Unique Normal Form Property of Higher-Order Rewriting Systems

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Abstract. Within the framework of Higher-Order Rewriting Systems proposed by van Oostrom, a sufficient condition for the unique normal form property is presented. This requires neither left-linearity nor termination of the system.

1 Introduction

Several frameworks of rewriting systems for higher-order expressions have been proposed [Klo80, Nip91, MN94, LS93, KvO95]. Van Oostrom and van Raamdonk proposed a framework of Higher-Order Rewriting Systems (HORSs) [vO94, vOvR94, vR96], capable of unifying the existing theory of rewriting, e.g., Combinatory Reduction Systems (CRSs) [Klo80], (another variation of) Higher-order Rewriting Systems (HRSs) by Nipkow [Nip91], and Term Rewriting Systems (TRSs). They also presented a sufficient condition for the Church-Rosser property of HORSs by introducing a notion corresponding to orthogonality (i.e., non-overlap and left-linearity) of TRSs.

The framework of HORSs is characterised by the clear separation of *replacement* with rewrite rules and *matching/substitution* for application of the rules. The latter is done by another rewriting system called *substitution calculus*, which is a parameter that determines an HORS together with a set of rewrite rules. In [vO94], some abstract conditions are first presented on rewrite rules and substitution calculi, and then properties such as the *Church-Rosser property* are derived for HORSs satisfying the conditions. In particular, a *non-overlapping left-linear pattern HORS* is shown to satisfy the conditions (see also [vOvR94], [vR96]).

It is well-known that a non-overlapping left-linear TRS has the Church-Rosser property. Without left-linearity and with a slight modification of the non-overlap requirement, some results [Che81, dV90, TO94, MO95] have concluded the *unique normal form property* of TRSs. The unique normal form property is a sufficient condition for consistency of the system [KdV89] and weaker than the Church-Rosser property.

Let us briefly introduce the methodology in [dV90] (see also [KdV90]). The following theorem concerning (abstract) rewriting systems is the key observation.

Theorem 1 [dV90]. *Suppose that rewriting systems* $\rightarrow_0$ *and* $\rightarrow_1$ *satisfy the following:*

1. $\rightarrow_0 \subseteq \rightarrow_1$,

2. $\rightarrow_1$ *has the Church-Rosser property, and*
3. the set of normal forms of $\rightarrow_1$ *contains those of* $\rightarrow_0$.

Then, $\rightarrow_0$ *has the unique normal form property.* ∎

In order to apply this to TRSs, the notion of (conditional) linearisation is introduced. Linearisation is a transformation from a TRS R to a *left-linear semi-equational conditional TRS* R^L and non-left-linearity of R is expressed in the condition part of R^L. Then, 1 of Theorem 1 apparently holds, and 3 is derived by assuming 2. Consequently, the unique normal form property of R is reduced to the Church-Rosser property of R^L. Using the result that *a non-overlapping left-linear semi-equational conditional TRS has the Church-Rosser property* [BK84], we obtain a sufficient condition for the unique normal form property of TRSs.

In this paper we extend this result to HORSs. The main theorem states that "a strongly non-overlapping pattern HORS has the unique normal form property".

The structure of this paper is as follows. Section 2 provides basic definitions and Section 3 explains HORSs. In Section 4, we introduce the notion of *Context-Conditional HORS* (CCHORS, for short), and give a sufficient condition for the Church-Rosser property of CCHORSs. Based on this result, we present the main result using *Context-Conditional Linearisation* in Section 5.

2 Definitions

The definitions in this subsection are based on [vO94].

2.1 Rewriting Systems

A *rewriting system* $\langle D, \rightarrow \rangle$ is a pair of the underlying domain D and the binary relation $\rightarrow$ called the *rewrite relation*. The domain D is often omitted when it is clear from the context. Suppose $a_1, a_2, \ldots \in D$. Each element $a_1 \rightarrow a_2$ of the rewrite relation is called a *rewrite step*. A sequence of rewrite steps $a_1 \rightarrow \cdots \rightarrow a_n$ is called a *rewrite*. The symmetric closure, the reflexive transitive closure and the reflexive transitive symmetric closure of $\rightarrow$ are written as $\leftrightarrow$, $\rightarrow^*$ and $\leftrightarrow^*$, respectively. If there is no a_2 such that $a_1 \rightarrow a_2$, then a_1 is a *normal form* of the reduction system. The set of normal forms of $\rightarrow$ is denoted by $NF_{\rightarrow}$. If $a_1 \rightarrow^* a_2$ and $a_2 \in NF_{\rightarrow}$, then a_2 is called a normal form of a_1, with notation $a_1 \rightarrow^{*!} a_2$.

A rewriting system $\rightarrow$ is *terminating* if there is no infinite sequence such as $a_1 \rightarrow a_2, \rightarrow \cdots$. A rewriting system $\rightarrow$ has the *unique normal form property* if for any normal forms a_1 and a_2, $a_1 \leftrightarrow^* a_2$ implies $a_1 \equiv a_2$. A rewriting system $\rightarrow$ has the *Church-Rosser property* if for any $a_1 \leftrightarrow^* a_2$ there exists a_3 such that $a_1 \rightarrow^* a_3$ and $a_2 \rightarrow^* a_3$. A rewriting system is said to be *complete* if it is terminating and has the Church-Rosser property.

2.2 Simply-typed preterms

In this subsection, we introduce higher-order expressions with so-called simple types. Note that the following definition distinguishes between the syntactic category of bound variables and that of free variables. Note also that the phrase "$(x, y \in)\ X$" introduces a set X with variables x, y ranging over X.

Definition 2. The set $(\delta, \tau \in)$ $\mathcal{T}$ of *Simple types* is inductively defined as follows:

1. the *base type* o is a simple type,
2. if δ and τ are simple types, then $\delta \to \tau$ is a simple type.

A function $\mathtt{arity} : \mathcal{T} \to \mathsf{N}$ is defined by (1) $\mathtt{arity}(o) \stackrel{\text{def}}{=} 0$, and (2) $\mathtt{arity}(\delta \to \tau) \stackrel{\text{def}}{=} \mathtt{arity}(\tau) + 1$.

Definition 3. An *alphabet* $(a, b, c \in)$ $\mathcal{A}$ is a countable set consisting of the following symbols:

1. *application* -(-),
2. *abstraction* -.-,
3. for each simple type τ,
   (a) *operator* (or *constant*) symbols $\mathtt{F}^{\tau}, \mathtt{G}^{\tau}, \mathtt{H}^{\tau}, \cdots$,
   (b) *bound variables* $\xi^{\tau}, \eta^{\tau}, \zeta^{\tau}, \cdots$,
   (c) *free variables* $x^{\tau}, y^{\tau}, z^{\tau}, \cdots$. Among these symbols, we have distinguished symbols $\Box_1^{\tau}, \Box_2^{\tau}, \cdots$ called *holes*. The hole $\Box_1^{\tau}$ is also written as $\Box^{\tau}$.

   The operators together with the free variables form the *rewrite alphabet* $\mathcal{A}_{\mathcal{R}}$.

The type information of the symbols are omitted if they are clear from the context. Therefore, ξ^{τ}, $\mathtt{F}^{\tau}$, x^{τ} and $\Box_i^{\tau}$ are often written as ξ, $\mathtt{F}$, x and $\Box_i$, respectively.

From the symbols in the alphabet, (simply-typed) raw preterms are built in the following way. Let Z be a set of bound variables. The set $(s, t, r \in)$ $\mathsf{RPT}(\mathcal{A})$ of *raw preterms* is inductively defined as follows:

1. operator $\mathtt{F}^{\tau}$, bound variable ξ^{τ} and free variable x^{τ} are raw preterms of type τ.
2. if s is a raw preterm of type $\delta \to \tau$ and t is a raw preterm of type δ, then $s(t)$ is a raw preterm of type τ,
3. if s is a raw preterm of type τ, then $\xi^{\delta}.s$ is a raw preterm of type $\delta \to \tau$.

