SAT Solving for Termination Analysis

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Term Rewriting

DEFINITION

- pair of terms $l \to r$ is rewrite rule if $l \notin \mathcal{V} \wedge \mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(l)$
- term rewrite system (TRS) is set of rewrite rules
- (rewrite relation) $s \to_{\mathcal{R}} t$ if $\exists l \to r \in \mathcal{R}$, context C, substitution σ. $s = C[l\sigma] \ \wedge \ t = C[r\sigma]$

EXAMPLE

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

rewriting

$$\begin{aligned} \mathsf{s}(0) \times \mathsf{s}(0) &\to_{\mathcal{R}} \mathsf{s}(0) \times 0 + \mathsf{s}(0) \\ &\to_{\mathcal{R}} 0 + \mathsf{s}(0) \\ &\to_{\mathcal{R}} \mathsf{s}(0 + 0) \\ &\to_{\mathcal{R}} \mathsf{s}(0) \qquad \text{normal form} \end{aligned}$$

Eratosthenes' sieve

$$
\begin{array}{rl}
1: & x - 0 \to x \\
2: & \mathsf{s}(x) - \mathsf{s}(y) \to x - y \\
3: & 0 \leqslant y \to \mathsf{true} \\
4: & \mathsf{s}(x) \leqslant 0 \to \mathsf{false} \\
5: & \mathsf{s}(x) \leqslant \mathsf{s}(y) \to x \leqslant y \\
6: & \mathsf{if}(\mathsf{true}, x, y) \to x \\
7: & \mathsf{if}(\mathsf{false}, x, y) \to y \\
8: & x \mid 0 \to \mathsf{true} \\
9: & \mathsf{s}(x) \mid \mathsf{s}(y) \to \mathsf{if}(x \leqslant y, \mathsf{s}(x) \mid (y - x), \mathsf{false})
\end{array}
$$

$$
\begin{array}{rl}
10: & \mathsf{filter}(x, \mathsf{nil}) \to \mathsf{nil} \\
11: & \mathsf{filter}(x, y : ys) \to \mathsf{if\_filter}(x \mid y, x, y : ys) \\
12: & \mathsf{if\_filter}(\mathsf{true}, x, y : ys) \to \mathsf{filter}(x, ys) \\
13: & \mathsf{if\_filter}(\mathsf{false}, x, y : ys) \to y : \mathsf{filter}(x, ys) \\
14: & \mathsf{sieve}(\mathsf{nil}) \to \mathsf{nil} \\
15: & \mathsf{sieve}(x : xs) \to x : \mathsf{sieve}(\mathsf{filter}(x, xs))
\end{array}
$$

$$
\begin{aligned}
\mathsf{sieve}(2 : 3 : 4 : 5 : 6 : \mathsf{nil}) &\to_{\mathcal{R}} 2 : \mathsf{sieve}(\mathsf{filter}(2, 3 : 4 : 5 : 6 : \mathsf{nil})) \\
&\to^{+}_{\mathcal{R}} 2 : \mathsf{sieve}(3 : 5 : \mathsf{nil})) \\
&\to_{\mathcal{R}} 2 : 3 : \mathsf{sieve}(\mathsf{filter}(3, 5 : \mathsf{nil})) \\
&\to^{+}_{\mathcal{R}} 2 : 3 : \mathsf{sieve}(5 : \mathsf{nil}) \\
&\to_{\mathcal{R}} 2 : 3 : 5 : \mathsf{sieve}(\mathsf{filter}(5, \mathsf{nil})) \\
&\to^{+}_{\mathcal{R}} 2 : 3 : 5 : \mathsf{sieve}(\mathsf{nil}) \\
&\to_{\mathcal{R}} 2 : 3 : 5 : \mathsf{nil}
\end{aligned}
$$

is this TRS terminating?

