# Call-by-Need Reduction for Membership Conditional Term Rewriting Systems

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**Abstract**

This paper proves that, for a membership conditional term rewriting system (MCTRS), (1) a reducible term has a needed redex if the MCTRS is nonoverlapping, and (2) whether a redex is nv-needed is decidable.

## 1 Introduction

A membership conditional term rewriting system (MCTRS) is a term rewriting system (TRS) in which substitutions are taken from some specific sets, typically, the set of normal forms. A normal MCTRS, which requires substitutions to be in normal form for each nonlinear variable, is a useful example; this system can specify the positive part of the equality class of functional programming (e.g., Haskell, ML) without type information. (The negative part of the equality class cannot be deduced without algebraic information about the construction of the type.)

A nonoverlapping normal MCTRS is the natural extension of an orthogonal TRS to nonlinearity, and it retains many nice properties, such as Parallel Move Lemma and confluence [14]. In addition to these properties, this paper investigates call-by-need reduction for a normal MCTRS. In general, a nonlinear TRS does not have needed redexes even if it is nonoverlapping; for instance,

$$\{d(x,x) \to a, f(y,z) \to b, c \to d\}$$

is a nonoverlapping (and also strongly normalizing, right-ground) nonlinear TRS and $d(f(c,z), f(d,z))$ does not have needed redexes. Thus, the membership restriction is essential; there seems to be no other choice when one explores the existence of needed redexes in nonlinear TRSs. In fact, the membership condition precisely corresponds to the proof techniques in [10].

The main results are:

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(i) A reducible term has a needed redex for a nonoverlapping normal MC-TRS.

(ii) Reachability and normalizability for a right-ground normal MCTRS are decidable.

(iii) Whether a redex is nv-needed is decidable for a normal MCTRS.

where nv-neededness approximates neededness by relaxing the rewrite relation such that variables in the right-hand-side of a rule may be instantiated by any terms.

It is worth remarking that, unlike left-linear TRSs, *modern* tree automata techniques [2,5,11] fail to produce decidability results of normal MCTRSs. This is because the set of normal forms is *not* regular; i.e., the set of normal forms of a normal MCTRS is same to that of the underlying TRS, and the set of normal forms of a nonlinear TRS is known to be not regular [8].

Section 2 presents basic notations and Section 3 introduces previous results on confluence of a normal MCTRS. Section 4 shows that a reducible term has a needed redex if a normal MCTRS is nonoverlapping, Section 5 shows that the reachability and normalizability of a right-ground normal MCTRS is decidable, and Section 6 shows that whether a redex is nv-needed is decidable for a normal MCTRS. Section 7 concludes the paper and discusses topics for future research.

## 2 Preliminaries

I assume that readers are familiar with rewriting terminology; for details, refer to [7]. This section explains our notations. Throughout the paper, we will consider only finite term rewriting systems (TRSs).

We will denote the set of function symbols by $\mathcal{F}$, the set of $n$-ary function symbols by $\mathcal{F}_n$, the set of variables by $\mathcal{V}$, and the set of terms over $\mathcal{F}$ and $\mathcal{V}$ by $T(\mathcal{F}, \mathcal{V})$. A term without variables is a ground term, and the set of ground terms is denoted by $T(\mathcal{F})$. A term $t$ is *linear* if each variable $x$ appears in $t$ at most once, and a variable $x$ in a term $t$ is *linear* if $x$ appears once in $t$. The set of variables that appear in a term $t$ is denoted by $\mathcal{V}ar(t)$, and the set of nonlinear variables that appear in a term $t$ is denoted by $\mathcal{V}ar^{nl}(t)$. For a (possible nonlinear) term $t$, $\bar{t}$ is a linearization of $t$, i.e., $\bar{t}$ is obtained by replacing all occurrences of nonlinear variables in $t$ by distinct fresh variables (thus, $\bar{t}$ is linear).

We denote the set of all positions in a term $t$ by $Pos(t)$, the subterm occurring at $p$ in $t$ by $t|_p$, and the head symbol of $t$ by $head(t) (\in \mathcal{F} \cup \mathcal{V})$.

For terms $t$, $s$ and position $p \in Pos(t)$, $t[s]_p$ is the term obtained from $t$ by replacing the subterm at $p$ with $s$. For positions $p, q$ and a set $U$ of positions, we denote $p < q$ if $p$ is a proper prefix of $q$, and $p \perp q$ if neither $p < q$, $p = q$, nor $p > q$, $U < q$ if $\exists p \in U\ p < q$, and $U \leq q$ if $\exists p \in U\ p \leq q$. For a term $t$ and a variable $x$, we denote the set of positions in $t$ by $\mathcal{P}os(t)$, the set of positions of function symbols in $t$ by $\mathcal{P}os_{\mathcal{F}}(t)$, the set of positions of variables in $t$ by $\mathcal{P}os_{\mathcal{V}}(t)$, the set of positions where $x$ occurs in $t$ by $\mathcal{P}os(t,x)$, and the number of positions in $\mathcal{P}os(t)$ (i.e., size of $t$) by $|t|$. If $s$ is a (proper) subterm of $t$, we denote $s \trianglelefteq t$ $(s \triangleleft t)$.