Since every raw preterm has a unique simple type, we can define $\mathtt{arity}(s)$ as the arity of the type of s.

Definition 4. Let $\mathcal{A}$ be an alphabet. The set $(\phi, \psi, \chi \in)$ $\mathtt{Pos} \stackrel{\text{def}}{=} \{0,1\}^*$ is called the set of *positions* (or *occurrences*). The concatenation of ϕ and ψ is denoted by $\phi;\psi$. We say ϕ is a *prefix* of ψ when there exists ϕ' such that $\psi = \phi;\phi'$. We write $\phi \preceq \psi$ when ϕ is a prefix of ψ, and $\phi \prec \psi$ when $\phi \preceq \psi$ and $\phi \neq \psi$. Two positions are *disjoint* if they are not prefixes of each other.

Positions in a raw preterm t are represented in $\mathtt{Pos}$ in the usual way (see also [MN94] and [LS93][3]). The set of all positions in t is denoted by $\mathtt{Pos}(t)$. The subterm at position ϕ in s is denoted by s/ϕ. Furthermore, the symbol at position ϕ in s is denoted by $s(\phi)$. For a symbol a, we define $a\mathtt{Pos}(s) \stackrel{\text{def}}{=} \{\phi \in \mathtt{Pos}(s) \mid s(\phi) = a\}$. If $a\mathtt{Pos}(s)$ is a singleton set, the element is also denoted by $a\mathtt{Pos}(s)$. We also define $\mathcal{R}\mathtt{Pos}(s) \stackrel{\text{def}}{=} \{\phi \in \mathtt{Pos}(s) \mid s(\phi) \in \mathcal{A}\}$. The function $\mathtt{Fvar}$ ($\mathtt{Bvar}$) maps a raw preterm to the set of free (bound) variables occurring in it.

For a term s and a position $\phi \in \mathtt{Pos}(s)$, we say ϕ is *in the scope of* ξ *in* s if there exists ψ satisfying $s/\psi = \xi.s'$ and $\psi \prec \phi$. If $s(\phi) = \xi$ and ϕ is not in the scope of ξ in s, then ϕ is called an *unbound occurrence* of ξ, and ξ is said to *occur unboundly in* s. The set of bound variables occurring unboundly in a raw preterm s is denoted by $\mathtt{UBvar}(s)$.

[3] In [MN94] and [LS93], $\{1,2\}^*$ is used instead of $\{0,1\}^*$ for $\mathtt{Pos}$.

Definition 5. Let $\mathcal{A}$ be an alphabet. A *preterm* is a raw preterm without any unbound occurrences of bound variables. From now on, we restrict s, t and r to range over the set $\mathsf{PT}(\mathcal{A})$ of preterms. If $\mathtt{Fvar}(s) = \emptyset$, then s is said to be *closed*.

An *instantiation* is a set $\theta \stackrel{\text{def}}{=} \{x_i := t_i \mid i = 1, \cdots, n\}$ of pairs consisting of a free variable and a preterm. For a preterm t, $t\theta$ denotes the term obtained by replacing all x_i with t_i $(i = 1, \ldots, n)$.

A preterm is an *m-ary precontext*, if the holes occurring in it are among $\Box_1, \ldots, \Box_m$. We use C, D and E to range over precontexts. We denote m-ary precontexts by $C[m]$, $D[m]$ and $E[m]$. We also write a unary precontext as $C[\,]$, and a binary precontext as $C[\,,\,]$. The term $C[m]\{\Box_i := s_i \mid i = 1, \cdots, m\}$ is denoted by $C[s_1, \cdots, s_m]$. An m-ary precontext is *linear* if every hole $\Box_1, \cdots, \Box_m$ occurs exactly once in it.

Definition 6. Let s be a preterm and x a free variable. Then $\xi.s'$ is called an *x-closure of s* if x does not appear in the scope of ξ in s and s' is obtained from s by replacing every occurrence of x with ξ. The set of all x-closures of s is denoted by $\mathtt{clos}_x(s)$. For a sequence $\sigma \stackrel{\text{def}}{=} x_1, \ldots x_m$ of free variables, a *σ-closure of s* is an x_1-closure of ... of an x_m-closure of s.

3 Higher-Order Rewriting Systems

As is mentioned at the beginning, the *actual* rewrite relation of an HORS is defined by two rewrite relations, that is, replacement with rewrite rules and another rewriting system called substitution calculus which performs matching and substitution. In other words, the rewrite relation of an HORS is defined modulo the substitution calculus. For example, let us consider an HORS which has the usual λ-calculus as its substitution calculus, and as rules $1 + 1 \rightarrow 2$, $2 + 2 \rightarrow 4$, $\xi.\xi/\xi \rightarrow \xi.1$ and $\xi.2 \times \xi \rightarrow \xi.\xi + \xi$. Then we have the following computation. Subexpressions to which rewrite rules are applied are indicated by $[\,]$.

$$\begin{array}{ll} \quad 2 \times (1/1 + 3/3) & \leftarrow_\beta^* 2 \times (\eta.(\eta(1) + \eta(3)))([\xi.\xi/\xi]) \\ \rightarrow 2 \times (\eta.(\eta(1) + \eta(3)))(\xi.1) & \rightarrow_\beta^* 2 \times ([1 + 1]) \\ \rightarrow 2 \times 2 & \leftarrow_\beta ([\xi.2 \times \xi])(2) \\ \rightarrow (\xi.\xi + \xi)(2) & \rightarrow_\beta [2 + 2] \\ \rightarrow 4 & \end{array}$$

In this paper, we are interested in a *pattern HORS*, not a general HORS. A pattern HORS has a simply-typed λ-calculus $\lambda_{\overline{\eta}}^{\rightarrow}$ with *restricted η-expansion* as its substitution calculus, and the left-hand side of each of its rewrite rules is a special term called a *pattern*. Various rewriting systems, e.g., CRSs [Klo80], HRSs [Nip91] and also TRSs, can be embedded into pattern HORSs [vO94].

Basically, the definitions in this section follow [vO94]. The theorems in this section without any notice appear in [vO94] and/or [vOvR94].

3.1 Simply-Typed λ-calculus as Substitution Calculus

Definition 7. The α-, β- and η-rewrite relations on $\mathsf{PT}(\mathcal{A})$ are denoted by $\rightarrow_\alpha$, $\rightarrow_\beta$ and $\rightarrow_\eta$, respectively [Bar84]. In order to indicate that the rewrite *contracts* the subterm at position ϕ, we also use $\rightarrow_{(\phi,\aleph)}$ for $\aleph = \alpha, \beta, \eta$. We define $\rightarrow_{(\phi,\overline{\eta})} \stackrel{\text{def}}{=} \leftarrow_{(\phi,\eta)}$

$-\leftarrow_\beta$ and $\rightarrow_{\bar\eta}$ is the union over all positions ϕ of $\rightarrow_{(\phi,\bar\eta)}$. The rewriting system $\langle \mathsf{PT}(\mathcal{A}), \rightarrow_\alpha \cup \rightarrow_\beta \cup \rightarrow_{\bar\eta}\rangle$ is denoted by $\lambda^{\rightarrow}_{\bar\eta}$.

A rewrite step $s \rightarrow_{(\phi,\aleph)} t$ is *below* ψ if $\phi \succeq \psi$. In order to indicate that a rewrite from s to t is a sequence of $\aleph$-rewrite steps below ψ, we write $s \rightarrow^*_{(\succeq\psi,\aleph)} t$.

The following property is a basic fact about typed λ-calculi, see e.g., Thm. 2.35 and Thm. 2.38 in [Wol93].

