Complete Axiomatization of an Algebraic Construction of Graphs

Mizuhito Ogawa[1]

Japan Advanced Institute of Science and Technology
1-1 Asahidai Tatsunokuchi Nomi Ishikawa, 923-1292 Japan
mizuhito@jaist.ac.jp

Abstract. This paper presents a complete (infinite) axiomatization for an algebraic construction of graphs, in which a finite fragment denotes the class of graphs with bounded tree width.

1 Introduction

A graph is a flexible relational structure for describing problems. However, solving graph problems can be difficult, partially because graphs lack an obvious recursive construction.

The algebraic construction of graphs opens the possibility for graph algorithms that could be applied:

- efficient programming methodologies, such as depth-first search, divide-and conquer, and dynamic programming, which would enable us to design a new graph algorithm, and
- program transformation techniques, which are well-developed in the functional programming community [FS96,Erw97,SHTO00].

This is especially true for graphs with bounded tree width [RS86]. The class of graphs with bounded tree width is limited, but still contains interesting application areas; for instance, the control flow graphs of GOTO-free C programs have tree widths of at most 6 [Tho98], and those of practical Java programs mainly have at most 3 [GMT02].

A notable feature is that many NP-hard graph problems for general graphs are reduced to linear-time for graphs with bounded tree width [Cou90,BPT92]. This corresponds to the fact that algebraic constructions become finitely generated for a class of graphs with bounded tree width [BC87,ACPS93,OHS03], though they are infinitely generated for general graphs.

However, the algebraic structures referred above are not *initial*, i.e., the same graph could have several different expressions. Clarifying such equivalence could lead

- a debugging opportunity of programs, i.e., programs must have no conflicts with axioms, and
- efficient algorithm design for graph properties, such as graph isomorphism.

Our ultimate aim is to give a complete (finite) axiomatization for graphs with bounded tree width. This is half done; this paper presents the complete (infinite) axiomatization for an algebraic construction of general graphs, in which a finite fragment denotes graphs with bounded tree width. The idea of the proof for ground cases comes from [BC87]; our work further extends the completeness result to non-ground cases.

This paper is organized as follows. Section 2 prepares basic notations. Section 3 presents an algebraic construction of graphs with infinite signatures, which is a variation of those in [ACPS93]. Section 4 gives the complete (infinite) axioms for ground terms, and Section 5 extends them to non-ground terms. Section 6 is a brief overview of related work, and Section 7 discusses future work.

2 Preliminaries

Let F be a set of *function symbols* and X a countably infinite set of *variables*. Each function symbol f is supposed to have its arity $ar(f)$. A function symbol c such that $ar(c) = 0$ is called a *constant symbol*. The set of all *terms*, denoted by $T(F, X)$, built from F and X is defined as follows:

1. Constant symbols in F and variables in X are terms.
2. If $t_1, \ldots, t_n$ are terms, and f is a function symbol in F such that $ar(f) = n$, then $f(t_1, \ldots, t_n)$ is a term.

$\mathcal{V}(t)$ denotes the set of variables occurring in a term t. A term without variables is called a *ground* term, and a term in which each variable occurs at most once is called a *linear* term. The set of ground terms is denoted by $T(F)$ for the set F of underlying function symbols.

Let $\Box$ be a fresh special constant symbol. A *context* $C[\ ]$ is a term built from $F \cup \Box$ and X. When $C[\ ]$ is a context with n $\Box$'s and $t_1, \cdots, t_n$ are terms, $C[t_1, \cdots, t_n]$ denotes the term obtained by replacing the i-th $\Box$ from the left in $C[\ ]$ with t_i for each $i = 1, \ldots, n$.

Definition 1. *A* term rewriting system *(TRS) is a set R of* rewrite rules*. A rewrite rule is a pair of terms denoted by $l \to r$ satisfying two conditions: (1) l is not a variable and (2) $\mathcal{V}(l) \supseteq \mathcal{V}(r)$.*

If $t = C[l\theta]$ and $s = C[r\theta]$ for $l \to r \in R$ and a substitution θ, $t \to_R s$ is a (one-step) reduction and $l\theta$ is called a redex.

A TRS R is terminating (or, strongly normalizing, **SN** *for short) if there are no infinite rewrite sequences $t_1 \to_R \cdots \to_R t_n \to_R \cdots$.*

Throughout the paper, we will use G, G' for (k-terminal) graphs, S for a set, X for a set of variables, s, t for terms, h, i, j, k, l for indices, and x, y for variables, s, t for terms, α, β for maps, θ for a substitution, and σ, τ for permutations. k is also often used for the number of terminals. l (resp. r) is sometimes used for the left-hand (resp. right-hand) side of a rewriting rule in a TRS.

3 Algebraic construction of graphs

In this paper we consider graphs with undirected edges, with at most one edge between any two vertices, and with no edge between a vertex and itself. (Extensions to multiple edges between vertices and to loops connecting a vertex to itself are easy, and sketched in Remark 2 of Section 4.) A k-terminal graph G is a graph with k distinguished vertices, called *terminals*, numbered 1 through k. The set of vertices of G is denoted $V(G)$, the set of edges of G is denoted by $E(G)$, and we write $G[i]$ for the i'th terminal of G, where $1 \leq i \leq k$. Ordinary graphs are obtained as 0-terminal graphs.

A k-terminal graph G is a pair of a graph and a tuple of its k distinct vertices, called *terminals*. The i-th terminal in a k-terminal graph G with $1 \leq i \leq k$ is denoted by $G[i]$ (like an array-like notation). Ordinary graphs are obtained as 0-terminal graphs after removal of terminals. For simplicity, we consider simple graphs (i.e., undirected and without multiple edged) without loops; but, the extensions to directed graphs, graphs with multiple edges, and/or graphs with loops are straightforward. The set of vertices of G is denoted by $V(G)$ and the set of edges of G is denoted by $E(G)$. The number of edges from a vertex v is denoted by $\#e(v)$.

Definition 2. *Let B_k be sorts for $k \geq 0$. Let $l_k^i, \oplus_k, r_k, \sigma_k^i, e^2, \mathbf{0}$ be function symbols with sorts below*

$$\begin{cases} e^2\colon B_2, & l_k^i : B_{k-1} \to B_k, \quad \oplus_k\colon B_k \times B_k \to B_k, \\ \mathbf{0} : B_0, & r_k\colon B_k \to B_{k-1}, \quad \sigma_k^j : B_k \to B_k. \end{cases}$$

where $i \leq k$, $j < k$, and $k \geq 0$ (For readability, $\oplus_k$ is an infix operation and the rest are prefix). Let $\mathcal{B}_n$ be the set of well-sorted ground terms in

$$T(\{\mathbf{0}, e^2, l_k^i, r_k, \oplus_k, \sigma_k^j \mid 1 \leq i \leq k \leq n,\ 1 \leq j < k\})$$

and $\mathcal{B}_\infty = \cup_{n=0}^{\infty} \mathcal{B}_n$.

A term $t \in \mathcal{B}_k$ is interpreted as a k-terminal graph (defined below) by interpreting function symbols $l_k^i, \oplus_k, r_k, \sigma_k^i, e^2, \mathbf{0}$ as following operations. This interpretation is denoted by $\psi(t)$.