We denote $\xrightarrow{p}_{\mathcal{R}}$ if a rewrite $\rightarrow_{\mathcal{R}}$ occurs at the position $p$; $\xrightarrow{>p}_{\mathcal{R}}$ if a rewrite $\rightarrow_{\mathcal{R}}$ occurs at the position larger than $p$; and $\xrightarrow{\geq p}_{\mathcal{R}}$ if a rewrite $\rightarrow_{\mathcal{R}}$ occurs at the position larger than or equal to $p$. For a set $P$ of positions, we denote $\xrightarrow{\geq P}_{\mathcal{R}}$ if a rewrite $\rightarrow_{\mathcal{R}}$ occurs at the position larger than or equal to some $p \in P$.

A term without redexes (of $\rightarrow_{\mathcal{R}}$) is a *normal form* (more specifically, $\mathcal{R}$-*normal form*), and the set of $\mathcal{R}$-normal forms is denoted by $NF_{\mathcal{R}}$. We will often omit the index $\mathcal{R}$ in $NF_{\mathcal{R}}$ and $\rightarrow_{\mathcal{R}}$ if they are apparent from the context.

## 3 Confluence of membership conditional TRS

A pair of rules $(l \rightarrow r, l' \rightarrow r')$ is *overlapping* if there exists a position $p$ such that

- $p \in \mathcal{P}os_{\mathcal{F}}(l)$,
- there exist substitutions $\sigma, \sigma'$ such that $l|_p\ \sigma = l'\sigma'$, and
- either $p \neq \epsilon$, or $l \rightarrow r$ and $l' \rightarrow r'$ are different rules.

A TRS is *nonoverlapping* if no pairs of rules in $\mathcal{R}$ are overlapping.

**Theorem 3.1 ([6])** *A left-linear nonoverlapping TRS is confluent.*

Without left-linearity, confluence may fail even for nonoverlapping TRSs. For instance, $\mathcal{R}_1$ below is nonoverlapping, but not confluent.

$$\mathcal{R}_1 = \left\{ \begin{array}{ll} d(x,x) & \rightarrow 0 \\ d(x,f(x)) & \rightarrow 1 \\ 2 & \rightarrow f(2) \end{array} \right\}$$

When a TRS is nonlinear, some restriction is required to recover confluence. A membership conditional TRS is such an example [14].

**Definition 3.2** A *membership conditional* TRS (MCTRS) $\mathcal{R}$ is a finite set of conditional rewrite rules

$$l \rightarrow r \Leftarrow C$$

where the condition $C$ is $\wedge_{x \in V} x \in T_x (\subseteq T(\mathcal{F},\mathcal{V}))$ for $V \subseteq \mathcal{V}ar(l)$.

An MCTRS $\mathcal{R}$ is *normal* if $C$ is $\wedge_{x \in V} x \in NF_{\mathcal{R}}$ and and $\mathcal{V}ar^{nl}(l) \subseteq V \subseteq \mathcal{V}ar(l)$.

**Remark 3.3** Since the membership condition is non-monotonic wrt the inclusion of rewrite relations, it looks contradictory. But, this is not true; by induction on the size of a term, the rewrite relation is well-defined (refer to Lemma 4.1 in [14]).

**Theorem 3.4** [14] *A nonoverlapping normal MCTRS is confluent.*

**Example 3.5** The normal MCTRS, $\mathcal{R}_1$ plus additional membership conditions,

$$\left\{ \begin{array}{lll} d(x,x) & \to 0 & x \in NF \\ d(x,f(x)) & \to 1 & x \in NF \\ 2 & \to f(2) & \end{array} \right\}$$

is confluent.

## 4 Needed redex of nonoverlapping normal MCTRS

A term may have several redexes. A *reduction strategy* is the choice of a redex to rewrite (i.e., a function from a term to a redex in a term). Two especially important issues are, the normalizing strategy, which guarantees reaching a normal form (if one exists), and the optimal strategy, which selects a needed redex.

A redex is *needed* if either itself or its descendant is contracted in every rewrite sequence to a normal form.

In general, needed redexes may not exist. However, a left-linear nonoverlapping TRS has a needed redex in a term that is not in normal form (although it may be *not* computable). The idea in [10] is; instead of a needed redex, a root-needed redex is considered. A redex is *root-needed* if either itself or some of its descendants are contracted in every rewrite sequence to a root-stable form (i.e., a term that cannot be reduced to a redex). Since a normal form is root-stable, a root-needed redex is a needed redex. In this section, similar to [10], we show that a reducible term has a needed redex if a normal MCTRS is nonoverlapping.

For a rewrite sequence $A:\ t_0 \to \cdots \to t_n$ and $0 \leq i \leq j \leq n$, $B:\ t_i \to \cdots \to t_j$ is a subsequence, $C:\ t_0 \to \cdots \to t_i$ is a *prefix sequence*, and $D:\ t_i \to \cdots \to t_n$ is a *suffix sequence*. For rewrite sequences $A:\ s \to^* t$ and $B:\ t \to^* u$, the concatenation is denoted by $A;B:\ s \to^* u$.

**Definition 4.1** A term $s(\trianglelefteq t)$ is *root-stable* if for any $s'$ with $s \to^* s'$, $s'$ is not a redex.

**Lemma 4.2 (Lemma 2.1 [10])** *If $\to$ is confluent, $\stackrel{>\epsilon}{\to}$ is also confluent.*

**Lemma 4.3 (Lemma 3.2 [10])** *If a term $s$ is root-stable, each term $t$ with $s \to^* t$ is also root-stable.*

Lemma 4.2 and 4.3 hold for any TRSs. The proof of Lemma 3.3 and 4.2 in [10] require the left-linear and nonoverlapping properties [2] to guarantee

- confluence, and
- if $s$ is a redex, $t$ with $s \stackrel{>\epsilon}{\to}^* t$ is also a redex.