Theorem 8. *$\rightarrow_{\beta\bar\eta}$ is complete modulo $\leftrightarrow^*_\alpha$.* ∎

Theorem 9. *Suppose $s \in NF_{\rightarrow\bar\eta}$. Then $s \rightarrow_\beta t$ implies $t \in NF_{\rightarrow\bar\eta}$. Moreover, if $u \in NF_{\rightarrow\bar\eta}$, then $s\{x := u\} \in NF_{\rightarrow\bar\eta}$.* ∎

Definition 10. A *term* (*context*, resp.) is a preterm (precontext) in $\rightarrow_{\beta\bar\eta}$-normal form. The set of $\mathcal{A}$-terms is denoted by $\mathsf{T}(\mathcal{A})$.

Remark. A term s' of arity m can always be written as $\xi_1, \cdots, \xi_m.a(s'_1)\cdots(s'_k)$, where k is the arity of the symbol a. Each s'_i $(i = 1, \cdots, k)$ is in this form again. That is, s' is the $x_1, \cdots, x_m$-closure of some term $s \stackrel{\text{def}}{=} a(s_1)\cdots(s_k)$. In this case, a is called the *head* of s' and its position is denoted by $\mathtt{head}(s')$.

The descendant relation defined below traces the occurrences of $\mathcal{A}_\mathcal{R}$-symbols along a rewrite.

Definition 11. For a rewrite step $u \stackrel{\text{def}}{=} s \rightarrow_{(\phi,\aleph)} t$ ($\aleph = \alpha, \beta, \eta$ or $\bar\eta$), the *descendant relation* $|\underline{u}|$: $\mathsf{Pos} \times \mathsf{Pos}$ is defined as follows. Let $\psi \in \mathcal{R}\mathsf{Pos}(s)$. If $\phi \not\preceq \psi$, then $\psi\ |\underline{u}|\ \psi$. Otherwise:

1. if $\aleph = \alpha$, then $\psi\ |\underline{u}|\ \psi$,
2. if $\aleph = \beta$ and $s/\phi \stackrel{\text{def}}{=} \xi.s_1(s_2)$, then
   (a) $\psi\ |\underline{u}|\ \phi;\psi_1$ if $\psi \stackrel{\text{def}}{=} \phi; 00; \psi_1$, and
   (b) $\psi\ |\underline{u}|\ \phi;\psi_1;\psi_2$ if $\psi \stackrel{\text{def}}{=} \phi; 1; \psi_2$ and $s_1(\psi_1) = \xi$,
3. if $\aleph = \eta$ and $\psi = \phi; 00; \psi'$, then $\psi\ |\underline{u}|\ \phi\psi'$,
4. if $\aleph = \bar\eta$, then $|\underline{u}| \stackrel{\text{def}}{=} |\underline{t \rightarrow_{(\phi,\eta)} s}|^{-1}$.

The descendant relation $|\underline{d}|$ associated to a rewrite $d : t_1 \rightarrow^*_{\beta\bar\eta} t_n$ is the concatenation of descendant relations corresponding to each rewrite step in d. If $\phi \in \mathcal{R}\mathsf{Pos}(t_1)$ and $\phi\ |\underline{d}|\ \psi$, then ψ is called a *descendant of* ϕ, and ϕ is called an *origin of* ψ.

3.2 Pattern HORS

Definition 12. Let l' of arity m be the $x_1, \cdots, x_m$-closure of a term l. Then l' is called a *pattern*, if it satisfies the following conditions:

1. the head of l is an operator symbol,
2. let x of arity k be among $x_1, \cdots, x_m$,
   (a) there is at least one occurrence of x in l,
   (b) for every occurrence $\phi; 0^k$ of x in l, we have that if $l \rightarrow^{*!}_{(\succeq\phi,\eta)} g$, then $g/\phi \equiv x_i(\xi_1)\cdots(\xi_k)$, where $\xi_1, \cdots, \xi_k$ is a list of pairwise distinct bound variables.

Moreover, in (a) above, if there is *precisely* one occurrence of x in l, then l' is called a *linear* pattern.

Intuitively, 1 of the above conditions corresponds to "the left-hand side of each rule is not a variable" on TRSs, and 2 to "every variable in the right-hand side of each rule appears in the left-hand side". For the precise discussion about the correspondence between a pattern HORS and a TRS, see Lemma 15.

Definition 13. A *pattern Higher-Order Rewriting System* (pattern HORS) is a triple $\langle \mathcal{A}, \lambda_{\overline{\eta}}^{\rightarrow}, \mathcal{R} \rangle$ consisting of an alphabet $\mathcal{A}$, a *substitution calculus* $\lambda_{\overline{\eta}}^{\rightarrow}$, and a set $\mathcal{R}$ of pattern rewrite rules. A *pattern rewrite rule* is a pair of terms of the form $l \to r$ satisfying the following conditions: (1) both l and r are closed, (2) l and r are of the same type and (3) l is a pattern.

We use $\aleph$ and $\beth$ to range over rewrite rules. The first and second components of a rule $\aleph$ are denoted by $\mathtt{lhs}(\aleph)$ and $\mathtt{rhs}(\aleph)$ and are its *left-* and *right-hand side*, respectively.

A pattern rewrite rule with a linear left-hand side is called a *left-linear pattern rewrite rule*, and a pattern HORS with only left-linear pattern rewrite rules is called a *left-linear pattern HORS*.

Let $\mathcal{H} \stackrel{\mathrm{def}}{=} \langle \mathcal{A}, \lambda_{\overline{\eta}}^{\rightarrow}, \mathcal{R} \rangle$ be a pattern HORS. We associate two rewriting systems to a pattern HORS. They are both defined on the set of preterms.

1. For a rule $\aleph \stackrel{\mathrm{def}}{=} l \to r \in \mathcal{R}$ and precontext $C[\,]$, we define the *replacement of* $\aleph$ *in* $C[\,]$ by $C[l] \to_{C[\aleph]} C[r]$. This is generalised to the *replacement of* $\aleph$ in an arbitrary precontext, by defining $\to_{\aleph} \stackrel{\mathrm{def}}{=} \bigcup_{C[\,]} \to_{C[\aleph]}$, and to *replacement* of an arbitrary rule, by defining $\to_{\mathcal{R}} \stackrel{\mathrm{def}}{=} \bigcup_{\aleph \in \mathcal{R}} \to_{\aleph}$.
2. The *rewrite relation* of $\mathcal{H}$ is obtained by $\to_{\mathcal{H}} \stackrel{\mathrm{def}}{=} \leftrightarrow^*_{\beta\overline{\eta}} ; \to_{\mathcal{R}} ; \leftrightarrow^*_{\beta\overline{\eta}}$.

Next, we present some basic definitions and properties related to pattern HORSs. In the following, we regard terms (i.e., $\to_{\beta\overline{\eta}}$-normal forms) as representatives of the $\leftrightarrow^*_{\beta\overline{\eta}}$-equivalence classes, and restrict, in principle, the rewrite relation $\to_{\mathcal{H}}$ to the relation on the terms. Then, the following property is useful for simplifying the arguments.

Theorem 14. *When we restrict the domain of the rewrite relation* $\to_{\mathcal{H}}$ *to* $\mathsf{T}(\mathcal{A})$*, we may assume the following without loss of generality:*

1. *the precontext* $C[\,]$ *in the definition of* $\to_{\mathcal{R}}$ *is a linear context,*
2. *we may use* $\leftarrow^*_{\beta\overline{\eta}} ; \to_{\mathcal{R}} ; \to^*_{\beta\overline{\eta}}$ *instead of* $\leftrightarrow^*_{\beta\overline{\eta}} ; \to_{\mathcal{R}} ; \leftrightarrow^*_{\beta\overline{\eta}}$ *in the definition of* $\to_{\mathcal{H}}$. ▮

In the second item, $\leftarrow^*_{\beta\overline{\eta}}$ and $\to^*_{\beta\overline{\eta}}$ are called the *expansion* and *reduction* of the rewrite step, respectively.

The following property is called '*head-definedness*' in [vO94].