Definition 3. *Let $\psi(e^2)$ be the edge with two terminals and $\psi(\mathbf{0})$ be the empty graph. We define operations among k-terminal graphs as*

- *$\psi(l_k^i(t))$ is a lifting for $1 \leq i \leq k$, i.e., insert a new isolated terminal (as a new vertex) to $\psi(t)$ at the i-th position in $k-1$ terminals.*
- *$\psi(r_k(t))$ removes the last terminal from $\psi(t)$.*
- *$\psi(s \oplus_k t)$ is a parallel composition for $k \geq 0$, i.e., fuse each i-th terminal in $\psi(s)$ and $\psi(t)$ for $1 \leq i \leq k$.*
- *$\psi(\sigma_k^i(t))$ is a permutation, i.e., permute the i-th terminal and the $i+1$-th terminal in $\psi(t)$ for $1 \leq i < k$.*

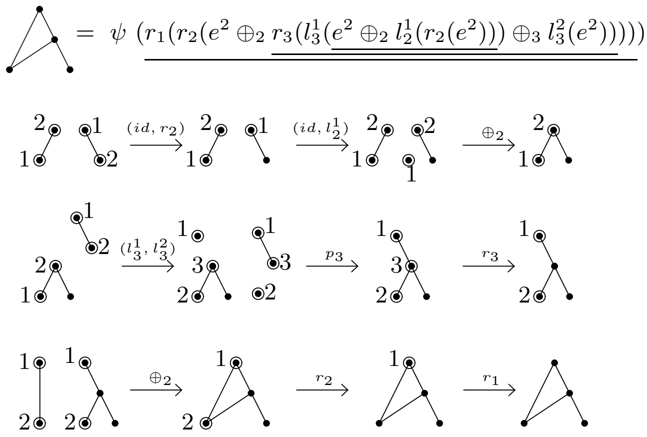


Fig. 1. An example of the algebraic construction

Example 1. Fig. 1 shows that the algebraic construction of a (0-terminal) graph. Each operation, underlined in $r_1(r_2(e^2 \oplus_2 r_3(l_3^1(e^2 \oplus_2 l_2^1(r_2(e^2))) \oplus_3 l_3^2(e^2))))$, is figured in lower columns.

Remark 1. Each permutation σ on $\{1, \cdots, k\}$ is generated from σ_k^i's. For instance, a circular permutation is generated as

$$\sigma_k^{j-1} \cdots \sigma_k^i = \begin{pmatrix} i & i+1 & \cdots & j \\ j & i & \cdots & j-1 \end{pmatrix}$$

for $1 \leq i < j \leq k$.

Although we do not show the definition of graphs with bounded tree width, the characterization of graphs with tree width at most k is given by the following theorem. This theorem is obtained similar to that in [ACPS93].

Theorem 1. *For $k \geq 0$, $\psi(\mathcal{B}_{k+1})$ is the set of graphs with tree width at most k (by neglecting terminals).*

4 Complete axiomatization of graphs : ground cases

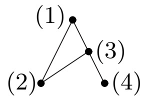
A k-terminal graph could be denoted by different algebraic expressions; for instance, see Example 2.

Example 2. Two terms below are equivalent and both denote the (0-terminal) graph in Fig. 1.

$$r_1(r_2(e^2 \oplus_2 r_3(l_3^1(e^2 \oplus_2 l_2^1(r_2(e^2))) \oplus_3 l_3^2(e^2)))))$$
$$r_1(r_2((e^2 \oplus_2 l_2^1(r_2(e^2))) \oplus_2 r_3(l_3^1(e^2) \oplus_3 l_3^2(e^2))))$$

In this section, we show that the (infinite) set of axioms $\mathcal{E}_\infty$ (in Fig. 3) is sound and complete for ground terms (Theorem 2 and 3). The key of the proof is the existence of a *canonical form* that denotes a graph in which all vertices are terminals (see Example 3). Then, canonical forms denoting an isomorphic graph are converted each other by the associativity and commutativity rules of the parallel composition $\oplus_k$'s (AC1 and AC2 in Fig. 3) and suitable permutations σ_k^i's among terminals.

Example 3. Fig. 3 shows a transformation to obtain a canonical form of the expression in Example 1, where R_1 will be defined in Definition 6. The underlined parts correspond to the rewrite steps. (The infix operation $\oplus_4$ has the commutative associative axioms, and we omit parenthesis in the last line for readability.)



$$
\begin{aligned}
& r_1(r_2(e^2 \oplus_2 r_3(l_3^1(e^2 \oplus_2 \underline{l_2^1(r_2}(e^2))) \oplus_3 l_3^2(e^2)))) \\
\rightarrow_{R_1}\ & r_1(r_2(e^2 \oplus_2 r_3(l_3^1(\underline{e^2 \oplus_2 r_3(l_3^1(e^2)))}, \oplus_3 l_3^2(e^2)))) \\
\rightarrow_{R_1}\ & r_1(r_2(e^2 \oplus_2 r_3(\underline{l_3^1(r_3}(l_3^3(e^2) \oplus_3 l_3^1(e^2))) \oplus_2 l_3^2(e^2)))) \\
\rightarrow_{R_1}\ & r_1(r_2(e^2 \oplus_2 r_3(r_4(\underline{l_4^1(l_3^3(e^2) \oplus_3 l_3^1(e^2)))}) \oplus_3 l_3^2(e^2)))) \\
\rightarrow_{R_1}\ & r_1(r_2(e^2 \oplus_2 r_3(\underline{r_4(l_4^1(l_3^3(e^2)) \oplus_4 l_4^1(l_3^1(e^2))) \oplus_3 l_3^2(e^2))})) \\
\rightarrow_{R_1}\ & r_1(r_2(\underline{e^2 \oplus_2 r_3(r_4(l_4^1(l_3^3(e^2)) \oplus_4 l_4^1(l_3^1(e^2)) \oplus_4 l_4^4(l_3^2(e^2)))))})) \\
\rightarrow_{R_1}^{+}\ & \underbrace{r_1(r_2(r_3(r_4(}_{R[\ ]}\underbrace{l_4^4(l_3^3}_{L_1[\ ]}\ \underbrace{(e^2)}\,)\ \oplus_4\ \underbrace{l_4^4(l_3^1}_{L_2[\ ]}\ \underbrace{(e^2)}\,)\ \underbrace{\oplus_4}_{P[\ ]}\ \underbrace{l_4^2(l_3^1}_{L_3[\ ]}\ \underbrace{(e^2)}\,)\ \oplus_4\ \underbrace{l_4^4(l_3^2}_{L_4[\ ]}\ \underbrace{(e^2)}\,)))))
\end{aligned}
$$

Increase the number of terminals

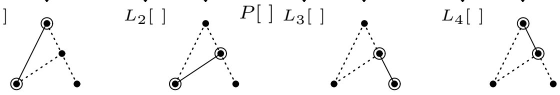


Fig. 2. Example of transformation to a canonical form (ground case)

Definition 4. *k-terminal graphs G_1, G_2 are isomorphic if there exists a one-to-one onto map $\alpha : V(G_1) \rightarrow V(G_2)$ such that*

- *For $v \in V(G_1)$, if v is the i-th terminal of G_1 with $1 \leq i \leq k$, then $\alpha(v)$ is the i-th terminal of G_2, and vice versa.*
- *For $v, v' \in V(G_1)$, if (v, v') is an edge of G_1, then $(\alpha(v), \alpha(v'))$ is an edge of G_2, and vice versa.*

