Hierarchies of Sets in Classical and Constructive Set Theories

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Creating the Universe in Three Simple Steps

1. Start with the empty set $V_0 = \emptyset$.
2. Take the powerset of what you have so far (i.e. take all subsets).
3. Go to step 2.

The Cumulative Hierarchy

$V_\alpha = \bigcup_{\beta < \alpha} P(V_\beta)$

$V = \bigcup_\alpha V_\alpha$
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Set Theory with Constructive Logic

- Classical Set Theory can serve as a framework for all classical mathematics
- The concept of set is just as compatible with constructivism
- Use set theory with constructive logic to serve as a framework for constructive mathematics
- For CZF, take same language and axioms as ZF
Classical Set Theory can serve as a framework for all classical mathematics

The concept of set is just as compatible with constructivism

Use set theory with constructive logic to serve as a framework for constructive mathematics

For CZF, take same language and axioms as ZF

But there is some ambiguity in how exactly to state the axioms: Classically equivalent formulations of the axioms can become constructively different.
The following are classically equivalent:

**Powerset**

\[ \forall a \exists b. \; b = \mathcal{P}(a) := \{x \mid x \subseteq a\} \]

**Binary Exponentiation**

\[ \forall a \exists b. \; b = a^2 := \{f \mid f : a \to 2\} \]
Powerset and Exponentiation

The following are classically equivalent:

**Powerset**

\[ \forall a \exists b. \ b = \mathcal{P}(a) := \{ x | x \subseteq a \} \]

**Binary Exponentiation**

\[ \forall a \exists b. \ b = ^a 2 := \{ f | f : a \to 2 \} \]

CZF instead includes the axiom

**Fullness**

\[ \forall A, B \exists C \forall R. \ \forall x \in A \exists y \in B (x, y) \in R \rightarrow \exists R' \in C.R' \subseteq R \wedge \forall x \in A \exists y \in B(x, y) \in R' \]
## ZF and CZF

<table>
<thead>
<tr>
<th>ZF</th>
<th>CZF</th>
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<tbody>
<tr>
<td>Extensionality</td>
<td>Extensionality</td>
</tr>
<tr>
<td>Foundation</td>
<td>∈-Induction</td>
</tr>
<tr>
<td>Pairing</td>
<td>Pairing</td>
</tr>
<tr>
<td>Union Axiom</td>
<td>Union Axiom</td>
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<tr>
<td>Infinity</td>
<td>Infinity</td>
</tr>
<tr>
<td>Separation</td>
<td>Separation for $\Delta_0$-formulae</td>
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<tr>
<td>Replacement</td>
<td>Strong Collection</td>
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<tr>
<td>Powerset</td>
<td>Fullness</td>
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The Cumulative Hierarchy in a Constructive Context

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}(V_\beta), \quad V = \bigcup_{\alpha} V_\alpha$$
The Cumulative Hierarchy in a Constructive Context

\[ V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}(V_\beta), \quad V = \bigcup_{\alpha} V_\alpha \]

- Constructively, the collection of all subsets (the "powerset") is too unstructured to be accepted as a set.
- Consequently, the \( V_\alpha \) are not sets but only classes.
- The description of the universe as \( \bigcup_\alpha V_\alpha \) still holds true.
- But it loses much of its power.
A Modified Hierarchy for Constructive Purposes

Definition

Let for $\alpha \in O_n$

$$\tilde{V}_\alpha = \bigcup_{\beta < \alpha} \{X \subseteq \tilde{V}_\beta\}$$
**A Modified Hierarchy for Constructive Purposes**

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Let for $\alpha \in O_n$

$$\tilde{V}_\alpha = \bigcup_{\beta < \alpha} \{X \subseteq \tilde{V}_\beta \mid \forall x \in \tilde{V}_\beta. \{0| x \in X\} \in \{0, 1\}\}$$
Definition

Let for $\alpha \in \mathbb{O}_n$

$$\tilde{V}_\alpha = \bigcup_{\beta < \alpha} \{ X \subseteq \tilde{V}_\beta \mid \forall x \in \tilde{V}_\beta. \{0|x \in X\} \in \{0, 1\} \cup \tilde{V}_\beta \}$$

be defined by recursion over the ordinals.
### Finite Stages

For $n$ finite, $\tilde{V}_n$ is also finite and has $2^n - 1$ elements:

<table>
<thead>
<tr>
<th>$\tilde{V}_0$</th>
<th>$\tilde{V}_1$</th>
<th>$\tilde{V}_2$</th>
<th>$\tilde{V}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>${\emptyset}$</td>
<td>${\emptyset, {\emptyset}}$</td>
<td>${\emptyset, {\emptyset}, {{\emptyset}}, {\emptyset, {\emptyset}}}$</td>
</tr>
</tbody>
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### Between 0 and 1

For $0 \leq \alpha \leq 1$, and $\alpha < \beta$: $\tilde{V}_\alpha = \{0|0 \in \alpha\} = \alpha \in \tilde{V}_\beta$
The Good Stuff

Theorem

1. For all $\alpha$, the class $\tilde{V}_\alpha$ is actually a set.
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Theorem

1. For all $\alpha$, the class $\tilde{V}_\alpha$ is actually a set.

2. $V = \bigcup_\alpha \tilde{V}_\alpha$.
   More precisely, there is a class function $\tilde{rk} : V \rightarrow O_n$ such that

   $\forall a. a \in \tilde{V}_{\tilde{rk}(a)}$
Quantifier Elimination

As an immediate consequence of this, all unbounded quantification in CZF can be replaced by bounded quantification and quantification over the ordinals.

Theorem

There is a definitional extension of CZF and a primitive recursive mapping $\phi \mapsto \phi^*$ of formulas, such that

1. All quantifiers in $\phi^*$ are either bounded by sets
   $(\forall x \in a, \exists x \in a)$ or range over the class of ordinals
   $(\forall \alpha \in O_n, \exists \alpha \in O_n)$

2. $\phi$ and $\phi^*$ are provably equivalent.
Large cardinals have become a central topic in classical set theory

The classical concept of cardinals does not fit well with constructive set theory

Instead of lifting the properties of a large cardinal $\kappa$ to a constructive setting, better lift the properties of the universe $V_\kappa$. 
Large Cardinals and Large Sets

- Large cardinals have become a central topic in classical set theory
- The classical concept of cardinals does not fit well with constructive set theory
- Instead of lifting the properties of a large cardinal $\kappa$ to a constructive setting, better lift the properties of the universe $V_\kappa$.

Inaccessible Sets

A set $I$ is called inaccessible iff

$$(I, \in) \models CZF_2$$
The modified hierarchy interacts very well with large sets:

- Inaccessible sets are closed under the mappings $\alpha \mapsto \tilde{V}_\alpha$ and $a \mapsto \tilde{rk}(a)$.
- Every inaccessible set $I$ is equal to some $\tilde{V}_\alpha$, in fact
  \[ I = \tilde{V}_{I \cap \alpha} = \tilde{V}_{rk(I)} = \tilde{V}_{\tilde{rk}(I)} \]
- So inaccessible sets are uniquely determined by the ordinals they contain.
- The class of all inaccessible sets is isomorphic to a subclass of the ordinals with the isomorphism just being $I \mapsto rk(I)$. 
Two Definitions of Mahlo

The constructive definition works with constructively powerful concepts like total relations and reflections:

**Constructive Definition of Mahloness**

An inaccessible set $M$ is called Mahlo if every total relation $R$ with $\forall a \in M \exists b \in M. aRb$ is reflected at an inaccessible point $I \in M$, i.e. $\forall a \in I \exists b \in I. aRb$. 

The classical definition uses classically successful concepts like stationary sets and clubs:

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**Classical Definition of Mahloness**

An inaccessible set $M$ is called Mahlo if the inaccessibles within $M$ are stationary, i.e. if every club has an inaccessible member.
It can be proved that a constructively useful definition of Mahlo sets is equivalent to the classical one:

**Theorem (DC)**

For an inaccessible set $M$, the following are equivalent:

1. $M$ is constructively Mahlo.
2. $M$ is classically Mahlo.

A similar result holds for the entire hierarchy of $\alpha$–Mahlo sets. There is also a choice free version of the theorem using the RRS-property.
The new hierarchy
- describes the structure of the set theoretic universe in a useable way
- lets constructive set theory make more fruitful use of ordinals as a tool for handling arbitrary sets
- can be applied to get new and interesting results about large sets in constructive set theory:
  - structure of inaccessible sets
  - characterisation of Mahlo sets
  - maybe also useful for weakly compact sets, 2-strong sets...