Theorem 15. *Let* $l \to r$ *be any pattern rewrite rule.*

1. *Let* $C[\,]$ *be a linear context with hole-position* ψ*. Then* $\psi;\phi$ *has a unique descendant* χ *along any reduction from* $C[l]$ *to its normal form* $C[l]\downarrow_{\beta\overline{\eta}}$*.*
2. *For a position* χ *in a term* s*, there is at most one linear context* $C[\,]$ *with hole-position* ψ*, such that* $\psi;\mathtt{head}(l)$ *is an origin of* χ *along the expansion from* s *to* $C[l]$*.* ▮

The head-definedness of a pattern HORS guarantees that "a rewrite step is determined by the position and the rewrite rule" which trivially holds in the case of TRSs. This makes it possible to give the following definition.

Definition 16. In the second item of the lemma above, if $C[]$ exists, the pair $(\chi, \aleph)$ is called a *redex* in s and any such expansion is called an *extraction of* $(\chi, \aleph)$ *from* s (*into* $C[]$). If the expansion of a rewrite step $w : s \rightarrow_{\mathcal{H}} t$ is an extraction of $(\phi, \aleph)$, we say w *rewrites* the redex $(\phi, \aleph)$, with notation $s \rightarrow_{(\phi,\aleph)} t$.

The descendant relation defined for $\lambda_{\overline{\eta}}^{\rightarrow}$ is extended to pattern HORSs in the following way.

Definition 17. Let $\mathcal{H}$ be a pattern HORS and $\aleph : l \rightarrow r$ a rewrite rule of $\mathcal{H}$. For a replacement $u : C[l] \rightarrow_{C[\aleph]} C[r]$, the descendant relation $|\underline{u}|$: $\mathcal{R}\mathtt{Pos}(C[l]) \times \mathcal{R}\mathtt{Pos}(C[r])$ is defined as follows:

$$\phi \, |\underline{u}| \, \psi \text{ iff } \phi = \psi \not\succeq \Box\mathtt{Pos}(C[]).$$

For a rewrite step $s \rightarrow_{\mathcal{H}} t$ in $\mathcal{H}$, the descendant relation $|\underline{s \rightarrow_{\mathcal{H}} t}|$ is the concatenation of the descendant relations corresponding to its expansion, replacement and reduction. The extension of the descendant relation to a rewrite $t_1 \rightarrow_{\mathcal{H}}^{*} t_n$ parallels the case of $\lambda_{\overline{\eta}}^{\rightarrow}$.

Definition 18. Let $\mathcal{H}$ be a pattern HORS and $u : s \rightarrow^{*} t$ a rewrite in $\mathcal{H}$ or $\lambda_{\overline{\eta}}^{\rightarrow}$. Suppose that $(\phi, \aleph)$ is a redex in s and that $\phi \, |\underline{u}| \, \phi'$. If $(\phi', \aleph)$ is a redex in t, it is called a *residual* of $(\phi, \aleph)$.

In the following, we outline the proof that "a non-overlapping left-linear pattern HORS has the Church-Rosser property" in [vO94] and [vOvR94]. The Church-Rosser property is derived from the Finite Developments theorem, which is an extension of the first-order case. In the case of a pattern HORS, the technical notion 'simultaneity' is introduced, and it mediates between the sufficient condition (i.e., non-overlap and left-linearity) and the Finite Developments theorem.

Definition 19. Let $\mathcal{H}$ be a pattern HORS and $\mathcal{U}$ a set of redexes in s_1. A rewrite $w : s_1 \rightarrow_{\mathcal{H}} \cdots \rightarrow_{\mathcal{H}} s_n$ is called a *development of* $\mathcal{U}$ if for each $w_i : t_i \rightarrow_{\mathcal{H}} t_{i+1}$, there exists $u_i \in \mathcal{U}$ and w_i rewrites a residual of u_i. If there are no residuals of $\mathcal{U}$ in s_n, w is called *complete*.

Definition 20. Let $\mathcal{H}$ be a pattern HORS. Let $\mathcal{U} \stackrel{\text{def}}{=} \{u_1, \cdots, u_m\}$ be a set of (pairwise distinct) redexes in a term s, where $u_i \stackrel{\text{def}}{=} (\chi_i, l_i \rightarrow r_i)$ $(i = 1, \cdots, m)$. An expansion $e : s \leftarrow_{\beta\overline{\eta}}^{*} C[l_1, \cdots, l_m]$ is called an *extraction of* $\mathcal{U}$ *from* s (*into* $C[m]$), if $C[m]$ is a linear context and for every $u_i \in \mathcal{U}$, we have that $\Box_i\mathtt{Pos}(C[m]); \mathtt{head}(l_i)$ is an origin of χ_i. If such an extraction exists for the set $\mathcal{U}$, then $\mathcal{U}$ is called a *simultaneous* set of redexes. A pattern HORS is *simultaneous* (*pairwise simultaneous*) if every set (pair) of redexes is simultaneous.

Definition 21. Let $u \stackrel{\text{def}}{=} (\phi; 0^m, \aleph)$ and $v \stackrel{\text{def}}{=} (\psi; 0^n, \mathtt{i})$ be distinct redexes in a term s, where m (n) is the arity of the head symbol of $\mathtt{lhs}(\aleph)$ $(\mathtt{lhs}(\mathtt{i}))$.

1. The redexes are said to be *disjoint* if ϕ and ψ are disjoint.
2. Otherwise we may assume without loss of generality that $\psi \succeq \phi$. Let $\mathtt{lhs}(\aleph)$ be the $x_1, \cdots, x_k$-closure of l. Two cases are distinguished:
   (a) the redex u is said to *nest* v, if $\phi; \chi; \omega = \psi$, where $\chi; 0^i$ is the position of some variable x of arity i in l, and x is among $x_1, \cdots, x_k$,
   (b) otherwise u is said to *overlap* with v.

A rule $\aleph$ *overlaps with* another rule $\mathtt{i}$, if some $\aleph$-redex overlaps with some $\mathtt{i}$-redex. A pattern HORS *has overlap* if some rule overlaps with another one, it is *non-overlapping* otherwise.

It is decidable whether an HORS has overlap or not.

The following theorem 22, 23 and 24 correspond to Lemma 3.9, Lemma 3.8 and Theorem 3.10 in [vOvR94], respectively.

Theorem 22. *Let $\mathcal{H}$ be a left-linear pattern HORS. Then $\mathcal{H}$ is non-overlapping iff $\mathcal{H}$ is pairwise simultaneous iff $\mathcal{H}$ is simultaneous.* ∎

Theorem 23 (Finite Developments). *For any left-linear pattern HORS, every complete development of a set of simultaneous redexes ends in the same term.* ∎

From these theorems, the following theorem is derived immediately.

Theorem 24. *A non-overlapping left-linear pattern HORS has the Church-Rosser property.* ∎

4 Context-Conditional HORS

In this section, we introduce Context-Conditional HORSs (CCHORSs). We then give a sufficient condition for the Church-Rosser property of CCHORSs.

4.1 Context Condition

Definition 25. A *Context-Conditional pattern rewrite rule* (CC rule, for short) is $\hat{\aleph} \stackrel{\text{def}}{=} \hat{l} \to \hat{r} \Leftarrow Q$ satisfying the following:

1. $\hat{l} \to \hat{r}$ is a pattern rewrite rule, and
2. Q is a sequence $N_1, \ldots, N_m$ of sets of natural numbers satisfying:
   (a) $\bigcup_{1 \le i \le m} N_i = \{1, \ldots, \mathtt{arity}(\hat{l})\}$, and
   (b) $N_i \cap N_j = \emptyset$ for $i, j = 1, \ldots, m$ and $i \neq j$.

We call Q the *condition part* of $\hat{\aleph}$, and $\hat{l} \to \hat{r}$ the *unconditional part* of $\hat{\aleph}$. A CC rule is *left-linear* if its unconditional part is left-linear. A *Context-Conditional HORS* (CCHORS) is a triple $\langle \mathcal{A}, \lambda_{\overline{\eta}}^{\rightarrow}, \hat{\mathcal{R}} \rangle$ consisting of an alphabet $\mathcal{A}$, a *substitution calculus* $\lambda_{\overline{\eta}}^{\rightarrow}$, and a set $\hat{\mathcal{R}}$ of CC rules.