Haskell-like program

$$\begin{aligned} \mathsf{id}\ x &\to x \\ \mathsf{add}\ 0\ y &\to y \\ \mathsf{add}\ (\mathsf{s}\ x)\ y &\to \mathsf{s}\ (\mathsf{add}\ x\ y) \end{aligned} \qquad \begin{aligned} \mathsf{map}\ f\ \mathsf{nil} &\to \mathsf{nil} \\ \mathsf{map}\ f\ (x : y) &\to (f\ x) : (\mathsf{map}\ f\ y) \end{aligned}$$

uncurried TRS

$$\begin{aligned} \mathsf{id}_1(x) &\to x \\ \mathsf{add}_2(0, y) &\to \mathsf{id}_1(y) \\ \mathsf{add}_2(\mathsf{s}_1(x), y) &\to \mathsf{s}_1(\mathsf{add}_2(x, y)) \\ \mathsf{map}_2(f, \mathsf{nil}) &\to \mathsf{nil} \\ \mathsf{map}_2(f, :_2(x, y)) &\to :_2(f \circ x, \mathsf{map}_2(f, y)) \end{aligned} \qquad \begin{aligned} : \circ x &\to :_1(x) \\ :_1(x) \circ y &\to :_2(x, y) \end{aligned} \qquad \begin{aligned} \mathsf{id} \circ x &\to \mathsf{id}_1(x) \\ \mathsf{add} \circ x &\to \mathsf{add}_1(x) \\ \mathsf{add}_1(x) \circ y &\to \mathsf{add}_2(x, y) \\ \mathsf{s} \circ x &\to \mathsf{s}_1(x) \\ \mathsf{map} \circ x &\to \mathsf{map}_1(x) \\ \mathsf{map}_1(x) \circ y &\to \mathsf{map}_2(x, y) \end{aligned}$$

APPLICATIONS

- verification for functional programming
- theorem proving
- code optimization in compilers
- symbolic computation in mathematics
- ...

Is Non-termination More Interesting?

DEFINITION

$\mathcal{R}$ is non-terminating if $\exists$infinite rewrite sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

- non-termination tools are useful for detecting bugs
- liveness property can be reduced to termination property

Termination Proof Methods

long history and many results

polynomial interpretation order, multiset path order, Knuth-Bendix order, recursive decomposition order, semantic path order, general path order, $\mathcal{C}_\mathcal{E}$-termination, hierarchical modular theorem, freezing, erasure method, size-change termination, lexicographic path order, match-bound, matrix interpretation order, dependency pair method, monotonic semantic path order, forward closure, semantic labeling, self labeling, predictive labeling uncurrying transformation, innermost recursive path order, ...

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- polynomial order and lexicographic path order (LPO)
- dependency pair method

This Talk

finding suitable

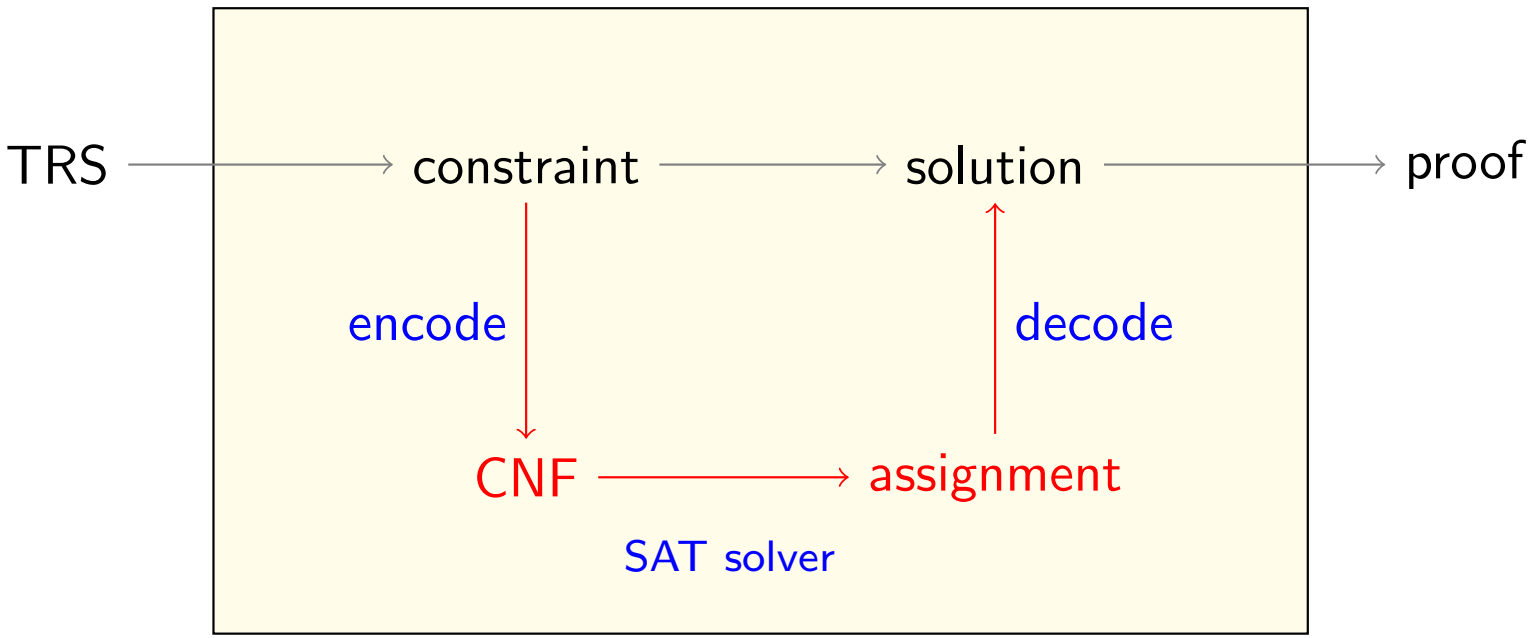
- LPO precedence
- polynomial interpretation
- argument filtering