Definition 5. *Two terms s, t of sort B_k are* equivalent *if the k-terminal graphs $\psi(s), \psi(t)$ are isomorphic.*

$\mathcal{E}_k$ in Fig. 3 is the set of axioms indexed by k. Let $\mathcal{E}_\infty = \cup_{k=1}^{\infty} \mathcal{E}_k$ and $\mathcal{E}_{\leq n} = \cup_{k=1}^{n} \mathcal{E}_k$. By regarding each equation (axiom) as a left-to-right rewrite

$$
\begin{array}{rclrl}
t_1 \oplus_k t_2 & = & t_2 \oplus_k t_1 & (\mathit{Commut.}) & \text{(AC1)} \\
(t_1 \oplus_k t_2) \oplus_k t_3 & = & t_1 \oplus_k (t_2 \oplus_k t_3) & (\mathit{Assoc.}) & \text{(AC2)} \\
l^j_k(l^i_{k-1}(t)) & = & l^i_k(l^{j-1}_{k-1}(t)) & 1 \le i < j \le k & (l\text{-Com}) \\
l^i_k(t_1 \oplus_{k-1} t_2) & = & l^i_k(t_1) \oplus_k l^i_k(t_2) & 1 \le i \le k & (l\text{-Dist}) \\
l^i_{k-1}(r_{k-1}(t)) & = & r_k(l^i_k(t)) & 1 \le i < k & \text{(E1)} \\
t_1 \oplus_{k-1} r_k(t_2) & = & r_k(l^k_k(t_1) \oplus_k t_2) & & \text{(E2)} \\
t \oplus_k l^k_k(\cdots l^1_1(\mathbf{0}))) & = & t & & \text{(E3)} \\
e^2 \oplus_2 e^2 & = & e^2 & & \text{(E4)} \\
\sigma^j_k(l^i_k(t)) & = & l^i_k(\sigma^{j-1}_{k-1}(t)) & 1 \le i < j < k & (\sigma\text{1-a}) \\
\sigma^i_k(l^i_k(t)) & = & l^{i+1}_k(t) & 1 \le i < k & (\sigma\text{1-b}) \\
\sigma^i_k(l^{i+1}_k(t)) & = & l^i_k(t) & 1 \le i < k & (\sigma\text{1-c}) \\
\sigma^j_k(l^i_k(t)) & = & l^i_k(\sigma^j_{k-1}(t)) & 1 < j+1 < i \le k & (\sigma\text{1-d}) \\
\sigma^1_2(e^2) & = & e^2 & & (\sigma 2) \\
\sigma^i_k(t_1 \oplus_k t_2) & = & \sigma^i_k(t_1) \oplus_k \sigma^i_k(t_2) & 1 \le i < k & (\sigma 3) \\
\sigma^i_{k-1}(r_k(t)) & = & r_k(\sigma^i_k(t)) & 1 \le i < k-1 & (\sigma 4) \\
r_{k-1}(r_k(\sigma^{k-1}_k(t))) & = & r_{k-1}(r_k(t)) & & (\sigma 5)
\end{array}
$$

Fig. 3. Axioms $\mathcal{E}_k$ of the algebraic construction of graphs

rule), its reflexive symmetric transitive closure (i.e., the finite application of axioms in $\mathcal{E}_\infty$) is denoted by $=_{\mathcal{E}_\infty}$.

It is easy to see that each axiom in $\mathcal{E}_\infty$ is sound.

Theorem 2. (Soundness for ground terms) *Let s, t be ground terms in $\mathcal{B}_\infty$. Then, s and t are equivalent if $s =_{\mathcal{E}_\infty} t$.*

Theorem 3. (Completeness for ground terms) *Let s, t be ground terms in $\mathcal{B}_\infty$. Then, $s =_{\mathcal{E}_\infty} t$ if s and t are equivalent.*

Definition 6. *For axioms in $\mathcal{E}_\infty$, let TRSs R_1 and R_2 be defined as*

$$
\begin{cases}
R_1 = \{(E1), (E2), (E2)', (l\text{-}Dist), (\sigma 3), (\sigma 4)\}, \\
R_2 = \{(\sigma 1), (\sigma 2),
\end{cases}
$$

where $(E2)'$ is $r_k(t_1) \oplus_{k-1} t_2 \to r_k(t_1 \oplus_k l^k_k(t_2))$ for each k.

Lemma 1. *R_1 and R_2 are terminating.*

Proof. Let $\delta(t, f)$ be the number of occurrences of a function symbol f in a term t, and let $\Delta(t, g, f)$ be the sum of all $\delta(s, f)$ where s is a subterm of t such that $root(s) = g$. We define the weight $\omega(t)$ of a term t by

$$\omega(t) = (\omega_{\oplus,r}(t),\ \omega_{l,r}(t) + \omega_{l,\oplus}(t) + \omega_{\sigma,r}(t) + \omega_{\sigma,\oplus}(t))$$

where

$$\begin{aligned}
\omega_{\oplus,r}(t) &= \Sigma_{j,k}\Delta(t,\oplus_k,r_j),\\
\omega_{l,r}(t) &= \Sigma_{i,j,i',j'}\Delta(t,l^i_j,r_{j'}),\\
\omega_{l,\oplus}(t) &= \Sigma_{i,j,k}\Delta(t,l^i_j,\oplus_k),\\
\omega_{\sigma,r}(t) &= \Sigma_{i,j,i',j'}\Delta(t,\sigma^i_j,r_{j'}),\\
\omega_{\sigma,\oplus}(t) &= \Sigma_{i,j,k}\Delta(t,\sigma^i_j,\oplus_k),
\end{aligned}$$

and define the lexicographic order on the weight. Then, for each reduction of R_1 the weight $\omega(t)$ decreases, and R_1 is **SN**. Similarly, each reduction of R_2 decreases the weight $\omega_{\sigma,l}(t) = \Sigma_{i,j,i',j'}\Delta(t,\sigma^i_j,l^{i'}_{j'})$, and R_2 is **SN**. ∎

Definition 7. *Let $t \in \mathcal{B}_\infty$ be a ground term of sort B_k, $n = |V(\psi(t))|$, and $m = |E(\psi(t))|$. t is a canonical form if either*

$$t = r_{k+1}(\cdots r_n(l^n_n(\cdots l^1_1(\mathbf{0})))),$$

or there exist

- $R_{n,k}[\ ] = r_{k+1}(\cdots r_n[\ ])$ *with* $0 \le k < n$,
- $P_n[\ ,\cdots,\ ]$ *consists of* $\oplus_n$*'s,*
- $L_i[\ ]$ *has the form* $l_n^{u_{i,n-2}}(\cdots l_3^{u_{i,1}}[\ ])$ *with* $u_{i,n-2} > \cdots > u_{i,1}$ *for* $1 \le i \le m$,

such that $t = R_{n,k}[P_n[L_1[e^2],\cdots,L_m[e^2]]]$.