[2] More precisely, the requirement of Lemma 3.3 in [10] can be relaxed to almost nonoverlapping.

A nonoverlapping normal MCTRS also satisfies these requirements, and the same proofs in [10] give the following Lemmas 4.4 and 4.5.

**Lemma 4.4** *For a nonoverlapping normal MCTRS $\mathcal{R}$, if a term $t$ is root-stable, each term $s$ with $s \overset{>\epsilon}{\rightarrow}{}^* t$ is also root-stable.*

**Lemma 4.5** *For a nonoverlapping normal MCTRS $\mathcal{R}$ and a term $t$, if there exist a rule $l \to r \in \mathcal{R}$ and a substitution $\sigma$ such that $t \overset{>\epsilon}{\rightarrow}{}^* l\sigma$ then $l$ is uniquely determined regardless of reduction sequences.*

Similar to Theorem 4.3 in [10], which states that for a left-linear nonoverlapping TRS a term not root-stable has a root-needed redex, the next theorem holds.

**Theorem 4.6** *For a nonoverlapping normal MCTRS, a term that is not root-stable has a root-needed redex.*

**Proof.** By induction on the size of a term. Without loss of generality, we can assume that $t$ is neither a root-stable term nor a redex. Assume $t \to^+ t'$ for some root-stable form $t'$. From lemma 4.4, a reduction sequence $A : t \to^* t'$ contains a reduction $A : t \overset{>\epsilon}{\rightarrow}{}^+ \Delta_A \overset{\epsilon}{\rightarrow} t'' \to^* t'$ at the position $\epsilon$. From lemma 4.5, a rule used in $\Delta_A$ is uniquely determined. We denote it by $l \to r \Leftarrow C$ where $C = \wedge_{x \in V} x \in NF$ with $\mathcal{V}ar^{nl}(l) \subseteq V \subseteq \mathcal{V}ar(l)$.

Let $P$ be the set of positions of proper non-root-stable subterms in $t$. There are two cases; (1) $P \cap \mathcal{P}os_{\mathcal{F}}(l) \neq \phi$, and (2) $P \cap \mathcal{P}os_{\mathcal{F}}(l) = \phi$. For (1) [3], let $p$ be a minimal position in $P \cap \mathcal{P}os_{\mathcal{F}}(l)$. Since $t|_p \subset t$, from induction hypothesis, there exists a root-needed redex $\Delta$ in $t|_p$. We claim that $\Delta$ is contracted in the subsequence $B : t \overset{>\epsilon}{\rightarrow}{}^+ \Delta_A$. From minimality of $p$, $t|_p$ must be rewritten to $\Delta_A|_p$, which is root-stable (because $\mathcal{R}$ is nonoverlapping). Thus, $\Delta$ is root-needed in $t$.

For (2), if each $p \in P$ is $p \in \mathcal{P}os(l, x)$ for some $x \in \mathcal{V}ar(l) \setminus V$, $t$ is a redex; thus, root-needed because $\mathcal{R}$ is nonoverlapping. Assume there exists $p \in \mathcal{P}os(l, x)$ for $x \in V$. Let $B$ be a subsequence of $A$ such that $B : t|_p \to^+ \Delta_A|_p$. Since $C = \wedge_{x \in V} x \in NF$, $\Delta_A|_p$ is a normal form. From induction hypothesis, $t|_p$ has a root-needed redex, and this is also root-needed in $t$. □

**Corollary 4.7** *For a nonoverlapping normal MCTRS, a reducible term has a needed redex.*

**Remark 4.8** Since a nonoverlapping normal MCTRS satisfies the Parallel Move Lemma, the same proofs in Section 5 in [10] work for a nonoverlapping normal MCTRS. Thus, the repeated reduction of root-needed redexes is a root-normalizing reduction strategy.

Further, since a nonoverlapping normal MCTRS is confluent, a context-free root-normalizing reduction strategy is a normalizing reduction strategy (refer to Theorem 6.5 in [10]), where a reduction strategy is context-free if the

[3] For (1), the proof is same as in theorem 4.3 in [10].

choice of a redex in a root-stable term is reduced to the choice of a redex in each direct subterm.

## 5 Decidable results for right-ground normal MCTRS

In [12], Oyamaguchi proved that reachability and joinability are decidable for a (possibly non-left-linear) right-ground TRS. Similarly, in this section, we show that reachability and normal joinability are decidable for a right-ground normal MCTRS. The main difference is that we use normal joinability $\{t_1, \cdots, t_n\}_N$ instead of joinability $\{t_1, \cdots, t_n\}_J$, as used in [12]. Otherwise, the translation of the proof in [12] is quite straight forward.

We say:

- For terms $s, t$, $s$ is *reachable* to $t$ if $s \to^* t$, and denoted by $(s, t)_R$.
- Terms $t_1, \cdots, t_n$ are *joinable in normal form* if there exists $t \in NF_R$ with $t_i \to^* t$ for each $i$, and denoted by $\{t_1, \cdots, t_n\}_N$.

We call $s \to^* t$ a witness of $(s, t)_R$, and the existence of $t \in NF_{\mathcal{R}}$ with $t_i \to^* t$ for each $i$, a witness of $\{t_1, \cdots, t_n\}_N$. Note that normalizability of a term $t$ is expressed as $\{t\}_N$.