is main bottleneck

THIS TALK

use SAT solver or Diophantine constraint solver to find them

Implementation of Termination Tools



Overview

- SAT
- Diophantine constraints
- precedence termination
- lexicographic path orders
- polynomial interpretation orders
- dependency pair method

SAT

SAT solvers

DEFINITION CNF

(CNF) $C ::= c_1 \wedge \cdots \wedge c_m$

(clause) $c ::= l_1 \vee \cdots \vee l_n$

(literal) $l ::= x \mid \neg x$

- SAT solvers finds assignment α such that $\alpha \models C$
- popular solvers: zChaff, Minisat, ...

EXAMPLE

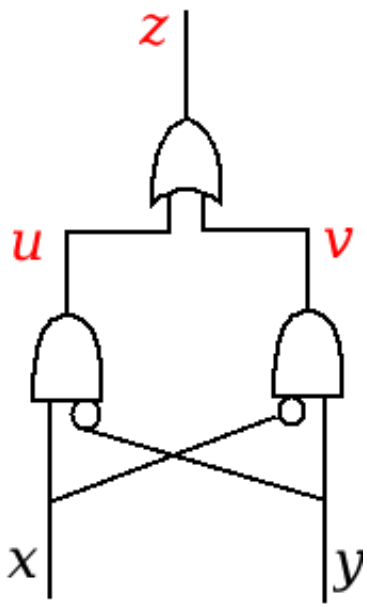
$$\models \bigwedge \left\{ \begin{array}{l} x_1 \vee \neg x_2, \\ \neg x_1 \vee x_2 \vee \neg x_3, \\ x_3 \vee \neg x_2, \\ \neg x_3 \end{array} \right\} ?$$

```
$ cat a.cnf
p cnf 3 #variables 4 #clauses
1 -2 0
-1 2 -3 0
3 -2 0
-3 0

$ minisat a.cnf a.ans
$ cat a.ans
SAT
1 -2 -3 0
```

Tseitin's transformation

$$x \leftrightarrow (y \wedge z) = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z \vee x)$$
$$x \leftrightarrow (y \vee z) = x \leftrightarrow \neg(\neg y \wedge \neg z)$$



$$\models (x \wedge \neg y) \vee (\neg x \wedge y)$$
$$\Leftrightarrow \quad \models z \text{ where } \begin{cases} z = u \vee v \\ u = x \vee \neg y \\ v = \neg x \vee y \end{cases}$$
$$\Leftrightarrow \quad \models z \wedge \bigwedge \left\{ \begin{array}{l} z \leftrightarrow (u \vee v), \\ u \leftrightarrow (x \vee \neg y), \\ v \leftrightarrow (\neg x \vee y) \end{array} \right\}$$