Lemma 2. *For any term s, there exists a canonical form $t \in \mathcal{B}_n$ such that $s =_{\mathcal{E}_{\le n}} t$ where $n = |V(\psi(t))|$.*

Proof. We first show that there exists t' in the form $t' = R_{n,k}[P'[L'_1[c_1],\cdots,L'_l[c_l]]]$ with $s =_{\mathcal{E}_{\le n}} t'$ where

- $R_{n,k}[\ ] = r_{k+1}\cdots r_n[\ ]$,
- $P'[\ ]$ consists of $\oplus_j$'s, and
- $L'_1[\ ],\cdots,L'_l[\ ]$ consist of l^i_j's and $\sigma^{i'}_{k'}$'s.
- c_i is either e^2 or $\mathbf{0}$,

From Lemma 1, s has an R_1-normal form t' of the form $R_{n,k}[P'[L'_1[c_1],\cdots,L'_l[c_l]]]$. Since all vertices in e^2 are terminals and l^i_j,σ^i_j preserves a set of terminals, all vertices of each $L'_i[e^2]$ are terminals. r_i and $\oplus_j$ do not change the number of vertices, thus each $\oplus_j$ in $P'[\ ]$ satisfies $j = n = |V(\psi(t))|$. Further, from Lemma 1 each $L'_i[c_i]$ has an R_2-normal form, i.e., a σ^j_k-free term.

If $|E(\psi(s))| = 0$, this means $\psi(s)$ consists of isolated vertices and all c_i's are $\mathbf{0}$. Thus, $L'_i[\ ] = l^k_k(\cdots(l^1_1[\ ]))$ by (*l-Com*) and s is reduced to a canonical form $R_{n,k}[L_1[\mathbf{0}]]$ by (AC1), (AC2), and ($E3$).

If $|E(\psi(s))| > 0$, we can sort each $L'_i[\ ]$ by (*l-Com*). Since there exists $c_i = e^2$, we can erase $\mathbf{0}$'s by (AC1), (AC2), and ($E3$). Thus we assume $c_i = e^2$ for each i. If $L'_i[c_i]$ and $L'_j[c_j]$ are equal, we can eliminate redundant $L'_i[c_i]$'s by (AC1), (AC2), and ($E4$). Since each $L'_i[c_i]$ corresponds to an edge in $\psi(s)$ (i.e., the number of $L'_i[c_i]$'s is the number of edges in $\psi(s)$), we obtain a canonical form $t = R_{n,k}[P_n[L_1[e^2],\cdots,L_m[e^2]]]$ by (*l-Com*) (from-right-to-left direction). ∎

Definition 8. *Let* $e(n,i,j) = l_n^n \cdots l_{j+1}^{j+1} \cdot (l_j^{j-1} \cdots l_{i+2}^{i+1} \cdot l_{i+1}^{i-1} \cdots l_3^1(e^2)$ *for* $1 \leq i < j \leq n$ *(here we omit apparent parenthesis for readability).*

Lemma 3. *Let* $s \in \mathcal{B}_\infty$. $\psi(s)$ *contains an edge between the* i*-th and the* j*-th vertices, if, and only if, a canonical form of* s *contains* $e(n,i,j)$.

Sketch of proof of Theorem 3 Let $s, t \in \mathcal{B}_\infty$ such that $\psi(s)$ and $\psi(t)$ are equivalent. Assume that an isomorphism $\alpha : V(\psi(s)) \to V(\psi(t))$ satisfies the conditions in Definition 4. If $|E(\psi(s))| = |E(\psi(t))| = 0$, they have the unique canonical form from Lemma 2 and obviously the theorem holds. We assume $|E(\psi(s))| = |E(\psi(t))| > 0$.

From Lemma 2, we can assume that both s and t are canonical. Let $s = R_{n,k}[P_n[L_1[e^2], \cdots, L_m[e^2]]]$ and $t = R_{n,k}[P'_n[L'_1[e^2], \cdots, L'_m[e^2]]]$ where $n = |V(\psi(s))| = |V(\psi(t))|$ and $m = |E(\psi(s))| = |E(\psi(t))|$. Thus, α can be regarded as the permutation σ on $\{k+1, \cdots, n\}$.

Non-trivial permutation needs at least two elements, so we can assume $k \leq n-2$. Then from $(\sigma 4)$ and $(\sigma 5)$, $r_{k+1}^{k+1}(\cdots r_n^n(\sigma_n^i(t)) = r_{k+1}^{k+1}(\cdots r_n^n(t))$ for $k+1 \leq i \leq n-1$. Since a permutation over $\{k+1, \cdots, n\}$ is generated by σ_n^i's for $k+1 \leq i \leq n-1$, $r_{k+1}^{k+1}(\cdots r_n^n(\sigma(t)) = r_{k+1}^{k+1}(\cdots r_n^n(t))$. Thus, it is enough to show

$$\sigma(P_n[L_1[e^2], \cdots, L_m[e^2]]) =_{\mathcal{E}_{\leq n}} P'_n[L'_1[e^2], \cdots, L'_m[e^2]].$$

Since $\psi(s)$ and $\psi(t)$ are isomorphic, if there is an edge between the i-th and j-th vertices of $\psi(s)$, there is an edge between the $\alpha(i)$-th and $\alpha(j)$-th vertices of $\psi(t)$, and vice versa. Thus, if there is an edge between the i-th and j-th vertices in $\psi(s)$, then, form Lemma 3, there uniquely exist $L_k[e^2]$ and $L'_k[e^2]$ such that $L_k[e^2] =_{\mathcal{E}_{\leq n}} e(n,i,j)$ and $L'_k[e^2] =_{\mathcal{E}_{\leq n}} e(n, \alpha(i), \alpha(j))$.

Since $\sigma(e(n,i,j)) = e(n, \alpha(i), \alpha(j))$,

$$\sigma(P_n[L_1[e^2], \cdots, L_m[e^2]]) =_{\mathcal{E}_{\leq n}} P'_n[L'_1[e^2], \cdots, L'_m[e^2]]$$

holds from $(AC1)$, $(AC2)$, $(\sigma 2)$, and $(\sigma 3)$. ∎

Remark 2. The extensions to directed graphs, graphs with multiple edges, and/or graphs with loops are as follows:

- The removal of $(E4)$ in Fig. 3 gives the sound and complete axioms for graphs with multiple edges.
- By adding a constant l^1 as a 1-terminal graph that consists of the unique terminal and the unique edge from the terminal to the terminal itself, we obtain the algebraic construction of graphs with loops. The axioms are preserved for this extension.
- For digraphs, instead of an edge e^2, we use e_+^2 and e_-^2, where e_+^2 is the directed edge from the first terminal to the second, and e_-^2 is opposite. Then, the replacement of $\sigma_2^1(e^2) = e^2$ $(\sigma 2)$ with $\sigma_2^1(e_+^2) = e_-^2$ and $\sigma_2^1(e_-^2) = e_+^2$ lead the sound and complete axioms for directed graphs.

5 Complete axiomatization of graphs : non-ground cases

In this section, we extend the result of soundness (Theorem 2) and completeness (Theorem 3) for ground terms to general terms. In this extension, we need additional axioms ($\Sigma 1$) and ($\Sigma 2$) in Fig. 4, which present the *defining relation* of the permutation group [Wey39].

