Find a weakest axiom α to prove Φ in S.

3. Very often, we can show (over S) that
   α and Φ are logically equivalent

Second Order Arithmetic (Hilbert Arithmetic)

A first order theory of natural numbers and sets of them.

Standard Model: $(\omega \cup \wp(\omega); +, \cdot, 0, 1, <, \in)$

Second order arithmetic Z_2

$$
\begin{aligned}
= & \quad \textit{Basic axioms for } (+, \cdot, 0, 1, <) \\
+ & \quad \textit{Comprehension (CA)}: \ \exists X \forall x (x \in X \leftrightarrow \varphi(x)) \\
+ & \quad \textit{Induction}: \ \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)
\end{aligned}
$$

Classifying Formulas

- Bounded formulas (Σ^0_0), only with $\forall x < t, \exists x < t$
- Arithmetical formulas (Σ^1_0) , with no set quantifiers

$$\Sigma^0_n : \exists\overrightarrow{x_1}\forall\overrightarrow{x_2}\cdots Q\overrightarrow{x_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{bounded}.$$

$$\Pi^0_n : \forall\overrightarrow{x_1}\exists\overrightarrow{x_2}\cdots Q\overrightarrow{x_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{bounded}.$$

- Analytical formulas:

$$\Sigma^1_n : \exists\overrightarrow{X_1}\forall\overrightarrow{X_2}\cdots Q\overrightarrow{X_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{arithmetic}.$$

$$\Pi^1_n : \forall\overrightarrow{X_1}\exists\overrightarrow{X_2}\cdots Q\overrightarrow{X_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{arithmetic}.$$

Examples.

$\Delta_1^0 = \Sigma_1^0 \cap \Pi_1^0$	Recursive, Computable, Decidable
Σ_1^0	r.e. (recursively enumerable), c.e., the set of theorems of a formal system
Π_1^0	co-r.e., finitistic assertions, the Gödel sentence, consistency, the Goldbach conjecture
Π_2^0	1-consistency, the twin prime conj., Paris-Harrington, $P \neq NP$

Big five subsystems

$$RCA_0 = \Delta^0_1\text{-}CA + \Sigma^0_1\text{-}ind$$

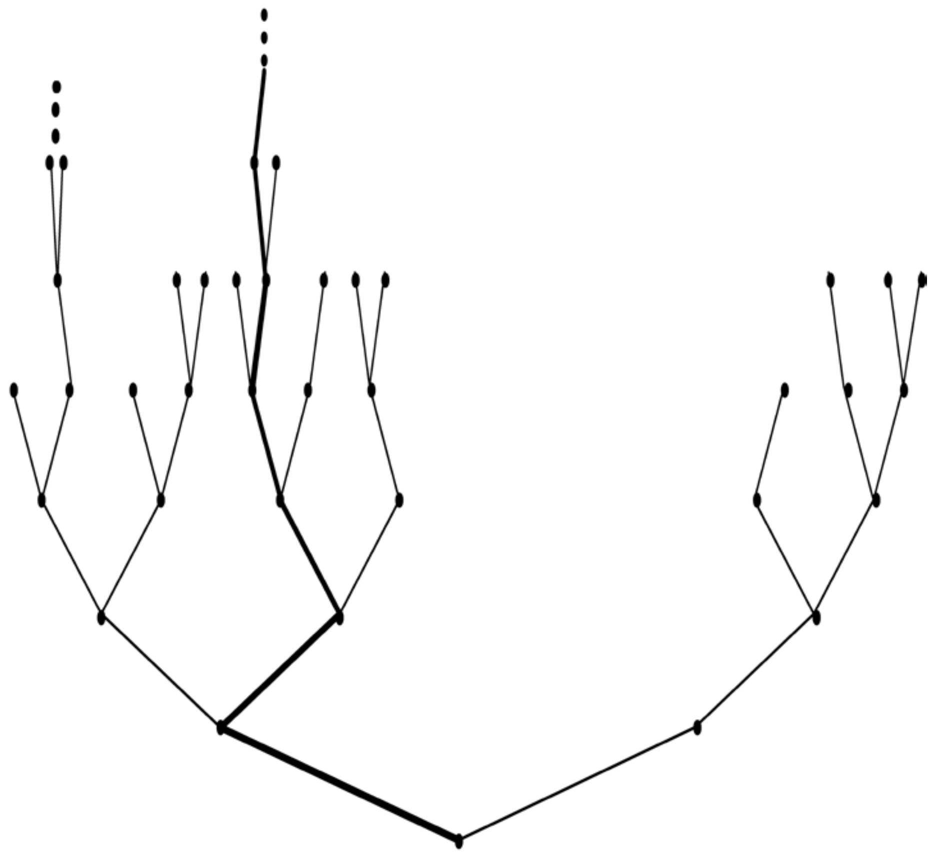
$$WKL_0 = RCA_0 + \ weak\ K\ddot{o}nig's\ lemma$$
$$(\Sigma^0_1 - \text{Separation})$$