Implementation in OCaml

```
type literal = int and cluase = int list and cnf = clause list

let nvars = ref 0
let fresh () = incr nvars; !nvars
let cnf = ref [[]]

let conj xs =
  let y = fresh () in
  cnf := (y :: [ -x | x <- xs ]) :: [ [-y; x] | x <- xs ] @ !cnf; y

let disj xs = - conj [ -x | x <- xs ]
let negate x = -x
```

EXAMPLE

CNF of $(x \wedge \neg y) \vee (\neg x \wedge y)$ is

```
let z = disj [conj [1; -2]; conj [-1; 2]] in [z] :: !cnf
```

Why Very Very Fast?

- decision:
  depth-first search with heuristics

- deduction:
  boolean constraint propagation
  two watched literals

- diagnosis (learning):
  conflict-directed backtrack (1-UIP)

Boolean Constraint Propagation

current assignment:

$$x = 1, y = 0$$

current decision: $z = 1$

$\neg x \vee z \vee b \vee c$	✓
$\neg x \vee y \vee \neg z \vee \neg w$	$w = 0$
$w \vee y \vee a$	–
$\neg a \vee z \vee b \vee y$	–
$\neg x \vee c \vee d \vee e$	–

Two Watched Literals

current assignment:

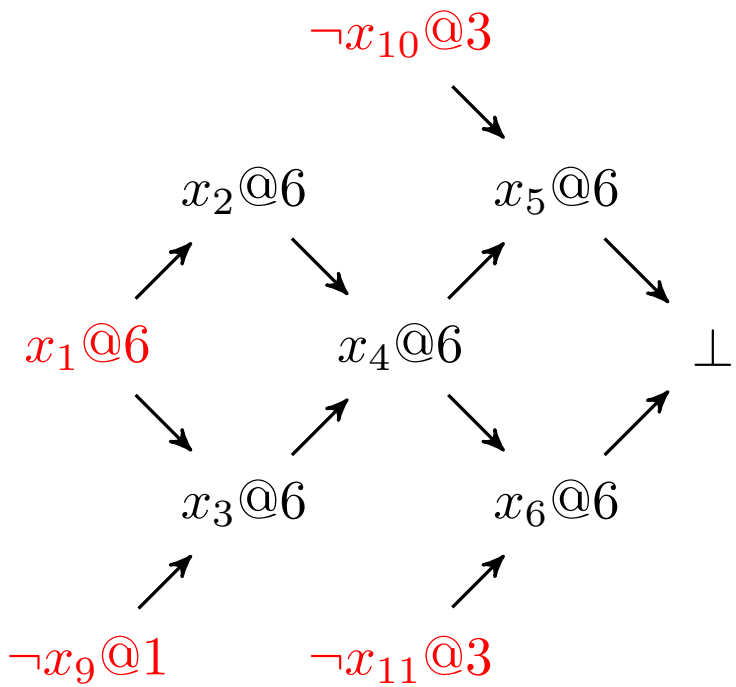
$$x = 1, y = 0$$

current decision: $z = 1$

$$\neg x \vee z \vee \boxed{b} \vee \boxed{c}$$
$$\neg x \vee y \vee \boxed{\neg z} \vee \boxed{\neg w}$$
$$\boxed{w} \vee \boxed{y} \vee a$$
$$\boxed{\neg a} \vee z \vee \boxed{b} \vee y$$
$$\neg x \vee \boxed{c} \vee \boxed{d} \vee e$$

Conflict-Directed Backtracking

current decision: $x_1 = 1@6$



- if $x_1 \neq 0$, back to level 3
- GRASP learns $\neg x_1 \vee x_9 \vee x_{10} \vee x_{11}$ (UIP is $\neg x_4$)