Lemma 4. *[Wey39] For any permutation σ and σ' that are expressed as products of σ_k^i's with $1 \leq i < k$, they are equivalent as a map if and only if $\sigma =_{\mathcal{G}_k} \sigma'$, where $\mathcal{G}_k$ consists of ($\Sigma 1$) and ($\Sigma 2$) axioms in Fig. 4.*

$$
\begin{array}{rlll}
\sigma_k^i \cdot \sigma_k^i(G) & = G & 1 \leq i < k & (\Sigma 1) \\
(\sigma_k^{i-1} \cdot \sigma_k^i)^3(G) & = G & 1 < i < k & (\Sigma 2)
\end{array}
$$

Fig. 4. Additional axioms $\mathcal{G}_k$ of the algebraic construction of graphs

Example 4. Consider the permutation of 1 and 3 among $\{1, 2, 3\}$

$$
\begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}
$$

which is represented as $\sigma_3^2 \cdot \sigma_3^1 \cdot \sigma_3^2$ or $\sigma_3^1 \cdot \sigma_3^2 \cdot \sigma_3^1$. This equivalence is obtained by $=_{\mathcal{G}_k}$ as

$$
\begin{array}{rl}
\sigma_3^2 \cdot \sigma_3^1 \cdot \sigma_3^2 & =_{\Sigma 2} \sigma_3^2 \cdot \sigma_3^1 \cdot (\sigma_3^1 \cdot \sigma_3^2)^3 \cdot \sigma_3^2 \\
& = \quad \sigma_3^2 \cdot (\sigma_3^1 \cdot \sigma_3^1) \cdot \sigma_3^2 \cdot \sigma_3^1 \cdot \sigma_3^2 \cdot \sigma_3^1 \cdot (\sigma_3^2 \cdot \sigma_3^2) \\
& =_{\Sigma 1} (\sigma_3^2 \cdot \sigma_3^2) \cdot \sigma_3^1 \cdot \sigma_3^2 \cdot \sigma_3^1 \\
& =_{\Sigma 1} \sigma_3^1 \cdot \sigma_3^2 \cdot \sigma_3^1
\end{array}
$$

Remark 3. For ground terms, ($\Sigma 1$) and ($\Sigma 2$) in Fig. 4 are not required, because the same can be performed by (σ1-d) and (σ2) in Fig. 3.

Let X_k be a set of variables with sort B_k. The i-th terminal of x is denoted by $x[i]$. Let $X = \cup_k X_k$. The set of well-sorted terms in

$$
T(\{\mathbf{0}, e^2, l_k^i, r_k, \oplus_k, \sigma_k^j \mid 1 \leq i \leq k \leq n,\ 1 \leq j < k\},\ X)
$$

is denoted by $\mathcal{B}_\infty(X)$. Define a substitution $\theta_{\mathbf{0}}$ by $x\theta_{\mathbf{0}} = l_k^k \cdots l_1^1(\mathbf{0})$ for each variable $x \in X_k$.

Definition 9. *For $s, t \in \mathcal{B}_\infty(X)$, s and t are equivalent if, for each ground substitution θ, $\psi(s\theta)$ and $\psi(t\theta)$ are isomorphic.*

The next theorem is immediate.

Theorem 4. (Soundness) *Let s, t be terms in $\mathcal{B}_\infty(X)$. Then s and t are equivalent if $s =_{\mathcal{E}_\infty \cup \mathcal{G}_\infty} t$.*

Difficult part is completeness.

Theorem 5. (Completeness) *Let s, t be terms in $\mathcal{B}_\infty(X)$. Then $s =_{\mathcal{E}_\infty \cup \mathcal{G}_\infty} t$ if s and t are equivalent.*

Similar to the ground case, we first consider a canonical form of a term t. The set of variables that appear in a term t in $\mathcal{B}_\infty(X)$ is denoted by $\mathcal{V}(t)$.

Definition 10. *Let t $(\in \mathcal{B}_\infty(X))$ be a term of sort B_k, $n = |V(\psi(t\theta_{\mathbf{0}}))|$, $m = |E(\psi(t\theta_{\mathbf{0}}))|$, and $\mathcal{V}(t) = \{x_1, \cdots, x_{m'}\}$. t is a canonical form if either*

$$t = r_{k+1}(\cdots r_n(l_n^n(\cdots l_1^1(\mathbf{0})))),$$

or there exist

- $R_{n,k}[\ ] = r_{k+1}(\cdots r_n[\ ])$,
- $P_n[\ ,\cdots,\ ]$ *consists of* $\oplus_n$*'s,*
- $L_i[\ ]$ *has the form* $l_n^{u_{i,n-2}}(\cdots l_3^{u_{i,1}}[\ ])$ *with* $u_{i,n-2} > \cdots > u_{i,1}$ *for* $1 \leq i \leq m$,
- $L_{m+i}[\ ]$ *has the form* $l_n^{u'_{i,n-d_i}}(\cdots l_{d_i+1}^{u'_{i,1}}[\ ])$ *with* $u'_{i,n-d_i} > \cdots > u'_{i,1}$ *for* $x_i \in X_{d_i}$ *and* $1 \leq i \leq m'$,
- G_i *is* $\sigma_i(x_i)$ *for some combination* σ_i *of* $\sigma_{d_i}^j$*'s for* $1 \leq i \leq m'$,

such that

$$t = R_{n,k}[P_n[L_1[e^2],\ \cdots,\ L_m[e^2],\ L_{m+1}[G_1],\ \cdots,\ L_{m+m'}[G_{m'}]]].$$

Define $Center(t) = \psi(R_{n,k}[P_n[L_1[e^2], \cdots, L_m[e^2]]])$. For a ground substitution θ, let $Inner(t,\theta) = V(Center(t))$ and $Outer(t,\theta) = V(\psi(t\theta)) \setminus Inner(t,\theta)$. We say a vertex is inner *if it is in $Inner(t,\theta)$, and* outer *otherwise.*

Lemma 5. *$Center(t)$ is isomorphic to $\psi(t\theta_{\mathbf{0}})$.*

Example 5. Fig. 5 shows the conversion of

$$t = r_2 \cdot p_2(e^2, r_3 \cdot p_3(l_3^1 \cdot p_2(e^2, l_2^1 \cdot r_2(e^2)), \sigma_3^2 \cdot \sigma_3^1 \cdot \sigma_3^2 \cdot l_3^2(x)))$$

to a canonical form. The circle expresses a substitution to a variable x, and the parenthesis for σ_k^i and the commutative associative operator $\oplus_4$ are omitted.

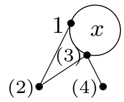
The next lemma is similarly proved as the proof of Lemma 2.

Lemma 6. *For any term $s \in \mathcal{B}_\infty(X)$, there exists a canonical form $t \in \mathcal{B}_n$ such that $s =_{\mathcal{E}_{\leq n}} t$ where $n = |V(Center(t))|$.*

When terms s and t are equivalent, without loss of generality, we can assume that s and t are canonical forms. Let us fix canonical forms s and t.