In the rest of this section, $\hat{\mathcal{H}} \stackrel{\text{def}}{=} \langle \mathcal{A}, \lambda_{\overline{\eta}}^{\rightarrow}, \hat{\mathcal{R}} \rangle$ denotes a CCHORS. The rewriting system associated to $\hat{\mathcal{H}}$ is defined as follows.

Definition 26. Let $\hat{\aleph} : \hat{l} \to \hat{r} \Leftarrow N_1, \cdots, N_m \in \hat{\mathcal{R}}$. The *replacement* $\to_{\hat{\aleph}^i}$ *with* $\hat{\aleph}$ *of rank* i $(i = 0, 1, \cdots)$ is inductively defined as follows:

$$\to_{\hat{\aleph}^0} = \emptyset,$$

$$\to_{\hat{\aleph}^{i+1}} = \{(C[\hat{l}], C[\hat{r}]) \,|\, \text{the precontext } C[\,] \text{ satisfies the } \leftrightarrow^*_{\hat{\mathcal{H}}^i}\text{-}\textit{context condition} \text{ of } \hat{\aleph}\},$$

where $\to_{\hat{\mathcal{R}}^i} \stackrel{\text{def}}{=} \bigcup_{\hat{\mathtt{i}} \in \hat{\mathcal{R}}} \to_{\hat{\mathtt{i}}^i}$, $\to_{\hat{\mathcal{H}}^i} \stackrel{\text{def}}{=} \leftrightarrow^*_{\beta\overline{\eta}}; \to_{\hat{\mathcal{R}}^i}; \leftrightarrow^*_{\beta\overline{\eta}}$, and we say $C[\,]$ satisfies the $\leftrightarrow^*_{\hat{\mathcal{H}}^i}$-context condition of $\hat{\aleph}$ if the following holds:

for any $\psi \in \Box\mathsf{Pos}(C[\,])$,

1. there exists ψ' such that $\psi = \psi'; 0^{\hat{m}}$ and $C[\hat{l}]/\psi' = \hat{l}(s_1)\cdots(s_{\hat{m}})$, where $\hat{m}$ is the arity of $\hat{l}$,
2. for any $j = 1, \cdots, m$, $\tilde{s}_k \leftrightarrow^*_{\hat{\mathcal{H}}^i} \tilde{s}_{k'}$ if $k, k' \in N_j$, where $\hat{l}(\tilde{s}_1)\cdots(\tilde{s}_{\hat{m}})$ is the preterm obtained by replacing all bound variables $\xi_1, \cdots, \xi_k$ occurring unboundly in the raw preterm $\hat{l}(s_1)\cdots(s_{\hat{m}})$ with fresh free variables $x_1, \cdots, x_k$.

We define the *replacement in* $\hat{\mathcal{R}}$ by $\rightarrow_{\hat{\mathcal{R}}} \stackrel{\mathrm{def}}{=} \bigcup_i \rightarrow_{\hat{\mathcal{R}}^i}$ and then the *rewrite relation* of $\hat{\mathcal{H}}$ by $\rightarrow_{\hat{\mathcal{H}}} \stackrel{\mathrm{def}}{=} \bigcup_i \rightarrow_{\hat{\mathcal{H}}^i}$

It is apparent from the definition that $\rightarrow_{\hat{\mathcal{H}}^0} \subseteq \rightarrow_{\hat{\mathcal{H}}^1} \subseteq \cdots \subseteq \rightarrow_{\hat{\mathcal{H}}}$. Thus $C[\,]$ satisfies the $\leftrightarrow^*_{\hat{\mathcal{H}}^i}$-context condition of $\hat{\aleph}$ for some i iff $C[\,]$ satisfies the $\leftrightarrow^*_{\hat{\mathcal{H}}}$-context condition of $\hat{\aleph}$. Hereafter, we call the $\leftrightarrow^*_{\hat{\mathcal{H}}}$-context condition simply the *context condition.*

Let $\bar{\mathcal{R}}$ be the set of unconditional parts of $\hat{\mathcal{R}}$, and $\bar{\mathcal{H}}$ the pattern HORS with the set $\bar{\mathcal{R}}$ of rewrite rules. Then, $\rightarrow_{\hat{\mathcal{R}}}$ and $\rightarrow_{\hat{\mathcal{H}}}$ are the subsets of $\rightarrow_{\bar{\mathcal{R}}}$ and $\rightarrow_{\bar{\mathcal{H}}}$ restricted to satisfying the context condition, respectively. In particular, the substitution calculus $\lambda_{\bar{\eta}}^{\rightarrow}$ does not change at all. Therefore, the above definitions related only to $\lambda_{\bar{\eta}}^{\rightarrow}$ can be extended to CCHORSs with no modification.

In order to extend other definitions and statements about a pattern HORS we must show that some 'operations' (*division*, *substitution* and *rewrite of the context*) preserve the context condition.

The following lemma directly follows from the definition of the context condition.

Lemma 27 (division). *Let* $C[\,]$ *be a context with* m *occurrences of* $\Box$. *Let* $D[m]$ *be the linear context such that* $D[\Box, \cdots, \Box] = C[\Box]$. *Then* $C[\,]$ *satisfies the context condition of* $\hat{\aleph} : \hat{l} \rightarrow \hat{r} \Leftarrow Q \in \hat{\mathcal{H}}$ *iff* $D[\underbrace{\hat{l}, \cdots, \hat{l}}_{i-1}, \Box, \hat{l}, \cdots, \hat{l}]$ *satisfies the context condition of* $\hat{\aleph}$ *for* $i = 1, \cdots, m$. ∎

Lemma 28 (substitution). *For any preterms* s, t, u, *and a free variable* x, $s \rightarrow_{\hat{\mathcal{H}}} t$ *implies* $s\{x := u\} \rightarrow_{\hat{\mathcal{H}}} t\{x := u\}$. *Moreover, if a precontext* $C[\,]$ *satisfies the context condition of* $\hat{\aleph}$, $C[\,]\{x := u\}$ *also satisfies it.*

Proof Using Theorem 9, it is proved by induction on the rank. ∎

Lemma 29 (rewrite of the context). *Suppose that* $C[\,]$ *satisfies the context condition of* $\hat{\aleph} \in \hat{\mathcal{R}}$.

1. *If* $C[\,] \rightarrow_{\hat{\mathcal{R}}} C'[\,]$, *then* $C'[\,]$ *satisfies the context condition of* $\hat{\aleph}$.
2. *If* $C[\,] \rightarrow_{\beta\bar{\eta}} C'[\,]$, *then* $C'[\,]$ *satisfies the context condition of* $\hat{\aleph}$. ∎

Then, we can prove the following lemma, which corresponds to Theorem 14.

Lemma 30. *When we restrict the domain of the rewrite relation* $\rightarrow_{\hat{\mathcal{H}}}$ *to* $\mathsf{T}(\mathcal{A})$, *we may assume the following without loss of generality:*

1. *the precontext* $C[\,]$ *in the definition of* $\rightarrow_{\hat{\mathcal{R}}^i}$ *is a linear context,*
2. *in the definition of* $\rightarrow_{\hat{\mathcal{H}}^i}$, *we may use* $\leftarrow^*_{\beta\bar{\eta}}; \rightarrow_{\hat{\mathcal{R}}^i}; \rightarrow^*_{\beta\bar{\eta}}$ *instead of* $\leftrightarrow^*_{\beta\bar{\eta}}; \rightarrow_{\hat{\mathcal{R}}^i}; \leftrightarrow^*_{\beta\bar{\eta}}$.

Proof The proof is done similarly to that of Propositions 3.1.17 and 3.1.22 in [vO94] using Lemma 27 and Lemma 29. ∎

Definition 31. The definition of the *redex* of a CCHORS is obtained by replacing

if $C[]$ exists

in Definition 16 with

if $C[]$ exists and it satisfies the context condition of $\hat{\aleph}$.