Diophantine Constraints

Diophantine Constraints

SYNTAX

$$
\begin{aligned}
c &::= \neg c \mid c_1 \vee c_2 \mid c_1 \wedge c_2 \\
&\quad \mid p_1 \geq p_2 \mid p_1 \leq p_2 \mid p_1 > p_2 \mid p_1 < p_2 \mid p_1 = p_2 \mid p_1 \neq p_2 \\
p &::= x \mid n \mid p_1 + p_1 \mid p_1 \times p_2
\end{aligned}
$$

where $x \in \mathcal{V}$ and $n \in \mathbb{Z}$

SEMANTICS

- $\alpha \vDash p_1 \geq p_2$ if $[\alpha]_{\mathbb{N}}(p_1) \geq [\alpha]_{\mathbb{N}}(p_2)$
- $\alpha \vDash \neg c$ if $\alpha \vDash c$ does not hold
- $\alpha \vDash c_1 \vee c_2$ if $\alpha \vDash c_1$ or $\alpha \vDash c_2$
- $\alpha \vDash c_1 \wedge c_2$ if $\alpha \vDash c_1$ and $\alpha \vDash c_2$

- $\vDash c$ if $\exists \alpha : \mathcal{V} \to \mathbb{Z}.\ \alpha \vDash c$

Diophantine Constraint Problems

1. $\models \quad (a > b \ \vee \ b > d) \ \wedge \ b > c \ \wedge \ c > a \quad ?$

satisfiable $a = 3, b = 2, c = 1, d = 4$

2. $\models \quad \neg(b > a) \ \wedge \ b > c \ \wedge \ c > a \quad ?$

unsatisfiable

3. $\models a^2 - b^2 = 5 \ \vee \ (2a + b \leq 12 \ \wedge \ ab \geq 15) \quad ?$

satisfiable $a = 3, b = 5$

NOTE

1 and 2 are precedence problem

How To Solve Diophantine Constraints?

- in general undecidable
- so search only from finite domain (incomplete)
- efficient easy way is to use SAT solver to implement Diophantine constraints

Encoding Arithmetics (I)

for $a \leq x \leq b$, introduce

$$x_a, \ldots, x_b$$

add uniqueness condition:

$$(x_a \vee x_{a+1} \vee \cdots \vee x_b) \wedge \bigwedge \{\neg x_i \vee x_j \mid a \leq i, j \leq b,\ i \neq j\}$$

for $z = x + y$ ($a \leq x \leq b$, $c \leq y \leq d$), introduce

$$z_{a+c}, \ldots, z_{b+d}$$

Encoding Arithmetics (II)

for $x + y = z$, introduce propositional variables

$$x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k), z = (z_1, \ldots, z_k)$$

implement k-bit ripple carry adder...

Precedence Termination

Precedence Termination

$>$ is precedence if $>$ is strict order on function symbols

DEFINITION

$l >_{\mathsf{prec}} r$ if $\mathcal{V}\mathsf{ar}(l) \supseteq \mathcal{V}\mathsf{ar}(r)$, $l = f(\ldots)$ and $\forall g \in \mathcal{F}\mathsf{un}(r).\ f > g$

THEOREM

$\mathcal{R}$ is terminating if there exists well-founded precedence $>$ such that $\mathcal{R} \subseteq >_{\mathsf{prec}}$

NOTATION

$\mathcal{R} \subseteq >$ if and only if $\forall l \to r \in \mathcal{R}.\ l > r$

REMARK

this constraint is Diophantine constraint

Example

prove termination of TRS

$$\begin{aligned}
\mathsf{not}(x) &\to \mathsf{if}(x, \mathsf{false}, \mathsf{true}) & \mathsf{if}(\mathsf{true}, x, y) &\to x\\
\mathsf{and}(x, y) &\to \mathsf{if}(x, y, \mathsf{false}) & \mathsf{if}(\mathsf{false}, x, y) &\to y\\
\mathsf{or}(x, y) &\to \mathsf{if}(x, \mathsf{true}, y)\\
\mathsf{equiv}(x, y) &\to \mathsf{if}(x, y, \mathsf{not}(y))
\end{aligned}$$