$$ACA_0 = RCA_0 + \Sigma^0_1\text{-}CA$$

$$ATR_0 = RCA_0 + \text{transfinite iteration } of\ \Sigma^0_1\text{-}CA$$
$$(\Sigma^1_1 - \text{Separation})$$

$$\Pi^1_1\text{-}CA_0 = RCA_0 + \Pi^1_1\text{-}CA$$

Weak König's Lemma for infinite binary trees



Some results of R. M.

Over RCA_0

$$\begin{aligned} WKL_0 &\leftrightarrow \textit{the maximum principle} \\ &\leftrightarrow \textit{the Cauchy-Peano theorem} \\ &\leftrightarrow \textit{Brouwer's fixed point theorem} \end{aligned}$$

$$\begin{aligned} ACA_0 &\leftrightarrow \textit{the Bolzano-Weierstrass theorem} \\ &\leftrightarrow \textit{the Ascoli lemma} \end{aligned}$$

$$\begin{aligned} ATR_0 &\leftrightarrow \textit{the Luzin separation theorem} \\ &\leftrightarrow \Sigma^0_1\textit{-determinacy} \end{aligned}$$

$$\begin{aligned} \Pi^1_1\text{-}CA_0 &\leftrightarrow \textit{the Cantor-Bendixson theorem} \\ &\leftrightarrow \Sigma^0_1 \wedge \Pi^0_1\textit{-determinacy} \end{aligned}$$

Mathematics in the Big Five

	RCA_0	WKL_0	ACA_0	ATR_0	$\Pi^1_1 - CA_0$
analysis (separable):					
differential equations	×	×			
continuous functions	×	×	×		
completeness, etc.	×	×	×		
Banach spaces	×	×	×		×
open and closed sets	×	×		×	×
Borel and analytic sets	×			×	×
algebra (countable):					
countable fields	×	×	×		
commutative rings	×	×	×		
vector spaces	×		×		
Abelian groups	×		×	×	×
miscellaneous:					
mathematical logic	×	×			
countable ordinals	×		×	×	
infinite matchings		×	×	×	
the Ramsey property			×	×	×
infinite games			×	×	×

Defining the real number system $\mathbb{R}$

The following definitions are made in RCA_0.

- Using the pairing function, we define $\mathbb{N}$ and $\mathbb{Q}$.
- The basic operations on $\mathbb{N}$ and $\mathbb{Q}$ are also naturally defined.
- A real number is an infinite sequence $\{q_n\}$ of rationals such that $|q_n - q_m| \leq 2^{-n}$ for all $m > n$.
- The operations on $\mathbb{R}$ are also defined so that the resulting structure is a real closed order field.

Some results

1. Sakamoto-T (2004) proved

$RCA_0 \mid\!- \ \forall\sigma\ (RCOF \mid\!- \sigma \ \Rightarrow\ \mathbf{R} \mid\!= \sigma)$

with the help of the fundamental theorem of algebra

(strong FTA) $RCA_0 \vdash \forall p(x) \in \mathbb{Q}[x] \exists \overrightarrow{\alpha} \in \mathbb{C}^{<\mathbb{N}} p(x) = \Pi_i(x - \alpha_i)$

2. Simpson-T.-Yamazaki (2002) proved

$$WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma$$

$$for\ \sigma \equiv \forall X \exists! Y \varphi(X,Y)\ with\ \varphi\ arith.$$

Thus, it suffices to show strong FTA in WKL_0.