We say that a rewrite sequence $s \to^* t$ is *top-invariant* if $s \overset{\geq\epsilon}{\to}{}^* t$. For a right-ground normal MCTRS $\mathcal{R} = \{l_i \to r_i \Leftarrow C_i\}$, we denote the set $\{l_i\}$ (resp.$\{r_i\}$) of the left-hand-sides (resp. right-hand-sides) of rules in $\mathcal{R}$ by $\mathcal{R}^l$ (resp. $\mathcal{R}^r$). From now on, throughout this section, $\mathcal{R}$ is a right-ground normal MCTRS.

**Definition 5.1** For a term $t$, $\delta_{\mathcal{R}}(t) = \{t' \mid t' \trianglelefteq t \ \vee \ t' \trianglelefteq r \in \mathcal{R}^r\}$. A substitution $\theta$ is a $\delta_{\mathcal{R}}(t)$-*substitution*, if for each variable $x$, $x\theta \in \delta_{\mathcal{R}}(t)$.

We start with an explicit construction of the search space, i.e., possible reduction of $(s, t)_R$ and $\{t_1, \cdots, t_n\}_N$ to "*smaller*" problems. During the construction, next Lemma 5.2 is the key.

**Lemma 5.2** *If a rewrite sequence $A: \ s \to^* t$ is not top-invariant, there exist $l \to r \Leftarrow C \in \mathcal{R}$, a substitution $\sigma$, and a $\delta_{\mathcal{R}}(s)$-substitution $\theta$ such that*

$$B: \ s \ \overset{\geq\epsilon}{\to}{}^* \ \bar{l}\theta \ \overset{\geq\mathcal{P}os_{\mathcal{V}}(l)}{\to}{}^* \ l\sigma \ \overset{\epsilon}{\to} \ r \ \to^* t$$

*with the same rewrite steps. (Recall the $\bar{l}$ is a linearization of $l$.)*

**Proof.** Since $A$ is not top-invariant, there exists a rewrite at the root $\epsilon$. Let $l\sigma \overset{\epsilon}{\to} r$ be the first such rewrite. Let $A'$ be the prefix sequence of $A$ from $s$ to $l\sigma$, and let $A''$ be the suffix sequence of from $l\sigma$ to $t$. Then, $A': \ s \overset{\geq\epsilon}{\to}{}^* l\sigma$. Let $\{p_1, \cdots, p_n\} = \mathcal{P}os_{\mathcal{V}}(l)$ and let $A_i$ be the maximum suffix sequence in $A'$ such that all rewrites are below or equal to $p_i$. Then, by interchanging the order of parallel rewrites, we can decompose $A'$ as $C; A_1; \cdots; A_n$. By construction of $A_1, \cdots, A_n$, there exists a substitution $\theta$ such that $C: \ s \to^* \bar{l}\theta$ and, for each $p_i$,

(i) either all rewrite steps in $C$ are parallel to $p_i$, or

(ii) the last rewrite step in $C$ that is *not* parallel to $p_i$ occurs above $p_i$.

Let $x_i \in \mathcal{V}ar(\bar{l})$ with $\{p_i\} = \mathcal{P}os(\bar{l}, x_i)$. For (i), $x_i\theta = s|_{p_i}$, and for (ii), $x_i\theta$ is a subterm of $r'$ for some $r' \in \mathcal{R}^r$ (Recall that $\mathcal{R}$ is right-ground). Thus, $\theta$ is a $\delta_{\mathcal{R}}(s)$-substitution. □

**Definition 5.3** Let $s, t, t_1, \cdots, t_n$ be terms and let $\theta$ be a $\delta_{\mathcal{R}}(s)$-substitution. Define $\Phi_R((s,t)_R) = \Phi_{R,1}((s,t)_R) \ \cup\ \Phi_{R,2}((s,t)_R)$ and $\Phi_N(\{t_1, \cdots, t_n\}_N) = \Phi_{N,1}(\{t_1, \cdots, t_n\}_N) \ \cup\ \Phi_{N,2}(\{t_1, \cdots, t_n\}_N)$ where:

$$\Phi_{R,1}((s,t)_R) \;=\; \{\ \{(s_i, t_i)_R \mid s = f(s_1, \cdots, s_n), t = f(t_1, \cdots, t_n)\}\ \} \qquad \text{if } \mathit{head}(s) = \mathit{head}(t)$$

$$\Phi_{R,2}((s,t)_R) = \left\{ \begin{array}{l|l} \{(s, \bar{l}\theta)_R, (r,t)_R\} \ \cup & l \to r \Leftarrow C \in \mathcal{R} \\ (\bigcup_{x \in \mathcal{V}ar^{nl}(l),\ p_i \in \mathcal{P}os(l,x)} \{\{\bar{l}|_{p_1}\ \theta, \cdots, \bar{l}|_{p_m}\ \theta\}_N\}) & \forall x. x\theta \in \delta_{\mathcal{R}}(s) \end{array} \right\}$$

$$\Phi_{N,1}(\{t_1, \cdots, t_n\}_N) \;=\; \{\ \{\{t_1|_j, \cdots, t_n|_j\}_N \mid 1 \le j \le \mathit{arity}(\mathit{head}(t_1))\}\ \} \qquad \text{if } \mathit{head}(t_1) = \cdots = \mathit{head}(t_n)$$

$$\Phi_{N,2}(\{t_1, \cdots, t_n\}_N) \;=\; \{\ \{(t_i, r)_R,\ \{t_1, \cdots, t_{i-1}, r, t_{i+1}, \cdots, t_n\}_N\} \mid r \in \mathcal{R}^r\}$$