$$\bigwedge \left\{ \begin{array}{rlrlrl} \mathsf{not}(x) & >_{\mathsf{prec}} \mathsf{if}(x, \mathsf{false}, \mathsf{true}), & \mathsf{if}(\mathsf{true}, x, y) & >_{\mathsf{prec}} x, \\ \mathsf{and}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, y, \mathsf{false}), & \mathsf{if}(\mathsf{false}, x, y) & >_{\mathsf{prec}} y, \\ \mathsf{or}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, \mathsf{true}, y), & & \\ \mathsf{equiv}(x, y) & >_{\mathsf{prec}} \mathsf{if}(x, y, \mathsf{not}(y)) & & \end{array} \right\}$$

$$\Leftrightarrow \left(\begin{array}{rlcrlcrlc} \mathsf{not} & > \mathsf{if} & \wedge & \mathsf{not} & > \mathsf{false} & \wedge & \mathsf{not} & > \mathsf{true} & \wedge \\ \mathsf{and} & > \mathsf{if} & \wedge & \mathsf{and} & > \mathsf{false} & \wedge & & & \\ \mathsf{or} & > \mathsf{if} & \wedge & \mathsf{or} & > \mathsf{true} & \wedge & & & \\ \mathsf{equiv} & > \mathsf{if} & \wedge & \mathsf{equiv} & > \mathsf{not} & & & & \end{array} \right)$$

Diophantine constraint

```
% diosat if.dio
SAT
not = -13967; if = -28612; false = -16738; true = -18786;
and = -1; or = -1; equiv = -370;
```

take precedence

$$\mathsf{and}, \mathsf{or} > \mathsf{equiv} > \mathsf{not} > \mathsf{false} > \mathsf{true} > \mathsf{if}$$

then $\mathcal{R} \subseteq >_{\mathsf{prec}}$, hence TRS is terminating

Example

prove termination of TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

REMARK

precedence termination does not hold:

$$\begin{aligned} & \mathcal{R} \subseteq >_{\mathsf{prec}} \\ \Leftrightarrow\ & \mathsf{s}(x) + y >_{\mathsf{prec}} \mathsf{s}(x + y) \ \wedge \ \cdots \\ \Leftrightarrow\ & + > \mathsf{s} \ \wedge \ + > + \ \wedge \ \cdots \end{aligned}$$

is unsatisfiable Diophantine constraint

Decidability

- Diophantine constraint problems are undecidable in general
- precedence constraint problems are decidable
- completeness threshold is number of function symbols

Lexicographic Path Orders

Lexicographic Path Order

DEFINITION LPO

for precedence $>$

$s >_{\mathsf{lpo}} t$ if $s = f(s_1, \ldots, s_m)$, and $t \in \mathcal{V}\mathrm{ar}(s)$ or $t = g(t_1, \ldots, t_n)$, and

- $\forall i \in \{1, \ldots, m\}.\ s_i >_{\mathsf{lpo}} t \ \vee \ s_i = t$,
- $f > g \ \wedge \ \forall i \in \{1, \ldots, n\}.\ s >_{\mathsf{lpo}} t_i$, or
- $f = g \ \wedge \ \exists i \in \{1, \ldots, n\}.$
  $s_1 = t_1 \ \wedge \cdots \wedge \ s_{i-1} = t_{i-1} \ \wedge \ t_i >_{\mathsf{lpo}} s_i \ \wedge \ s >_{\mathsf{lpo}} t_{i+1}, \ldots s >_{\mathsf{lpo}} t_n$

THEOREM Kamin and Levy, 1980

finite TRS $\mathcal{R}$ is terminating if $\exists$well-founded precedence $>$ such that $\mathcal{R} \subseteq >_{\mathsf{lpo}}$

REMARK

constraints are Diophantine constraints (precedence constraints)

Example

prove termination of TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

$$\mathcal{R} \subseteq >_{\mathsf{lpo}}$$
$$\Leftrightarrow\ (+ > \mathsf{s} \ \vee\ \bot) \ \wedge\ (\times > 0 \ \vee\ \top) \ \wedge\ (\times > + \ \vee\ (\mathsf{s} > + \ \wedge\ \mathsf{s} > \times \ \wedge\ \bot))$$