Lemma 7. *If s and t are equivalent, $\mathcal{V}(s) = \mathcal{V}(t)$.*

$$
\begin{aligned}
& r_2(e^2 \oplus_2 r_3(l_3^1(e^2 \oplus_2 \underline{l_2^1(r_2}(e^2))) \oplus_3 \sigma_3^2 \cdot \sigma_3^1 \cdot \underline{\sigma_3^2(l_3^2}(x)))) \\
\to^+\ & r_2(e^2 \oplus_2 r_3(l_3^1(\underline{e^2 \oplus_2 r_3(l_3^1(e^2))}) \oplus_3 \sigma_3^2 \cdot \underline{\sigma_3^1(l_3^3}(x)))) \\
\to^+\ & r_2(e^2 \oplus_2 r_3(\underline{l_3^1(r_3}(l_3^3(e^2) \oplus_3 l_3^1(e^2))) \oplus_3 \underline{\sigma_3^2(l_3^3}(\sigma_2^1(x))))) \\
\to^+\ & r_2(e^2 \oplus_2 r_3(r_4(\underline{l_4^1(l_3^3(e^2) \oplus_3 l_3^1(e^2))}) \oplus_3 l_3^2(\sigma_2^1(x)))) \\
\to\ \ & r_2(e^2 \oplus_2 r_3(\underline{r_4(l_4^1(l_3^3(e^2)) \oplus_4 l_4^1(l_3^1(e^2))) \oplus_3 l_2^3(\sigma_2^1(x))})) \\
\to\ \ & r_2(\underline{e^2 \oplus_2 r_3(r_4(l_4^1(l_3^3(e^2)) \oplus_4 l_4^1(l_3^1(e^2)) \oplus_4 l_4^4(l_3^2(\sigma_2^1(x))))})) \\
\to^+\ & \underbrace{r_2(r_3(r_4(}_{R[\,]}\underbrace{l_4^4(l_3^3}_{L_1[\,]}\underbrace{(e^2))}\oplus_4 \underbrace{l_4^4(l_3^1}_{L_2[\,]}\underbrace{(e^2))}\underbrace{\oplus_4}_{P[\,]} \underbrace{l_4^2(l_3^1}_{L_3[\,]}\underbrace{(e^2))} \oplus_4 \underbrace{l_4^4(l_3^2}_{L_4[\,]}\underbrace{(\sigma_2^1(x))}))))
\end{aligned}
$$



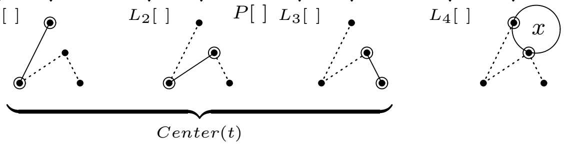


Fig. 5. Example of transformation to a canonical form (non-ground case)

Proof. Assume $\mathcal{V}(s) \neq \mathcal{V}(t)$. Without loss of generality, we can assume that $x \in \mathcal{V}(s)$ and $x \notin \mathcal{V}(t)$. From Lemma 5, $Center(s)$ and $Center(t)$ are isomorphic. Let $n = |V(Center(s))| = |V(Center(t))|$.

Consider a ground substitution θ that substitutes a term denoting K_{n+1} (complete graph with $n+1$ vertices) to x, and $l_k^k \cdots l_1^1(\mathbf{0})$ otherwise (for those that in X_k). Then, $|V(\psi(s\theta))| > |V(\psi(t\theta))| = n$, and the contradiction. ■

Lemma 8. *If s and t are equivalent, each variable x occurs the same times both in s and t.*

Proof. Assume that x occurs in s more than in t. Similar to Lemma 7, consider a ground substitution θ that substitutes a term denoting K_{n+1} (complete graph with $n+1$ vertices) to x, and $l_k^k \cdots l_1^1(\mathbf{0})$ otherwise (for those that in X_k). Then, $|V(\psi(s\theta))| > |V(\psi(t\theta))|$, and the contradiction. ■

For notational clarity, we consider conditional linearization of a term.

Definition 11. *Conditional linearization of a term t is obtained by renaming different occurrences of the same variable x to distinct variables $x', x'', \cdots$, associated with the side condition $\mathcal{C} = \{x' = x'' = \cdots\}$.*

Example 6. Conditional linearization of a term $p_3(l_3^1(p_2(x,y)), l_3^2(x))$ is

$$p_3(l_3^1(p_2(x',y)), l_3^2(x'')) \text{ with } \{x' = x''\}.$$

From now on, we consider conditional linearization of canonical forms s and t. Let us fix $\mathcal{V}(s)(= \mathcal{V}(t))$ as $\{x_1, \cdots, x_m\}$ with the side condition $\mathcal{C} : \{x_i = x_j\}$. Note that from Lemma 7 and 8, such $\mathcal{C}$ is well-defined.

Next we define $x_i[t,j]$, which is the vertex in $Center(t)$ that corresponds to the j-th terminal in $\psi(x\theta)$ for each ground substitution θ.

Definition 12. *We borrow the notation from Definition 10. Let* t *be a canonical form* $t = R_{n,k}[P_n[L_1[e^2], \;\cdots,\; L_m[e^2],\; L_{m+1}[G_1],\; \cdots,\; L_{m+m'}[G_{m'}]]]$ *and let* $(v_1, v_2, \cdots, v_n)$ *be the tuple of terminals of*

$$\psi(P_n[L_1[e^2], \;\cdots,\; L_m[e^2],\; L_{m+1}[G_1],\; \cdots,\; L_{m+m'}[G_{m'}]]\; \theta_{\mathbf{0}}).$$

Assume that a variable x_i *in* t *is of the sort* B_{d_i} *and let*

$$L_{m+i}[G_i] = l_n^{u_{n-d}}(\cdots(l_{d_i+1}^{u_1}[\sigma_i(x_i)]))$$

with $u_{n-d_i} > \cdots > u_1$*. Define* $x_i[t,j] = v_{\sigma_i^{-1}(w_j)}$ *where*

$$\{w_1, \cdots, w_{d_i}\} = \{1, \cdots, n\} \setminus \{u_1, \cdots, u_{n-d_i}\}$$

with $w_1 < \cdots < w_{d_i}$*.*

Example 7. In Example 5, $x[t,1] = v_3$ and $x[t,2] = v_1$.

Below, we define a *marker substitution* $\theta_{\mathcal{M}}$, which distinguishes each terminal $x_i[t,j]$ by the pair of its outer neighborhoods; these neighborhoods are distinguished each other by the number of edges in $\psi(t\;\theta_{\mathcal{M}})$.

Since the number of edges and the neighborhood relation are preserved by an isomorphism, an isomorphism between $\psi(st\;\theta_{\mathcal{M}})$ and $\psi(t\;\theta_{\mathcal{M}})$ induces the isomorphism between $Center(s)$ and $Center(t)$ that maps $x_i[s,j]$ to $x_{i'}[t,j]$ with $x_i = x_{i'} \in C$.

Definition 13. *Let* $term_1, \cdots, term_d$ *be vertices, and let* $ch_0, \cdots, ch_d$ *be their children. A rooted tree with the root vertex* v *and its* m *children is denoted by* $br(v,m)$*. For* $d \le h$*, a* marker forest $MF(h,d)$ *is a* d*-terminal graph such that*

$$\begin{aligned} & V(MF(h,d))) \\ & = \begin{cases} \phi & \text{if } d = 0 \\ V(br(ch_0, h-d)) \;\cup\; (\bigcup_{1\le i\le d} V(br(ch_i, h+2i-2)) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \cup\; \{term_i\}) & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & E(MF(h,d)) \\ & = \begin{cases} \phi & \text{if } d = 0 \\ E(br(ch_0, h-d)) \;\cup\; (\bigcup_{1\le i\le d} E(br(ch_i, h+2i-2))) & \\ \qquad\qquad \cup\; \{(term_i, ch_{i-1}), (term_i, ch_i), (ch_0, ch_i)\} & \text{otherwise} \end{cases} \end{aligned}$$