The intuition for $\Phi_R((s,t)_R)$ and $\Phi_N(\{t_1, \cdots, t_n\}_N)$ is the set of candidates of the reduction of the problem. For instance, $\Phi_{R,1}((s,t)_R)$ corresponds to the case that the witness $s \to^* t$ of $(s,t)_R$ is top-invariant, and $\Phi_{R,2}((s,t)_R)$ corresponds to the case that it is not top-invariant. They enumerate all possible reductions, based on Lemma 5.2. Similarly, $\Phi_{N,1}(\{t_1, \cdots, t_n\}_N)$ corresponds to the case that the witness, *for some* $t \in NF_{\mathcal{R}}$, $t_i \to^* t$ *for each* $i$, is top-invariant for each $i$. $\Phi_{N,2}(\{t_1, \cdots, t_n\}_N)$ corresponds to the case that some $t_i \to^* t$ is not top-invariant. We assume that redundancy in $\Phi_R$ and $\Phi_N$ is removed as

- to eliminate $(s,s)_R$, and
- to reduce $\{\cdots, t, t, \cdots\}_N$ to $\{\cdots, t, \cdots\}_N$.

Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$. Next, we define the search path $\Psi_\alpha(\rho)$ for the sequence $\alpha$ of pairs of integers.

**Definition 5.4** Let

$$\Phi(\rho) = \begin{cases} \Phi_R(\rho) & \text{if } \rho = (s,t)_R, \\ \Phi_N(\rho) & \text{if } \rho = \{t_1, \cdots, t_n\}_N. \end{cases}$$

and let $\alpha$ be a sequence of pairs of integers. Then, a search path $\Psi_\alpha(\rho)$ is inductively defined as:

$$\begin{aligned}
\Psi_{\epsilon}(\rho) \quad &= \{\rho\} \\
\Psi_{\alpha.(i,j)}(\rho) &= \{\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \cdots, \tau_m\} \ \cup \ \bar{\tau}_j \\
&\qquad \text{where } \{\tau_1, \cdots, \tau_m\} = \Psi_{\alpha}(\rho) \text{ and } \{\bar{\tau}_1, \cdots, \bar{\tau}_k\} = \Phi(\tau_i)
\end{aligned}$$

We will show that to decide $\rho$, it is enough to check on finitely many $\Psi_{\alpha}(\rho)$.

**Definition 5.5** Let $s, t, t_1, \cdots, t_n$ be terms. Assume that $(s,t)_R$ and $\{t_1, \cdots, t_n\}_N$ have witness. We denote the minimal (sum of) rewrite steps of the witness of $(s,t)_R$ and $\{t_1, \cdots, t_n\}_N$ by $\mathit{step}((s,t)_R)$ and $\mathit{step}(\{t_1, \cdots, t_n\}_N)$, respectively. Define *weight* $\omega$ by

$$\begin{aligned}
\omega((s,t)_R) &= (\mathit{step}((s,t)_R), |s|) \\
\omega(\{t_1, \cdots, t_n\}_N) &= (\mathit{step}(\{t_1, \cdots, t_n\}_N), |t_1| + \cdots + |t_n|)
\end{aligned}$$

and the lexicographical order over weight is

$$(i,j) > (i',j') \ \Leftrightarrow \ i > i' \ \vee \ (i = i' \ \wedge \ j > j').$$

The next lemma is immediate.

**Lemma 5.6** *Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$, and let $\bar{\tau} \in \Phi(\rho)$. If each $\tau \in \bar{\tau}$ has a witness and they give $\rho$ a witness, then $\omega(\tau) < \omega(\rho)$.*

We denote the maximum multiplicity of (nonlinear) variables in $l$ in $\mathcal{R}^l$ by $a_{\mathcal{R}}$, and $\{\bar{l}\theta \mid l \in \mathcal{R}^l, \theta \text{ is a } \delta_{\mathcal{R}}(s)\text{-substitution}\}$ by $\Delta_{\mathcal{R}}(s)$.

**Definition 5.7** Let $s, t, t_1, \cdots, t_n$ be terms.

$$\begin{aligned}
S_R((s,t)_R) \ &= \{(s',t')_R \mid s' \trianglelefteq s \ \vee \ s' \trianglelefteq r \in \mathcal{R}^r \text{ and} \\
&\qquad\qquad t' \trianglelefteq u \in \Delta_{\mathcal{R}}(s) \ \vee \ t' \trianglelefteq t \ \vee \ t' \trianglelefteq r \in \mathcal{R}^r\} \\
S_N((s,t)_R) &= \{\{t_1, \cdots, t_k\}_N \mid 1 \leq k \leq a_{\mathcal{R}} \text{ and } t_i \trianglelefteq s \ \vee \ t_i \trianglelefteq r \in \mathcal{R}^r\}
\end{aligned}$$

$$\begin{aligned}
S_R(\{t_1, \cdots, t_n\}_N) \ &= \{(s',t')_R \mid s' \trianglelefteq t_i \ \vee \ s' \trianglelefteq r \in \mathcal{R}^r \text{ and} \\
&\qquad\qquad t' \trianglelefteq u \in \cup_{1 \leq i \leq n} \Delta_{\mathcal{R}}(t_i) \ \vee \ t' \trianglelefteq r \in \mathcal{R}^r\} \\
S_N(\{t_1, \cdots, t_n\}_N) &= \{\{t'_1, \cdots, t'_k\}_N \mid 1 \leq k \leq max(n, a_{\mathcal{R}}) \text{ and} \\
&\qquad\qquad (\vee_{1 \leq i \leq n} \ t'_j \trianglelefteq t_i) \ \vee \ t'_j \trianglelefteq r \in \mathcal{R}^r\}
\end{aligned}$$