The *descendant relation* and the *residual* of a CCHORS can be defined completely in the same way as a pattern HORS.

Definition 32. The definition of the *simultaneity* of redexes of a CCHORS is obtained by modifying Definition 20 as follows:

1. replace $\mathcal{H}$ and $\aleph_i : l_i \rightarrow r_i$ with $\hat{\mathcal{H}}$ and $\hat{\aleph}_i : \hat{l}_i \rightarrow \hat{r}_i \Leftarrow Q_i$, respectively,
2. add the following:

   3. *each occurrence of holes in* $C[m]$ *satisfies the context condition*, that is, $C[\hat{l}_1, \cdots, \hat{l}_{i-1}, \square, \hat{l}_{i+1}, \cdots, \hat{l}_m]$ satisfies the context condition of $\hat{\aleph}_i$ for $i = 1, \cdots, m$.

   to the condition for simultaneity of redexes.

Definition 33. Let $\hat{\aleph}$ and $\hat{\beth}$ be CC rules with unconditional parts $\bar{\aleph}$ and $\bar{\beth}$, respectively. Let $\hat{u} \stackrel{\text{def}}{=} (\phi, \hat{\aleph})$ and $\hat{v} \stackrel{\text{def}}{=} (\psi, \hat{\beth})$ be distinct redexes in a term s. We say u and v are *disjoint* (*u nests v*, *u overlaps v*, resp.) if $\bar{u} \stackrel{\text{def}}{=} (\phi, \bar{\aleph})$ and $\bar{v} \stackrel{\text{def}}{=} (\psi, \bar{\beth})$ are disjoint ($\bar{u}$ nests $\bar{v}$, $\bar{u}$ overlaps $\bar{v}$).

A CC rule $\hat{\aleph}$ *overlaps with* another CC rule $\hat{\beth}$, if some $\hat{\aleph}$-redex overlaps with some $\hat{\beth}$-redex. A CCHORS *has overlap* if some rule overlaps with another one; it is *non-overlapping* otherwise.

4.2 Church-Rosser Property of CCHORS

A sufficient condition for the Church-Rosser property of CCHORSs is derived in a similar way to pattern HORSs.

The following two statements concerning pattern HORSs are used to derive properties of non-overlapping left-linear CCHORSs.

Theorem 34. *Let* $\mathcal{U} \stackrel{\text{def}}{=} \{(\phi_i, \aleph_i) \mid i = 1, \cdots, n\}$ *be a simultaneous redex in a term* s *of a left-linear pattern HORS* $\mathcal{H}$. *For any extractions* $w_1 : s \leftarrow^*_{\beta\bar{\eta}} C_1[l_1, \cdots, l_n]$ *and* $w_2 : s \leftarrow^*_{\beta\bar{\eta}} C_2[l_1, \cdots, l_n]$, *we have* $C_1[n] \equiv C_2[n]$.

Proof The result is immediate from Lemma 3.1.40 in [vO94]. ∎

Lemma 35. *Let* $\aleph : l' \rightarrow r'$ *be a pattern rewrite rule. Suppose that* l' *is an* $x_1, \cdots, x_m$-*closure of* $l \stackrel{\text{def}}{=} \mathtt{F}(u_1) \cdots (u_k)$. *Suppose also that* $u : s \leftarrow^*_{\beta\bar{\eta}} C[l']$ *is an extraction of a redex* $(\phi; 0^k, \aleph)$ *in a term* s. *Then, for any* x_i *of arity* m_i *and* χ *s.t.* $\chi; 0^{m_i} \in x_i\mathtt{Pos}(l)$ $(i = 1, \cdots, m)$, *the following holds:*

1. $s \leftarrow^*_{(\succeq\phi,\beta)} C[l']$,
2. $s/\phi; \chi$ *and* $C[]/\phi; 0^{m-i}; 1; 0^{m_i}$ *are the same modulo renaming of bound variables,*

3. if $\phi;\chi;\omega \in \mathcal{R}\texttt{Pos}(s)$*, then* $\phi;\chi;\omega\ |\underline{u}|\ \phi;0^{m-i};1;0^{m_i};\omega$*,*
4. $\Box\texttt{Pos}(C[\,]) = \phi;0^m$*.* ∎

Lemma 36. *A non-overlapping left-linear CCHORS is simultaneous.*

Proof First, we will show that a left-linear CCHORS $\hat{\mathcal{H}}$ is pairwise simultaneous if it is non-overlapping. Let $\hat{u} \stackrel{\text{def}}{=} (\phi;0^k,\hat{\aleph})$ and $\hat{v} \stackrel{\text{def}}{=} (\psi;0^j,\hat{\beth})$ be pairwise distinct redexes in a term s, where k (j, resp.) is the arity of the head symbol of $\hat{\aleph}$ ($\hat{\beth}$).

Let $\bar{\aleph}$ ($\bar{\beth}$) be the unconditional part of $\hat{\aleph}$ ($\hat{\beth}$). From Theorem 22, there exists an extraction $w : s \leftarrow^*_{\beta\bar{\eta}} D[\texttt{lhs}(\bar{\aleph}), \texttt{lhs}(\bar{\beth})]$ of redexes $\bar{u} \stackrel{\text{def}}{=} (\phi;0^k,\bar{\aleph})$ and $\bar{v} \stackrel{\text{def}}{=} (\psi;0^j,\bar{\beth})$. Therefore, it is enough to show that $D[\Box,\texttt{lhs}(\bar{\beth})]$ and $D[\texttt{lhs}(\bar{\aleph}),\Box]$ satisfy the context conditions of $\hat{\aleph}$ and $\hat{\beth}$, respectively.

1. If $\hat{u}$ and $\hat{v}$ are disjoint, the result is apparent since the extractions of them take place below the disjoint positions ϕ and ψ, respectively; so they do not influence each other.

2. Otherwise, we can assume $\hat{u}$ nests $\hat{v}$ without loss of generality. Let $\hat{l}' \stackrel{\text{def}}{=} \texttt{lhs}(\hat{\aleph})$ of arity $\hat{m}$ be an $x_1,\cdots,x_{\hat{m}}$-closure of $\hat{l}$. Let m_i be the arity of x_i ($i = 1,\cdots,\hat{m}$). Also let $u \stackrel{\text{def}}{=} s \leftarrow^*_{\beta\bar{\eta}} C[\hat{l}']$ be an extraction of $\bar{u}$ in s. From the definition, there exist x_i and $\chi \in x_i\texttt{Pos}(\hat{l})$ such that $\phi;\chi;\omega = \psi$. Thus, from 2 and 3 of Lemma 35, we have that $\psi\ |\underline{u}|\ \phi;0^{\hat{m}-i};1;0^{m_i};\omega$ and that $s/\phi;\chi$ and $C[\,]/\phi;0^{\hat{m}-i};1;0^{m_i}$ are equal modulo renaming of bound variables. So the redex $\bar{v}' \stackrel{\text{def}}{=} (\phi;0^{\hat{m}-i};1;0^{m_i};\omega,\bar{\beth})$ in $C[\,]$ is the descendant of $\bar{v}$ and we can construct an extraction of $\bar{v}'$ from $C[\,]$ by simulating one of $\hat{v}$. Then, the resulting context is $D[\,,\,]$ from Theorem 34. Therefore, $D[\texttt{lhs}(\bar{\aleph}),\Box]$ satisfies the context condition of $\hat{\beth}$.

Suppose $C[\,]/\psi \stackrel{\text{def}}{=} \Box(s_1)\cdots(s_{\hat{m}})$. Then, the extraction $C[\,] \leftarrow^*_{\beta\bar{\eta}} D[\Box,\texttt{lhs}(\bar{\beth})]$ of $\hat{v}'$ takes place 'inside' s_i. Since $C[\,]$ satisfies the context condition of $\hat{\aleph}$, so does $D[\Box,\texttt{lhs}(\bar{\beth})]$.

