Hierarchies of Sets in Classical and Constructive Set Theories

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The Cumulative Hierarchy

$$V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta)$$

$$V = \bigcup_{\alpha} V_\alpha$$

Set Theory with Constructive Logic

- Classical Set Theory can serve as a framework for all classical mathematics
- The concept of set is just as compatible with constructivism
- Use set theory with constructive logic to serve as a framework for constructive mathematics
- For CZF, take same language and axioms as ZF

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But...

But there is some ambiguity in how exactly to state the axioms: Classically equivalent formulations of the axioms can become constructively different.

Powerset and Exponentiation

The following are classically equivalent:

Powerset

$$\forall a \exists b.\ b = \mathcal{P}(a) := \{x | x \subseteq a\}$$

Binary Exponentiation

$$\forall a \exists b.\ b = {}^{a}2 := \{f | f : a \to 2\}$$

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CZF instead includes the axiom

Fullness

$$\forall A, B \exists C \forall R. \quad \forall x \in A \exists y \in B\ (x, y) \in R \to \exists R' \in C. R' \subseteq R \wedge \forall x \in A \exists y \in B (x, y) \in R'$$

ZF and CZF

ZF	CZF
Extensionality	Extensionality
Foundation	$\in$-Induction
Pairing	Pairing
Union Axiom	Union Axiom
Infinity	Infinity
Separation	Separation for Δ_0-formulae
Replacement	Strong Collection
Powerset	Fullness

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- Constructively, the collection of all subsets (the "powerset") is too unstructured to be accepted as a set.
- Consequently, the V_α are not sets but only classes.
- The description of the universe as $\bigcup_\alpha V_\alpha$ still holds true.
- But it loses much of its power.

A Modified Hierarchy for Constructive Purposes

Definition

Let for $\alpha \in \mathrm{O_n}$

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$$\widetilde{V}_\alpha = \bigcup_{\beta<\alpha} \{X \subseteq \widetilde{V}_\beta \mid \forall x \in \widetilde{V}_\beta.\ \{0|x \in X\} \in \{0,1\} \cup \widetilde{V}_\beta\}$$

be defined by recursion over the ordinals.

Exploring the Modified Hierarchy

Finite Stages

For n finite, $\widetilde{V}_n$ is also finite and has $2^n - 1$ elements:

$$\begin{aligned}
\widetilde{V}_0 &= \emptyset \\
\widetilde{V}_1 &= \{\emptyset\} \\
\widetilde{V}_2 &= \{\emptyset, \{\emptyset\}\} \\
\widetilde{V}_3 &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\dots
\end{aligned}$$

Between 0 and 1

For $0 \leq \alpha \leq 1$, and $\alpha < \beta$: $\widetilde{V}_\alpha = \{0 | 0 \in \alpha\} = \alpha \in \widetilde{V}_\beta$

The Good Stuff

Theorem

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Theorem

1. For all α, the class $\widetilde{V}_\alpha$ is actually a set.
2. $V = \bigcup_\alpha \widetilde{V}_\alpha$.
   More precisely, there is a class function $\widetilde{rk} : V \to \mathrm{O_n}$ such that

$$\forall a . a \in \widetilde{V}_{\widetilde{rk}(a)}$$

Quantifier Elimination

As an immediate consequence of this, all unbounded quantification in CZF can be replaced by bounded quantification and quantification over the ordinals.

Theorem

There is a definitional extension of CZF and a primitive recursive mapping $\phi \mapsto \phi^*$ of formulas, such that

1. *All quantifiers in ϕ^* are either bounded by sets ($\forall x \in a, \exists x \in a$) or range over the class of ordinals ($\forall \alpha \in \mathrm{O_n}, \exists \alpha \in \mathrm{O_n}$)*
2. *ϕ and ϕ^* are provably equivalent.*

Large Cardinals and Large Sets

- Large cardinals have become a central topic in classical set theory
- The classical concept of cardinals does not fit well with constructive set theory
- Instead of lifting the properties of a large cardinal κ to a constructive setting, better lift the properties of the universe V_κ.

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Inaccessible Sets

A set I is called inaccessible iff

$$(I, \in) \models CZF_2$$

The Structure of Large Sets

The modified hierarchy interacts very well with large sets:

- Inaccessible sets are closed under the mappings $\alpha \mapsto \widetilde{V}_\alpha$ and $a \mapsto \widetilde{rk}(a)$.
- Every inaccessible set I is equal to some $\widetilde{V}_\alpha$, in fact

$$I = \widetilde{V}_{I \cap \mathrm{On}} = \widetilde{V}_{rk(I)} = \widetilde{V}_{\widetilde{rk}(I)}$$

- So inaccessible sets are uniquely determined by the ordinals they contain.
- The class of all inaccessible sets is isomorphic to a subclass of the ordinals with the isomorphism just being $I \mapsto rk(I)$.

Two Definitions of Mahlo

The constructive definition works with constructively powerful concepts like total relations and reflections:

Constructive Definition of Mahloness

An inaccessible set M is called Mahlo if every total relation R with $\forall a \in M \exists b \in M.aRb$ is reflected at an inaccessible point $I \in M$, i.e. $\forall a \in I \exists b \in I.aRb$.

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The classical definition uses classically successful concepts like stationary sets and clubs:

Classical Definition of Mahloness

An inaccessible set M is called Mahlo if the inaccessibles within M are stationary, i.e. if every club has an inaccessible member.

A Characterization of Mahlo Sets

It can be proved that a constructively useful definition of Mahlo sets is equivalent to the classical one:

Theorem (DC)

For an inaccessible set M, the following are equivalent:

1. M is constructively Mahlo.
2. M is classically Mahlo.

A similar result holds for the entire hierarchy of $\alpha-$Mahlo sets. There is also a choice free version of the theorem using the RRS-property.

Summary and Outlook

The new hierarchy

- describes the structure of the set theoretic universe in a useable way
- lets constructive set theory make more fruitful use of ordinals as a tool for handling arbitrary sets
- can be applied to get new and interesting results about large sets in constructive set theory:
  - structure of inaccessible sets
  - characterisation of Mahlo sets
  - maybe also useful for weakly compact sets, 2-strong sets...