**Lemma 5.8** *Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$. Then, for each $\alpha$, $\Psi_{\alpha}(\rho) \ \subseteq \ S_R(\rho) \cup S_N(\rho)$.*

**Proof.** Since $s \in \delta_{\mathcal{R}}(t)$ implies $\delta_{\mathcal{R}}(s) \subseteq \delta_{\mathcal{R}}(t)$, by induction on the length of $\alpha$, $\Phi_{\alpha}(\rho) \ \subseteq \ S_R(\rho) \cup S_N(\rho)$. $\square$

Now, we show that it is enough to consider a search path $\Phi_\alpha(\rho)$ with the upper bound for the length of $\alpha$.

**Lemma 5.9** *Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$. There exists an upper bound $M_L$ such that if $\Phi_\alpha(\rho) \neq \phi$ with $|\alpha| > M_L$, then, for any $\beta$ that contains $\alpha$ as a prefix, $\Phi_\beta(\rho)$ does not give a witness of $\rho$.*

**Proof.** Since $\Delta_{\mathcal{R}}(s)$, $\Delta_{\mathcal{R}}(t_1), \cdots, \Delta_{\mathcal{R}}(t_n)$ are finite, $S_R(\rho)$ and $S_N(\rho)$ are finite by construction. Let $M_L = 2^{|S_R(\rho) \cup S_N(\rho)|}$. Assume that $\Phi_\beta(\rho)$ gives a witness of $\rho$ for some $\beta$ that contains $\alpha$ as a prefix. Without loss of generality, we assume $\beta = \alpha$. Then, $\Phi_\alpha(\rho)$ also gives $\Phi_{\alpha'}(\rho)$ a witness for each prefix $\alpha'$ of $\alpha$.

Let $\ll$ be the multiset extension of $<$. Then, from Lemma 5.6, for each prefix $\alpha'$ and $\alpha''$ of $\alpha$, if $\alpha''$ is a proper prefix of $\alpha'$, then $\Psi_{\alpha'}(\rho) \ll \Psi_{\alpha''}(\rho)$. Thus, $\Phi_{\alpha'}(\rho) \neq \Phi_{\alpha''}(\rho)$. However, from Lemma 5.8, this is a contradiction to $|\alpha| > M_L$. $\Box$

At last, we give an upper bound $M_W$ for branching of a search path.

**Lemma 5.10** *Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$. There is an upper bound $M_W$ such that, for each $\tau \in \bar{\tau} \in \Psi_\alpha(\rho)$, $|\Phi(\tau)| \leq M_W$.*

**Proof.** Let $v_{\mathcal{R}} = max\ \{|\mathcal{V}ar(l)| \mid l \in \mathcal{R}^l\}$ and $m = max\ \{|\delta_{\mathcal{R}}(s)|,\ |\delta_{\mathcal{R}}(t_1)|,\ \cdots,\ |\delta_{\mathcal{R}}(t_n)|\}$. Define $M_W = m^{v_{\mathcal{R}}} \cdot |\mathcal{R}| + 1$. Since $s \in \delta_{\mathcal{R}}(t)$ implies $\delta_{\mathcal{R}}(s) \subseteq \delta_{\mathcal{R}}(t)$, if $\tau = (s,t)_R)$, $|\Phi_R((s,t)_R)| \leq m^{v_{\mathcal{R}}} \cdot |\mathcal{R}| + 1$, and if $\tau = \{t_1, \cdots, t_n\}_N$, $|\Phi_N(\{t_1, \cdots, t_n\}_N)| \leq |\mathcal{R}| + 1$. Thus, $|\Phi(\tau)| \leq M_W$. $\Box$

**Theorem 5.11** *For a right-ground normal MCTRS, reachability and normal joinability are decidable.*

**Proof.** Let either $\rho = (s,t)_R$ or $\rho = \{t_1, \cdots, t_n\}_N$. From Lemma 5.9, it is enough to consider $\Phi_\alpha(\rho)$ with $|\alpha| \leq M_L$. The number of candidates for the next $\Phi_{\alpha.(i,j)}(\rho)$ is at most $|S_R(\rho) \cup S_N(\rho)| \times M_W$, because the possible choice of $i$ is at most $|S_R(\rho) \cup S_N(\rho)|$ from Lemma 5.8, and that of $i$ is at most $M_W$ from Lemma 5.10. Thus, the set of search paths to check is finite, and the theorem follows. $\Box$