In a similar way to the pattern HORS's case, we can show that pairwise simultaneity implies simultaneity by induction on the cardinality of the set of redexes using Lemma 28. ∎

Lemma 37 (Finite Developments of CCHORS). *Every complete development of a set of simultaneous redexes of a left-linear CCHORS* $\hat{\mathcal{H}}$ *ends in the same term.*

Proof The proof is done in the same way as the pattern HORS's case, but we must supplement the argument about the context condition. In [vOvR94], the Finite Developments theorem is proved in the following way:

Let $\mathcal{U} = \mathcal{V} \cup \{u\}$ be a set of redexes. Let L (R, resp.) be the set of the left- (right-) hand sides of the rules of $\mathcal{V}$, and let l (r, resp.) be the left- (right-) hand side of the rule of u. Let us consider any complete $\mathcal{U}$-development.

1. Suppose u is chosen in the first rewrite step of the development. Then, the rewrite step $s \leftarrow^*_{\beta\bar{\eta}} C[l] \rightarrow_{C[l\rightarrow r]} C[r] \rightarrow^*_{\beta\bar{\eta}} s'$ can be simulated by $s \leftarrow^*_{\beta\bar{\eta}} D[L,l]$ $\rightarrow_{D[L,l\rightarrow r]} D[L,r] \rightarrow^*_{\beta\bar{\eta}} s'$, which also extracts $\mathcal{V}$ simultaneously.

2. Suppose that $D[\,,r] \rightarrow^{*!}_{\beta\bar{\eta}} E[\,]$. Let $D'[\,]$ be the linear context such that $D'[\Box,\cdots,\Box] \equiv E[\Box]$. Then, $s' \leftarrow^*_{\beta\bar{\eta}} D'[L']$ is the extraction of the residuals of $\mathcal{V}$. Let R' be the right-hand sides corresponding to L'. We define an order $\preceq$ on the set of pairs of terms and natural numbers as the lexicographic order of $\leftarrow^*_{\beta\bar{\eta}}$ and $\leq$. When $(t,i) \preceq (t',i')$ and $(t,i) \neq (t',i')$, we write $(t,i) \prec (t',i')$. If holes appear n and n'

times in D and D', respectively, then we have $(D[R,r],n) \succ (D'[R'],n')$. Since $\succeq$ is a well-founded order, termination of the development follows.

3. By induction on $\preceq$, we have that any complete $\mathcal{U}$-development can be simulated by $s \leftarrow^*_{\beta\bar{\eta}} D[L,l] \rightarrow_{D[L\rightarrow R, l\rightarrow r]} D[R,r] \rightarrow^{*!}_{\beta\bar{\eta}} t$, so any resulting term is equal to t.

In the case of CCHORSs, we must ascertain that $s' \leftarrow^*_{\beta\bar{\eta}} D'[L']$ in the second item is an extraction of a set of redexes. To do that, it is enough to show that each occurrence of holes in $D'[\,]$ satisfies the context condition (see Definition 32).

From the definition, each occurrence of holes in $D[\,,l]$ satisfies the context condition. Successively, in $D[\,,r]$ from 1 of Lemma 29, in $E[\,]$ from 2 of Lemma 29 and in $D'[\,]$ from Lemma 27, each occurrence of holes satisfies the context condition. ∎

Remark. As is pointed out in [vO94], to be rigorous, we must show that the result does not depend on the choice of the first rewrite step in 1 in the above proof. In fact, from Lemma 29, $D[L,l]$ can reach $D[R,r]$ by the replacements in any order.

Theorem 38. *A non-overlapping left-linear CCHORS has the Church-Rosser property.*

Proof By Lemma 36 and Lemma 37. ∎

5 Unique Normal Form Property of HORS

In this section we present a new sufficient condition for the unique normal form property of a pattern HORS based on the result in the previous section. This condition requires neither left-linearity nor termination of the pattern HORS.

Definition 39. Suppose that a free variable x appears n times in a term l $(n \geq 1)$. Let $y_1, \cdots, y_n$ be a list of pairwise distinct free variables satisfying $\{y_1, \cdots, y_n\} \cap (\mathtt{Fvar}(l) - \{x\}) = \emptyset$ and $x \in \{y_1, \cdots, y_n\}$. Then, the *$x$-linearisation of l with* $y_1, \cdots, y_n$ is obtained by replacing each occurrence of x with $y_1, \cdots, y_n$ from left to right. For a sequence $\sigma \stackrel{\mathrm{def}}{=} x_1, \cdots, x_m$ of free variables and sequences $Y_1, \ldots, Y_m$ of lists of free variables satisfying $x_i \in Y_i$ $(i = 1, \cdots, m)$, the *σ-linearisation of l with* $Y_1, \cdots, Y_m$ is the x_1-linearisation with Y_1 of ... of x_m with Y_m of l.

The following lemma is used to define the context-conditional linearisation of a pattern HORS. The proof is routine.

Lemma 40. *Let $l' \rightarrow r'$ be a pattern rewrite rule. Let l' of arity m be an $x_1, \ldots, x_m$-closure of a term l. Then, there exists a unique term r such that r' is an $x_1, \ldots, x_m$-closure of r.* ∎

Definition 41. Let $\aleph \stackrel{\mathrm{def}}{=} l' \rightarrow r'$ be a pattern rewrite rule. If $\hat{l}'$, $\hat{r}'$ and Q satisfy the following conditions, then $\hat{\aleph} \stackrel{\mathrm{def}}{=} \hat{l}' \rightarrow \hat{r}' \Leftarrow Q$ is called a *Context-Conditionally Linearised rule* (CCL rule, for short) of $\aleph$.

1. Let l' of arity m be an $x_1, \cdots, x_m$-closure of l. Then, $\hat{l}'$ is a $\hat{\sigma}$-closure of $\hat{l}$, where $\hat{l}$ is the $x_1, \cdots, x_m$-linearisation of l with some sequences $Y_1, \cdots, Y_m$ of lists of free variables, and $\hat{\sigma}$ is the concatenation of $Y_1, \cdots, Y_m$.
2. Let r be the one in Lemma 40. Then, $\hat{r}'$ is a $\hat{\sigma}$-closure of r.
3. Re-define $\hat{\sigma} \stackrel{\mathrm{def}}{=} y_1, \cdots, y_{\hat{m}}$. Then, Q is the sequence $N_1, \cdots, N_m$ defined by:

$$\text{if } Y_i = y_j, \cdots, y_{j+j'}, \text{ then } N_i = \{j, \cdots, j+j'\} \ (1 \leq i \leq m) \ .$$

Note that $\hat{r}'$ is a closed term since x_i appears in Y_i for $i = 1, \cdots, m$.

Let $\mathcal{H} \stackrel{\text{def}}{=} \langle \mathcal{A}, \lambda_{\overrightarrow{\eta}}, \mathcal{R}\rangle$ be a pattern HORS. Then, $\hat{\mathcal{H}} \stackrel{\text{def}}{=} \langle \mathcal{A}, \lambda_{\overrightarrow{\eta}}, \hat{\mathcal{R}}\rangle$ is called a *Context-Conditional Linearisation* (CCL) of $\mathcal{H}$ if for any $\aleph \in \mathcal{R}$, there exists a CCL rule $\hat{\aleph} \in \hat{\mathcal{R}}$ of $\aleph$.

Any CCL is apparently left-linear from the definition.

Example 1. For a pattern rewrite rule $\aleph : \xi.\mathtt{D}(\xi)(\xi) \to \xi.\mathtt{E}(\xi)$, the following rules are CCL rules of $\aleph$:

$$\xi.\xi'.\mathtt{D}(\xi)(\xi') \to \xi.\xi'\mathtt{E}(\xi) \Leftarrow \{1, 2\},$$

$$\xi.\xi'.\mathtt{D}(\xi)(\xi') \to \xi.\xi'\mathtt{E}(\xi') \Leftarrow \{1, 2\}.$$

In the rest of this paper, $\hat{\mathcal{H}} \stackrel{\text{def}}{=} \langle \mathcal{A}, \lambda_{\overrightarrow{\eta}}, \hat{\mathcal{R}}\rangle$ denotes a CCL of a pattern HORS $\mathcal{H} \stackrel{\text{def}}{=} \langle \mathcal{A}, \lambda_{\overrightarrow{\eta}}, \mathcal{R}\rangle$.