A marker term $Mt(h,d)$ *is a term that denote* $MF(h,d)$*.*

Lemma 9. *In* $MF(h,d)$*,* $h+1 \le \#e(ch_i) \le h+2d+1$ *for each* $0 \le i \le d$ *and* $\#e(ch_i) < \#e(ch_j)$ *if* $i < j$*. More precisely,* $\#e(ch_i) = h+2i+1$ *for* $0 \le i < d$ *and* $\#e(ch_d) = h+2d$*.*

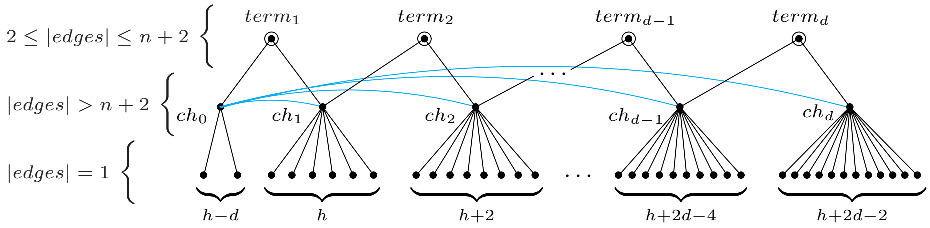


Fig. 6. d-terminal graph $MF(h, d)$

Definition 14. *Without loss of generality, we can assume that $x_1, \cdots, x_l$ are the representatives under the side condition C of t (i.e., $x_1, \cdots, x_l$ are mutually distinct and for each $x \in \mathcal{V}(t)$ there exists some x_i such that C contains $x = x_i$ with $1 \leq i \leq l$). Let $x_i \in X_{d_i}$.*

Let $n = |V(\psi(t\ \theta_{\mathbf{0}}))|$. The marker substitution $\theta_{\mathcal{M}}$ (see Fig. 6) is a ground substitution such that

$$\begin{cases} x_1\ \theta_{\mathcal{M}} & = Mt(n+2,\ d_1) \\ x_{i+1}\ \theta_{\mathcal{M}} & = Mt(n+2+\Sigma_{j=1}^{i}\ d_j,\ d_{i+1}) \qquad \text{for } 1 \leq i < l. \end{cases}$$

Example 8. In Example 5, $x\theta_{\mathcal{M}} = Mt(6, 2)$ (see Fig 7).

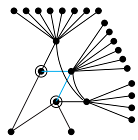


Fig. 7. Substitute $MF(6, 2)$ to x in Example 5

Lemma 10. *Let $v \in \psi(t\theta_{\mathcal{M}})$ and $n = |V(Center(t))|$. If v is inner, $2 \leq \#e(v) \leq n+2$. If v is outer, either $\#e(v) = 1$ or $\#e(v) > n+2$.*

Lemma 11. *If s and t are equivalent, an isomorphism α between $\psi(s\ \theta_{\mathcal{M}}))$ and $\psi(t\ \theta_{\mathcal{M}}))$ satisfies :*

- *α is an isomorphism between $Center(s)$ and $Center(t)$.*
- *For each x_i, there exists $x_{i'}$ with $x_i = x_{i'} \in \mathcal{C}$, $\alpha(\psi(x_i\theta_{\mathcal{M}})) = \psi(x_{i'}\theta_{\mathcal{M}})$, and $\alpha(x_i[s, j]) = x_{i'}[t, j]$.*

Proof. From Lemma 10, $\alpha(V(Center(s)) = V(Center(t))$.

Let $n = |V(Center(s))|$. For ch_0 in $\psi(x_i\theta_{\mathcal{M}})$, there exists $x_{i'}$ and with $x_i = x_{i'} \in \mathcal{C}$ and $\psi(x_{i'}\theta_{\mathcal{M}})$ such that $\alpha(ch_0) = ch_0'$ for ch_0' in $\psi(x_{i'}\theta_{\mathcal{M}})$ by construction. Since the unique neighborhood of ch_0 satisfying $2 \leq \#e(ch_0) \leq n+2$ is $term_1$, $\alpha(term_1) = term_1'$ with $term_1'$ in $\psi(x_{i'}\theta_{\mathcal{M}})$. Since ch_1 is the unique neighborhood of ch_0 that has more then $n+2$ edges, $\alpha(ch_1)$ must be ch_1'. Repeating similar construction, Lemma is proved. ∎

Sketch of proof of Theorem 5 By using the isomorphism α in Lemma 11, similar to the proof of Theorem 3, we obtain the proof of Theorem 5. ∎

6 Related Work

There are many works on algebraic constructions of graphs, including

- [FS96,Erw97] for functional programming,
- [CS92,Has97] from the categorical view point,
- [MSvE94,AA95] for term graphs,
- [Gib95] for directed acyclic graphs, and
- [BC87,ACPS93,OHS03] for graphs with bounded tree width.

Among them, only [BC87,ACPS93,OHS03] characterize the class of graphs with bounded tree width. Bauderon and Courcelle presented the complete axiomatization for ground terms [BC87,Cou90] in their formalization. Their algebraic construction consists of the function symbols

$$\begin{cases} \oplus_{m,n} : B_m \times B_n \to B_{m+n}, & e^2 : B_2 \quad \text{(edge)}, \\ \theta_{i,j,n} : B_n \to B_n, & \mathbf{1} : B_1 \quad \text{(vertex)}, \\ \sigma_\alpha \quad : B_m \to B_n, & \mathbf{0} : B_0 \quad \text{(empty)}, \end{cases}$$

where their interpretation ψ is

- $\psi(\oplus_{m,n}(t_1, t_2))$ is a disjoint union of $\psi(t_1)$ and $\psi(t_2)$,
- $\psi(\theta_{i,j,n}(t))$ fuses i-th and j-th terminals for $1 \leq i < j \leq n$, and
- $\psi(\sigma_\alpha(t))$ renumbers $\alpha(i)$-th terminal as i-th terminal for $\alpha : [1..m] \to [1..n]$.

and their complete axiomatization is shown in Fig. 8.

This paper gives the complete axiomatization for the variation of the algebraic construction given in [ACPS93]. Our choice of formalization comes from its compatibility with SP Term, since SP Term seems the most suitable data structure for programming on graphs with bounded tree width [OHS03]. The idea for the proof of the completeness for ground cases (Section 4) comes from [BC87]; this paper further extends the result to non-ground cases (Section 5).