**Corollary 5.12** *For a right-ground normal MCTRS, normalizability is decidable.*

**Remark 5.13** Since a normal form may not be preserved when adding context, even if $\Phi_\alpha(\rho)$ has a witness, this does not mean $\rho$ has a witness. We need to check further that $\Phi_\alpha(\rho)$ actually gives $\rho$ a witness. For instance, the witness of $\{t_{1,1}, t_{2,1}\}_N$, $\{t_{1,2}, t_{2,2}\}_N$ that *there exist $t_1, t_2 \in NF_{\mathcal{R}}$ such that $t_{1,1}, t_{2,1} \to^* t_1$ and $t_{1,2}, t_{2,2} \to^* t_2$*, does not mean that $f(t_{1,1}, t_{1,2}), f(t_{2,1}, t_{2,2}) \to^* f(t_1, t_2)$ is a witness of $\{f(t_{1,1}, t_{1,2}), f(t_{2,1}, t_{2,2})\}_N$, because it may be $f(t_1, t_2) \notin NF_{\mathcal{R}}$. However, this search path is produced for a top-invariant case, and $\{f(t_{1,1}, t_{1,2}), f(t_{2,1}, t_{2,2})\}_N$ will be analyzed in another search path for a not top-invariant case (note that $t_1, t_2 \in NF_{\mathcal{R}}$ implies $f(t_1, t_2)$ is a redex). Thus, we simply judge this search path fails.

## 6 NV-needed redex of normal MCTRS

In this section, we provide an alternative definition of a *needed* redex as in [5], and show that whether a redex is nv-needed is decidable for a normal MCTRS. The equivalence of two definitions for a nonoverlapping normal MCTRS is obtained similar to Lemma 4.1 in [5].

**Definition 6.1** Let $\bullet \notin \mathcal{F}$, $s \in T(\mathcal{F}, \mathcal{V})$. A redex $s|_p$ (in $s$) is *needed* if $s[\bullet]_p$ does not rewrite to a normal form without $\bullet$.

**Definition 6.2** Let $\Omega(\notin \mathcal{F})$ be a fresh constant and let $t$ be a term. $t_\Omega$ is a term obtained by replacing each variable in a term $t$ with $\Omega$.

For terms $t, u \in T(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$, we denote $t \preceq u$ if $t$ is obtained from $u$ by replacing the subterms in $u$ by $\Omega$'s.

**Definition 6.3** Let $\mathcal{R}$ be a normal MCTRS, and $s, t$ be terms. Let $p \in \mathcal{P}os(s)$. If $s|_p$ is a redex of $l \to r \Leftarrow C \in \mathcal{R}$,

$$s \to_{nv} t \Leftrightarrow t = s[u]_p \text{ where } r_\Omega \preceq u \in T(\mathcal{F})$$

**Definition 6.4** A redex is *nv-needed* if it is needed under $\to_{nv}$.

From now on, we concentrate on rewrite sequences starting from ground terms, and we assume $\mathcal{F}_0 \neq \phi$. This restriction does not lose generality, because for a rewrite sequence starting from a non-ground term $t$, we can regard the variables in $t$ as additional constants.

**Definition 6.5** Assume $\mathcal{F}_0 \neq \phi$. For a normal MCTRS $\mathcal{R}$, we define right-ground normal MCTRSs $\mathcal{R}_\Omega$, $\mathcal{R}^1_\Omega$, and $\mathcal{R}^2_\Omega$ as follows.

$$\begin{aligned}
\mathcal{R}_\Omega &= \mathcal{R}^1_\Omega \ \cup \ \mathcal{R}^2_\Omega \\
\mathcal{R}^1_\Omega &= \{l \to r_\Omega \Leftarrow C \mid l \to r \Leftarrow C \in \mathcal{R}\} \\
\mathcal{R}^2_\Omega &= \{\Omega \to f(\underbrace{\Omega, \cdots, \Omega}_{n}) \mid f \in \mathcal{F}_n,\ n \geq 0\}
\end{aligned}$$

We denote $\to_{\mathcal{R}_\Omega}$ (resp. $\to_{\mathcal{R}^1_\Omega}$, $\to_{\mathcal{R}^2_\Omega}$) by $\to_\Omega$ (resp. $\to^1_\Omega$, $\to^2_\Omega$).

**Lemma 6.6** *Let $s, t$ be ground terms. Assume $\mathcal{F}_0 \neq \phi$. If $s \to^*_\Omega t$ and $\Omega \not\trianglelefteq s$, then, for each term $t'$ with $t \preceq t'$ and $\Omega \not\trianglelefteq t'$, $s \to^*_{nv} t'$.*

**Proof.** By induction on the number $m$ of the occurrences of $\to^1_\Omega$'s in $s \to^*_\Omega t$. Since $\mathcal{F}_0 \neq \phi$, it is easy for $m = 1$.

Assume $m > 1$ and let $s \to^*_\Omega u \xrightarrow{p}{}^{1}_{\Omega} v\ (\to^2_\Omega)^*\ t$. Without loss of generality, we can assume that every rewrite in $v\ (\to^2_\Omega)^*\ t$ occurs at a position larger-than-or-equal-to $p$.

Since $u|_p$ is a redex of $\mathcal{R}^1_\Omega$, for each $u'$ with $u|_p \preceq u'$ and $\Omega \not\trianglelefteq u'$, $u'$ is a redex of $\mathcal{R}^1_\Omega$. (Note that $\Omega \notin NF_{\mathcal{R}_\Omega}$; thus $\Omega$ does not appear below nonlinear variable positions in the rewrite $u|_p \to v|_p$.) Thus, we can modify the rewrite sequence as $s \to_\Omega^* u \ (\to_\Omega^2)^* \ u[u']_p \xrightarrow{p\ 1}_\Omega v \ (\to_\Omega^2)^* \ t$.