3. Strong FTA can be proved by a non-standard method in WKL_0.

Satisfaction on $\mathbb{R}$

- Simpson-T.-Yamazaki

  $Sat_{\mathbb{R}}(\ulcorner\varphi(\overrightarrow{x})\urcorner, \overrightarrow{\xi})$ *can be defined as a* Δ^0_2 *formula.*

  In RCA_0, $Sat_{\mathbb{R}}$ *satisfies the Tarski clauses for the standard formulas.*

- Sakamoto-T. (2004)

  In RCA_0, $Sat_{\mathbb{R}}$ *satisfies the Tarski clauses for all the formulas. In particular,*

  $$Sat_{\mathbb{R}}(\ulcorner\exists\overrightarrow{x}\varphi(\overrightarrow{x},\overrightarrow{y})\urcorner,\overrightarrow{\beta}) \leftrightarrow \exists\overrightarrow{\alpha}Sat_{\mathbb{R}}(\ulcorner\varphi(\overrightarrow{x},\overrightarrow{y})\urcorner,\overrightarrow{\alpha},\overrightarrow{\beta})$$

- The following fact (called *strong FTA*) is essential:

  $$RCA_0 \vdash \forall p(x) \in \mathbb{Q}[x]\exists\overrightarrow{\alpha} \in \mathbb{C}^{<\mathbb{N}} p(x) = \Pi_i(x-\alpha_i)$$

Applications of Sakamoto-T's result

$$RCA_0 \vdash \ Hilbert's\ Nullstellensatz:$$

$$p_1, \cdots, p_m \in \mathbb{C}[\overrightarrow{x}]\ have\ no\ common\ zeros$$

$$\Rightarrow \exists q_1 \cdots \exists q_m \in \mathbb{C}[\overrightarrow{x}]\ p_1 q_1 + \cdots + p_m q_m = 0$$

$$RCA_0 \vdash \ strong\ FTA$$

Conservation results

- Shoenfield:

$$ZF + V = L \vdash \sigma \Rightarrow ZF \vdash \sigma \;\; for \;\; \sigma \in \Sigma^1_2 \cup \Pi^1_2$$

- Barwise-Schlipf:

$$\Sigma^1_1\text{-}AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma \;\; for \;\; \sigma \in \Pi^1_2$$

- Harrington:

$$WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \;\; for \;\; \sigma \in \Pi^1_1$$

- Simpson-T.-Yamazaki (2002):

$$WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma$$

$$for \;\; \sigma \equiv \forall X \exists ! Y \varphi(X, Y) \;\; with \;\; \varphi \;\; arith.$$

Application of Simpson-T.-Yamazaki's result

The fundamental theorem of algebra (FTA):

Any complex polynomial of a positive degree has a unique factorization into linear terms.

By the STY result, we have

WKL_0 |- (strong) FTA ⇒ RCA_0 |- (strong) FTA.

By the usual mathematical argument, we have

WKL_0 |- FTA (for any particular standard polynomial).

Thus, we have

RCA_0 |- FTA (standard),

which is not enough for our purpose.

Non-Standard Models

Theorem (H. Friedman, Kirby-Paris)

$$Suppose\ M \models\ PRA,\ countable.$$

$$Suppose\ b \ll_M c\ (i.e.,\ f(b) <_M c\ for\ all\ prim.\ rec.\ f).$$

$$Then\ \exists I \subseteq_e M\ \ s.t.\ b \in I, c \notin I\ \ and\ I \models \mathrm{I}\Sigma_1$$

$$Moreover,\ if\ C(M) = \{X \subseteq M : \exists a \in M\ codes\ X\},$$

$$(I, C(M) \upharpoonright I) \models WKL_0.$$

Theorem (T.) A converse to the above holds.

$$Suppose\ \ (M, S) \models WKL_0,\ countable,\ \ M \neq \omega.$$

$$Then\ \exists^* M \supseteq_e M\ \ s.t.\ {}^*M \models \mathrm{I}\Sigma_1\ and\ S = C({}^*M) \upharpoonright M.$$

Self-Embedding Theorems

Thm. (self-embedding for WKL_0 , T. 1997)

$$Suppose\ (M,S) \models WKL_0,\ countable,\ M \neq \omega.$$

$$Then\ \exists I \subsetneq_e M\ s.t.\ (M,S) \simeq (I, S\Gamma I).$$

- History of self embedding results.

  $H.Friedman\ (1970's)\ for\ PA.$
  $Ressayre,\ Dimitracopoulous\ and\ Paris$
  $(1980's)\ for\ \mathrm{I}\Sigma_1.$

(Proof) By a back-and-forth argument.