7 Conclusion and Future Work

This paper presents the complete axiomatization for the variation of the algebraic construction given in [ACPS93]. Compared to the original algebraic construction in [ACPS93], we add σ_k^i (which is needed for completeness; the parallel composition p_k has the different infix notation $\oplus_k$ for readability), and omit s_k, which is defined as

$$s_k(t_1, \cdots, t_k) = \begin{cases} r_2(e^2 \oplus_2 l_2^1(t_1)) & \text{if } k = 1, \\ r_{k+1}^{k+1}(l_{k+1}^1(t_1) \oplus_{k+1} \cdots \oplus_{k+1} l_{k+1}^k(t_k)) & \text{if } k \geq 2. \end{cases}$$

Our final goal is to give the complete (finite) axiomatization of SP Term SP_k [OHS03], which precisely denotes graphs with tree width at most k. SP Term would be the most desirable algebraic construction for writing a functional

$$
\begin{array}{rcll}
(s \oplus t) \oplus u & = & s \oplus (t \oplus u) & \text{(R1)} \\
\sigma_\beta \cdot \sigma_\alpha(t) & = & \sigma_{\alpha\cdot\beta}(t) & \text{(R2)} \\
\sigma_{id}(t) & = & t & \text{(R3)} \\
\theta_{i,j,n} \cdot \theta_{i',j',n}(t) & = & \theta_{i',j',n} \cdot \theta_{i,j,n}(t) & \text{(R4-1)} \\
\theta_{i,j,n} \cdot \theta_{j,k,n}(t) & = & \theta_{i,j,n} \cdot \theta_{i,k,n}(t) & \text{(R4-2)} \\
\theta_{i,j,n} \cdot \theta_{j,k,n}(t) & = & \theta_{i,k,n} \cdot \theta_{j,k,n}(t) & \text{(R4-3)} \\
\theta_{i,i,n}(t) & = & t & \text{(R5)} \\
\sigma_\alpha(s) \oplus \sigma_{\alpha'}(t) & = & \sigma_{(\rightharpoondown_m \cdot \alpha) \oplus (\alpha' \cdot \leftharpoonup_p)}(t \oplus s) & \text{(R6)} \\
 & & \text{if } \alpha : [p] \to [n], \alpha' : [p'] \to [m] & \\
\theta_{i,j,m}(s) \oplus \theta_{i',j',n}(t) & = & \theta_{i,j,m} \cdot \theta_{m+i',m+j',m+n}(s \oplus t) & \text{(R7)} \\
\theta_{i,n+1,n+1}(t \oplus \mathbf{1}) & = & \sigma_{id\downarrow_{[n]} \oplus (n+1 \mapsto i)}(t) & \text{(R8)} \\
\theta_{i,j,n} \cdot \sigma_\alpha(t) & = & \sigma_\alpha \cdot \theta_{\alpha(i),\alpha(j),n}(t) \quad \text{if } \alpha : [n] \to [n] & \text{(R9)} \\
\sigma_\alpha \cdot \theta_{i,j,n}(t) & = & \sigma_\beta \cdot \theta_{i,j,n}(t) & \text{(R10)} \\
 & & \text{if } \alpha(m), \beta(m) \in \{i, j\} \text{ or } \alpha(m) = \beta(m) \text{ for each } m. & \\
t \oplus \mathbf{0} & = & t & \text{(R11)}
\end{array}
$$

where $\alpha \cdot \leftharpoonup_p (i + p) = \alpha(i)$ and $\rightharpoondown_m \cdot \alpha(j) = m + \alpha(j)$.

Fig. 8. Axioms of algebraic construction of graphs in [BC87,Cou90]

program on graphs with bounded tree width, because it has only 2 functional constructors: the series composition s_k and the parallel composition $\oplus_k$ (though it has relatively many constants $e_k(i,j)$ and $\mathbf{k}$, which can be treated in a homogeneous way). We will use two approaches, one from rewriting and another from graph theory.

- We already know the complete axioms on $\mathcal{B}_\infty$, which consist of terms constructed from $l_k^i, \oplus_k, r_k, \sigma_k^i, e^2, \mathbf{0}$. We can define $s_k, e_k(i,j), \mathbf{k}$ like "macros". *Can we deduce equations on "macros" from equations on terms constructed from original function symbols?*
- Minimal separator of a graph is essential for graphs with bounded tree width. We hope that the Menger-like property [Tho90] would help.

Acknowledgments

The author thanks Aart Middeldorp and Masahiko Sakai for information about references, and Philip Wadler for indicating an extension to non-ground cases. Thanks also to Jeremy Gibbons, Zhenjiang Hu, and Aki Takano for fruitful discussions. This research is supported by Special Coordination Funds for Promoting Science and Technology by Ministry of Education, Culture, Sports, Science and Technology, PRESTO by Japan Science and Technology Agency, and Kayamori Foundation of Informational Science Advancement.

References

[AA95] Z.M. Ariola and Arvind. Properties of a first-order functional language with sharing. *Theoretical Computer Science*, 146(1-2):69–108, 1995.

[ACPS93] S. Arnborg, B. Courcelle, A. Proskurowski, and D. Seese. An algebraic theory of graph reduction. *Journal of the Association for Computing Machinery*, 40(5):1134–1164, 1993.

[BC87] M. Bauderon and B. Courcelle. Graph expressions and graph rewritings. *Mathematical System Theory*, 20:83–127, 1987.

[BPT92] R.B. Borie, R.G. Parker, and C.A. Tovey. Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families. *Algorithmica*, 7:555–581, 1992.

[Cou90] B. Courcelle. Graph rewriting: An algebraic and logic approach. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, chapter 5, pages 194–242. Elsevier Science Publishers, 1990.

[CS92] V.-E. Căzănescu and G. Stefănescu. A general result on abstract flowchart schemes with applications to study of accessibility, reduction and minimization. *Theoretical Computer Science*, 99:1–63, 1992.

[Erw97] M. Erwig. Functional programming with graphs. In *Proc. 1997 ACM SIGPLAN International Conference on Functional Programming, ICFP '97*, pages 52–65. ACM Press, 1997. SIGPLAN Notices 32(8).

[FS96] L. Fegaras and T. Sheard. Revisiting catamorphisms over datatypes with embedded functions *(or, programs from outer space)*. In *Proc. 23rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL'96*, pages 284–294. ACM Press, 1996.

[Gib95] J. Gibbons. An initial-algebra approach to directed acyclic graphs. In B. Moller, editor, *Mathematics of Program Construction, MPC'95*, pages 282–303, 1995. Lecture Notes in Computer Science, Vol. 947, Springer-Verlag.

[GMT02] J. Gustedt, O.A. Mæhle, and A. Telle. The treewidth of Java programs. In *Proc. 4th Workshop on Algorithm Engineering and Experiments, ALENEX 2002*, pages 86–97, 2002. Lecture Notes in Computer Science, Vol. 2409, Springer-Verlag.

[Has97] M. Hasegawa. Models of sharing graphs, 1997. PhD thesis, University of Edinburgh.

[MSvE94] M.J.Plasmeijer M.R. Sleep and M.C.J.D. van Eekelen, editors. *Term Graph Rewriting, Theory and Practice*. Wiley, 1994.

[OHS03] M. Ogawa, Z. Hu, and I. Sasano. Iterative-free program analysis. In *Proc. 8th ACM SIGPLAN International Conference on Functional Programming, ICFP'03*, pages 111–123. ACM Press, 2003. Uppsala, Sweden.

[RS86] N. Robertson and P.D. Seymour. Graph minors II. algorithmic aspects of tree-width. *Journal of Algorithms*, 7:309–322, 1986.

[SHTO00] I. Sasano, Z. Hu, M. Takeichi, and M. Ogawa. Make it practical: A generic linear-time algorithms for solving maximum-weightsum problems. In *Proc. 5th ACM SIGPLAN International Conference on Functional Programming, ICFP'00*, pages 137–149. ACM Press, 2000. Montreal, Canada.

[Tho90] R. Thomas. A Menger-like property of tree-width: The finite case. *Journal of Combinatorial Theory, Series B*, 48:67–76, 1990.

[Tho98] M. Thorup. All structured programs have small tree width and good register allocation. *Information and Computation*, 142:159–181, 1998.

[Wey39] H. Weyl. *Classical groups*. Princeton University Press, 1939.