For each $t'$ with $t'$ with $t \preceq t'$ and $\Omega \not\trianglelefteq t'$, $u[u']_p \preceq t'[u']_p$ and $\Omega \not\trianglelefteq t'[u']_p$. Thus, from induction hypothesis, $s \to_{nv}^* t'[u']_p$, and $u' \to_{nv} t'|_p$. This concludes $s \to_{nv}^* t'$. □

**Lemma 6.7** *Let $s, t$ be ground terms with $\Omega \not\trianglelefteq s$. If $s \to_{nv}^* t$, then $s \to_\Omega^* t$.*

**Proof.** By induction on the number $m$ of rewrite steps of $s \to_{nv}^* t$. For $m = 1$, the proof is easy. (Note that $s \to_{nv} t$ implicitly implies $\mathcal{F}_0 \neq \phi$.) Assume $m > 1$. Let $s \to_{nv}^* u \to_{nv} t$. Since the reduction of $\to_{nv}$ does not produce $\Omega$, $\Omega \not\trianglelefteq u$. Thus, induction hypothesis implies $s \to_\Omega^* u \to_\Omega^* t$. □

**Lemma 6.8** *For a term $t$ with $\Omega \not\trianglelefteq t$, $t$ is normalizable wrt $\to_{nv}$, if, and only if, $t$ is normalizable wrt $\to_\Omega$.*

**Proof.** A term without $\Omega$ is a normal form wrt $\to_{nv}$ if, and only if, a normal form wrt $\to_\Omega$. Thus, from Lemma 6.6 and 6.7. □

Since $R_\Omega \cup \{\bullet \to \bullet\}$ is a right-ground normal MCTRS, Theorem 6.9 follows immediately from Corollary 5.12 and Lemma 6.8.

**Theorem 6.9** *For a normal MCTRS, whether a redex is nv-needed is decidable.*

**Remark 6.10** $\to_{nv}$ approximates $\to$ (i.e., $NF_\to = NF_{\to_{nv}}$ and $\to \subseteq \to_{nv}$), thus Lemma 4.5 in [5] shows that nv-needed redexes are really needed. Thus, from the remark at the end of Section 4, the repeated reduction of nv-needed redexes is a normalizing strategy (if nv-need redexes exist in each reducible term).

**Remark 6.11** As pointed out in Example 5.1 [9], the repeated reduction of nv-needed redexes is *not* root-normalizing.

# 7 Conclusion

This paper investigated call-by-need reductions for a normal membership conditional term rewriting system (MCTRS). Its main results are:

(i) A reducible term has a needed redex for a nonoverlapping normal MCTRS.

(ii) Reachability and normalizability for a right-ground normal MCTRS are decidable.

(iii) Whether a redex is nv-needed is decidable for a normal MCTRS.

For the first result, there seems to be no other choice when one explores the existence of needed redexes in nonlinear TRSs; in fact, the membership condition precisely corresponds to the proof techniques in [10].

For the second and the third result, I expect that reachability and normalizability of a shallow [2,3] and right-linear normal MCTRS would be decidable, and that nv-neededness could be extended to shallow-neededness.

Note that, unlike left-linear TRSs, growing neededness is undecidable for normal MCTRSs, because Post's Correspondence Problem (PCP) is described as a reachability (or normalizability) problem of a growing and right-linear normal MCTRS (either with or without membership conditions). Let $\{(\alpha_i, \beta_i) \mid 1 \leq i \leq n\}$ be the set of $n$-pairs of finite sequences. Then, PCP is equivalent to whether $A$ is reachable to $B$ where

$$\left\{ \begin{array}{ll} a \rightarrow d(\alpha_1(c), \beta_1(c)), & d(x,y) \rightarrow d(\alpha_1(x), \beta_1(y)), \\ \quad \cdots & \qquad\qquad\quad \cdots \\ a \rightarrow d(\alpha_n(c), \beta_n(c)), & d(x,y) \rightarrow d(\alpha_n(x), \beta_n(y)), \\ & d(x,x) \rightarrow b \ \Leftarrow x \in NF. \end{array} \right\}$$

Here, we assume that the symbols $a, b, c, d$ do not appear in $\alpha_i$ and $\beta_i$, and we regard a finite sequence $\alpha$ as applications of monadic function symbols (such as $\alpha(x) = f(g(h(x)))$ for $\alpha = fgh$).

This demonstrates the difficulty of proving decidability results for normal MCTRSs. Another difficulty arises because modern tree automata techniques [2,5,11] fail for normal MCTRSs. This is because the set of normal forms of a normal MCTRS is *not* regular; i.e., the set of normal forms of a nonlinear TRS is *not* regular [8], and the reducibility of a normal MCTRS is equivalent to that of the underlying TRS (assume a term matches to the left-hand-side of some rule at some position. If all nonlinear variables are instantiated with normal forms, then it is reducible; otherwise, there is a redex below some nonlinear variable position).

There remains another decidability problem: *in a normal MCTRS, whether every reducible term has a nv-needed redex is decidable.* A similar result is found in [13] in the context of sequentiality by a classical method. However, the proof of Assertion 1 in Theorem 6.6 in [13] does not work (at least directly), and the gap has not yet been fulfilled.

If a normal MCTRS has only shallow nonlinear variables (i.e., each occurrence of a nonlinear variable in the left-hand-side of a rule in a normal TRS is at depth 1), tree automata with brotherhood equality [1] would work similar to [5], which is in principle the same as a classical tree automata. I expect reduction automata [4] would further help for this direction.

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