Cor. $Suppose\ (M,S)\ \models WKL_0,\ countable,\ M \neq \omega.$

$$Then\ \exists^* M \supsetneq_e M, \exists^* S\ s.t.\ (^*M, {}^*S) \models WKL_0$$
$$and\ S = {}^*S \restriction M.$$

Application (the maximum principle)

$$WKL_0 \vdash \ Any\ cont.\ function\ f : [0,1] \to [0,1]\ has\ a\ max.$$

(Proof)

$V = (M, S)$	${}^*V = ({}^*M, {}^*S)$
$f: \ [0,1] \cap \mathbb{Q} \to [0,1]$ $\Longrightarrow$	${}^*f : \{q_i\}_{i<a} \to 2^b$
$[0,1]\cap\mathbb{Q} = \{q_i\}_{i\in M}$, $[0,1] = 2^M$	$(a, b \in {}^*M - M, f = {}^*f \cap M)$
	$\Downarrow$
${}^*m \cap M\ \ is\ \ \sup f$ $\Longleftarrow$	${}^*m = \max\{{}^*f(q_i)\}_{i<a}$

Application

$$WKL_0 \vdash \ Strong\ FTA.$$

(Proof) $V = (M, S)$ | ${}^*V = ({}^*M, {}^*S)$

$f : \mathbb{Q}[x] \to (\mathbb{C} \cap \mathbb{Q}^2)^{<\mathbb{N}}$ $\Longrightarrow$ ${}^*f : \{p_i\}_{i<a} \to ({}^*\mathbb{C} \cap {}^*\mathbb{Q}^2)^{<b}$

$||$

$\{p_i\}_{i \in M}$ *with infinite repetition*

$(a, b \in {}^*M - M, f = {}^*f \cap M)$

s.t. $f(p_i)$ *is a list of rational approximations of the roots of* p_i *with error* $< 2^{-i}$.

$\Downarrow$

${}^*f(p_{j_i}) \upharpoonright M$ *is the list of roots of* p_i. $\Longleftarrow$ $\{p_i\}_{i \in M} = \{p_{j_i}\}_{j_i \notin M, i \in M}$

Other applications

$WKL_0 \vdash$ *The Cauchy–Peano theorem* (*Tanaka*, 1997)

$WKL_0 \vdash$ *The existence of Haar measure*

for a compact group (*Tanaka-Yamazaki*, 2000)

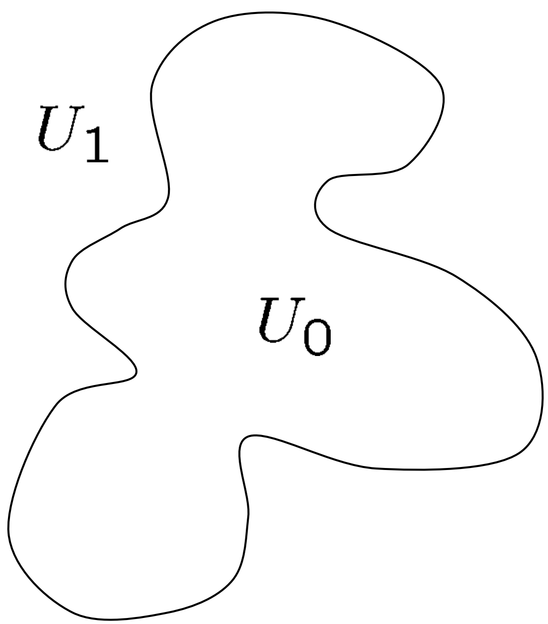
$WKL_0 \vdash$ *The Jordan curve theorem*

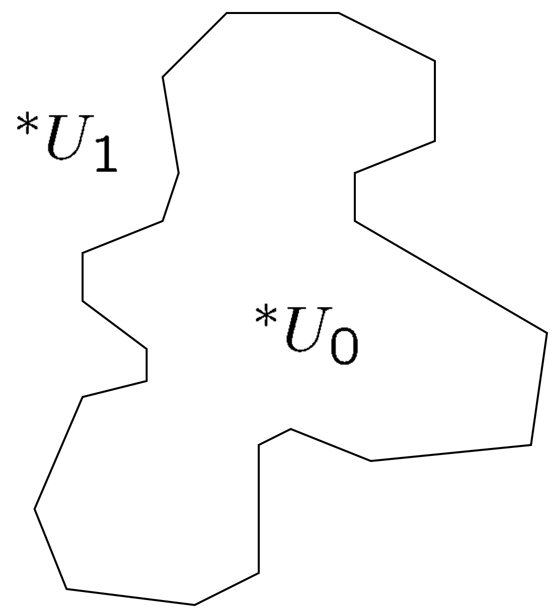
(*Sakamoto-Yokoyama*, 2007)