Lemma 42. $\to_{\mathcal{H}} \subseteq \to_{\hat{\mathcal{H}}}$.

Proof If $\hat{\aleph} : \hat{l}' \to \hat{r}' \Leftarrow Q \in \hat{\mathcal{H}}$ is a CCL rule of $\aleph : l' \to r' \in \mathcal{H}$, then $l' \leftarrow^*_{\beta\overline{\eta}} ; \to_{\hat{\aleph}} ; \to^*_{\beta\overline{\eta}} r'$ from the definition. Thus, the result is clear. ∎

Lemma 43. *Suppose that $\hat{\mathcal{H}}$ has the Church-Rosser property. Then, $NF_{\mathcal{H}} \subseteq NF_{\hat{\mathcal{H}}}$.*

Proof Assume $NF_{\mathcal{H}} \not\subseteq NF_{\hat{\mathcal{H}}}$. Let $\mathtt{size}(t)$ be the number of occurrences of symbols in term t. Let s be a term minimal wrt. $\mathtt{size}(s)$ satisfying $s \in NF_{\mathcal{H}} - NF_{\hat{\mathcal{H}}}$.

Let $\hat{u} \stackrel{\text{def}}{=} (\phi; 0^k, \hat{\aleph})$ be an $\hat{\mathcal{H}}$-redex in s, where $\hat{\aleph} \stackrel{\text{def}}{=} \hat{l}' \to \hat{r}' \Leftarrow N_1, \cdots, N_m \in \hat{\mathcal{R}}$ is a CCL rule of $\hat{u}$ and k is the arity of the head symbol of $\hat{l}'$. Also, let $\hat{m}$ be the arity of $\hat{l}'$. Suppose that $s \leftarrow^*_{\beta\overline{\eta}} C[\hat{l}']$ is the extraction of $\hat{u}$. From 1 of Lemma 35, $s \leftarrow^*_{(\succeq\phi,\beta)} C[\hat{l}']$. Let s' and $C'[]$ be the terms obtained from s/ϕ and $C[]/\phi$, respectively, by replacing all unbound occurrences of bound variables with fresh free variables. Then, we have $s' \leftarrow^*_{\succeq\epsilon,\beta} C'[\hat{l}']$, and $C'[]$ satisfies the context condition of $\hat{\aleph}$. From minimality of s, we have $\phi = \epsilon$.

Therefore, we can suppose that $C[] \stackrel{\text{def}}{=} \Box(s_1)\cdots(s_{\hat{m}})$ for some $s_1, \ldots, s_{\hat{m}}$ from 4 of Lemma 35. Note that there is no unbound occurrence of a bound variable in $s_1, \cdots, s_{\hat{m}}$, and that $\Box$ does not occur in $s_1, \cdots, s_{\hat{m}}$ since $C[]$ is a linear context. Since $C[]$ satisfies the context condition of $\hat{\aleph}$,

$$\text{for any } N_j \ (j = 1, \cdots, m), \text{ if } N_j = \{i, \cdots, i+i'\}, \text{ then } s_i \leftrightarrow^*_{\hat{\mathcal{H}}} \cdots \leftrightarrow^*_{\hat{\mathcal{H}}} s_{i+i'}.$$

Moreover, $s_i \in NF_{\to_{\hat{\mathcal{H}}}}$ for $i = 1, \cdots, \hat{m}$ by 2 of Lemma 35 and minimality of s. Thus, from the Church-Rosser property of $\hat{\mathcal{H}}$,

$$\text{for any } N_j \ (j = 1, \cdots, m), \text{ if } N_j = \{i, \cdots, i+i'\}, \text{ then } s_i \equiv \cdots \equiv s_{i+i'}.$$

Then, $(0^k, \aleph)$ is a redex of $\mathcal{H}$ in s from the definition of CCL rules, which contradicts the assumption. ∎

Definition 44. A pattern HORS $\mathcal{H}$ is *strongly non-overlapping* if there exists a non-overlapping CCL of $\mathcal{H}$.

Theorem 45. *A strongly non-overlapping pattern HORS has the unique normal form property.*

Proof From Theorem 1 using Lemma 42, Lemma 43 and Theorem 38. ∎

Example 2. The untyped λ-calculus can be translated into a pattern HORS in the following way [vO94]. First, new constant symbols $\mathtt{abs} : (o \to o) \to o$ and $\mathtt{app} : o \to (o \to o)$ are introduced in order to embed untyped λ-terms into $\mathsf{T}(\mathcal{A})$. Then, the β-rule is translated into the following pattern rule:

$$\mathtt{beta} : \xi.\eta.\mathtt{app}(\mathtt{abs}(\zeta.\xi(\zeta)))(\eta) \to \xi.\eta.\xi(\eta)$$

Moreover, let us consider the following pattern rule:

$$\mathtt{D} : \xi.\mathtt{app}(\mathtt{app}(\mathtt{D})(\xi))(\xi) \to \xi.\xi$$

which is a translation of the rewrite rule $Dxx \to E$ appearing [Klo80]. With either of CC rules of $\mathtt{D}$ in example 1, we can easily verify that a pattern HORS $\mathcal{H}$ with $\mathcal{R} = \{\mathtt{beta}, \mathtt{D}\}$ is strongly non-overlapping. Therefore, $\mathcal{H}$ has the unique normal form property.

6 Conclusion

We presented a sufficient condition for the unique normal form property of pattern HORSs. The condition requires neither left-linearity nor termination of the HORS. In order to investigate it, we introduced the Context-Conditional HORS and the Context-Conditional Linearisation of HORSs, and gave a sufficient condition for the Church-Rosser property of CCHORSs.

There were some results about the unique normal form property of individual non-left-linear and non-terminating higher-order rewriting systems, e.g. [KdV89]. We presented a decidable class of HORSs having the unique normal form property. Concerning higher-order conditional rewriting, [KvO95] derived the Church-Rosser property from 'orthogonality', which assumed some *closure property* of the conditions. Also, [LS93] extended the results in [Nip91] to the conditional version.

As a future work, we would like to relax the condition of strong non-overlap. We expect that the Church-Rosser property of CCHORSs will hold even if we weaken the non-overlap requirement to admit *trivial overlap* [vO94]. This will enable us to treat $\{\mathtt{beta}, \mathtt{eta}, \mathtt{D}\}$, where the pattern rewrite rule $\mathtt{eta}$ is a translation of untyped η-rule (see example 2). But this does *not* imply that any HORS only with trivial overlap has the unique normal form property. Consider a pattern HORS with the following rules:

$$\left\{ \begin{array}{l} \xi.\eta.\mathtt{app}(\mathtt{app}(\mathtt{app}(\mathtt{C})(\mathtt{T}))(\xi))(\eta) \to \xi.\eta.\xi \\ \xi.\eta.\mathtt{app}(\mathtt{app}(\mathtt{app}(\mathtt{C})(\mathtt{F}))(\xi))(\eta) \to \xi.\eta.\eta \\ \zeta.\xi.\mathtt{app}(\mathtt{app}(\mathtt{app}(\mathtt{C})(\zeta))(\xi))(\xi) \to \zeta.\xi.\xi \end{array} \right\}$$

which is called *Parallel Conditional* [KdV90]. Though this system has only trivial overlap, *any CCL of this system has non-trivial overlap.*

It would be solved either by a model-theoretic approach [dV90, KdV90], or by extending the notion of *compatibility* of TRSs [Che81, MO95]. The former was also used to conclude the unique normal form property of untyped λ-calculus with *Surjective Pairing* in [KdV89], whereas the latter would give a decidable condition.

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