Application (Sakamoto, Yokoyama)

$$WKL_0 \vdash \ The\ Jordan\ Curve\ Theorem$$

(Proof) $V = (M, S)$ | ${}^*V = ({}^*M, {}^*S)$





Outer model method for ACA_0

$Suppose\ (M,S) \models ACA_0,\ countable,\ M \neq \omega.$

$Then\ \exists^* M \supsetneq_e M \exists^* S$

$$s.t.\ ({}^*M, {}^*S) \models ACA_0, S = {}^*S \restriction M$$

$and\ \exists * : S \to {}^*S\ \forall \varphi(x, X) \in \Sigma^1_1 \cup \Pi^1_1$

$$(M,S) \models \varphi(m, A) \leftrightarrow ({}^*M, {}^*S) \models \varphi(m, {}^*A)$$

This easily follows from

Theorem (Gaifman): $Every\ model\ M\ of\ PA\ has\ a\ conservative\ extension\ K,\ i.e.,\ (the\ sets\ definable\ in\ K) \restriction M\ =\ the\ sets\ definable\ in\ M.$

Applications

$$ACA_0 \vdash \textit{Any Cauchy sequence converges.}$$

(Proof) $V = (M, S)$ | $^*V = (^*M, ^*S)$

$\{a_i\}_{i \in M}$ *a Cauchy seq.* $\Longrightarrow$ $^*(\{a_i\}_{i \in M}) = \{(^*a)_i\}_{i \in ^*M}$.

Pick $j \in {^*M} - M$.

$\forall n \in M \exists m \in M \forall k > m$

$\forall n \exists m \forall k > m |a_k - b| < 2^{-n}$ $\Longleftarrow$ $|(^*a)_k - (^*a)_j| < 2^{-n}$.

$^*b \approx (^*a)_j$.

- $ACA_0 \vdash$ *The Riemann mapping theorem.*

(*Yokoyama*)

Outer model method II for ACA_0

$Suppose\ (M,S) \models \Sigma^1_1\text{-}\mathsf{AC}_0,\ countable.$

$Then\ \exists {}^*M \supsetneq_e M \exists {}^*S$

$$s.t.\ ({}^*M, {}^*S) \models \Sigma^1_1\text{-}\mathsf{AC}_0, S = {}^*S \restriction M$$

$and\ \exists * : S \to {}^*S\ \forall \varphi(x,X) \in \Sigma^1_2 \cup \Pi^1_2$

$$(M,S) \models \varphi(m,A) \leftrightarrow ({}^*M, {}^*S) \models \varphi(m, {}^*A)$$

This can be used with

$$\Sigma^1_1\text{-}AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma\ \ for\ \ \sigma \in \Pi^1_2$$

Some general results (due to Schmerl)

Suppose $(M,S) \models \Sigma^1_n\text{-}\mathsf{AC}_0$, countable.

$$Then\ \exists^* M \supsetneq_e M\ \exists^* S\ \ s.t.\ \ S = {}^*S \upharpoonright M$$

$$and\ \exists * : S \to {}^*S\ \forall \varphi(x,X) \in \Sigma^1_{n+1} \cup \Pi^1_{n+1}$$

$$(M,S) \models \varphi(m,A) \leftrightarrow ({}^*M, {}^*S) \models \varphi(m, {}^*A)$$

Suppose $(M,S) \models \Pi^1_n\text{-}\mathsf{CA}_0 + \Sigma^1_n\text{-}\mathsf{AC}_0$, countable.

$$Then\ \exists^* M \supsetneq_e M\ \exists^* S\ \ s.t.\ \ S = {}^*S \upharpoonright M$$

$$({}^*M, {}^*S) \models \Pi^1_n\text{-}\mathsf{CA}_0 + \Sigma^1_n\text{-}\mathsf{AC}_0$$

$$and\ \exists * : S \to {}^*S\ \ as\ above.$$

Suppose $(M,S) \models \Pi^1_n\text{-}\mathsf{CA}_0 + \Sigma^1_n\text{-}\mathsf{AC}_0$

$$Then\ \exists^* M \supsetneq M\ \exists^* S\ \ s.t.\ ({}^*M, {}^*S) \models \Sigma^1_{n+1}\text{-}\mathsf{AC}_0$$

$$and\ \exists * : S \to {}^*S\ \ as\ above.$$

Other nonstandard methods

- Comparing nonstandard arithmetic with second-order arithmetic (Keisler, et al.)
- Nonstandardizing second-order arithmetic (Yokoyama)
- Analyzing the strength of transfer principles over very weak arithmetic (Impens, Sanders)
- Relating the existence of cuts or end-extentions of a nonstandard model of arithmetic to second order principles (Kaye, Wong)

THANK YOU