PROOF

take precedence $\times > + > \mathsf{s} > 0$

$\mathcal{R} \subseteq >_{\mathsf{lpo}}$ holds, hence TRS is terminating

Polynomial Interpretation Orders

Polynomial Constraints (with Unknown Coefficients)

SYNTAX

$$c ::= r \mid \neg c \mid c_1 \vee c_2 \mid c_1 \wedge c_2$$
$$r ::= p_1 \geq p_2 \mid p_1 \leq p_2 \mid p_1 > p_2 \mid p_1 < p_2 \mid p_1 = p_2 \mid p_1 \neq p_2$$
$$p ::= x \mid X \mid n \mid p_1 + p_1 \mid p_1 \times p_2$$

where $x \in \mathcal{V}$, $X \in \mathcal{X}$, and $n \in \mathbb{Z}$

SEMANTICS

- $\sigma \vDash p_1 \geq p_2$ if $\forall \alpha : \mathcal{X} \to \mathbb{Z}.\ \ [\alpha]_{\mathbb{Z}}(p_1\sigma) \geq [\alpha]_{\mathbb{Z}}(p_2\sigma)$
- $\sigma \vDash \neg c$ if $\sigma \vDash c$ does not hold
- $\sigma \vDash c_1 \vee c_2$ if $\sigma \vDash c_1$ or $\sigma \vDash c_2$
- $\sigma \vDash c_1 \wedge c_2$ if $\sigma \vDash c_1$ and $\sigma \vDash c_2$

- $\vDash c$ if $\exists \sigma : \mathcal{V} \to \mathbb{Z}.\ \sigma \vDash c$

Polynomial Constraint Problems

1. $\models \quad aX + bY > bX + 2b \ \wedge \ a \geq 0 \ \wedge \ b \geq 0 \quad ?$

unsatisfiable

2. $\models \quad aX + b > aX^2 + c \ \wedge \ a > 0 \quad ?$

unsatisfiable

3. $\models aX + aY + bY + c > aX + Y + 3 \quad ?$

satisfiable $a = 1, b = 1, c = 4$

Polynomial Interpretation Orders

Lankford 1979

DEFINITION interpretation order

$s >_A t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments $\alpha : \mathcal{V} \to A$

DEFINITION strict monotonicity

$f(a_1, \ldots, a_i, \ldots, a_n) > f(a_1, \ldots, a'_i, \ldots, a_n)$ if $a_i > a'_i$

THEOREM

TRS $\mathcal{R}$ is terminating if there exists well-founded strictly monotone algebra $(\mathcal{A}, >)$ such that $\mathcal{R} \subseteq >_A$

Example

TRS $\mathcal{R}$

$$\begin{array}{llll} x + 0 & \to & x & \qquad x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\ x \times 0 & \to & 0 & \qquad x \times \mathsf{s}(y) & \to & x \times y + x \end{array}$$

take well-founded strictly monotone interpretations

$$\begin{array}{rlrl} 0_{\mathbb{N}} &= 1 & \qquad \times_{\mathbb{N}}(X, Y) &= 2XY + Y + 1 \\ \mathsf{s}_{\mathbb{N}}(X) &= X + 1 & \qquad +_{\mathbb{N}}(X, Y) &= X + 2Y \end{array}$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\bigwedge \left\{ \begin{array}{llcll} [\alpha]_{\mathbb{N}}(x + 0) & = \alpha(x) + 2 & > & x & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = \alpha(x) + 2\alpha(y) + 2 & > & \alpha(x) + 2\alpha(y) + 1 & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ [\alpha]_{\mathbb{N}}(x \times 0) & = 2\alpha(x) + 2 & > & 1 & = [\alpha]_{\mathbb{N}}(0) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{s}(y)) & = \ldots & > & \ldots & = [\alpha]_{\mathbb{N}}(x \times y + x) \end{array} \right\}$$

for all $\alpha : \mathcal{V} \to \mathbb{N}$. this holds, hence $\mathcal{R}$ is terminating

how to check constraint automatically?

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = 1 \qquad \times_{\mathbb{N}}(X, Y) = 2XY + Y + 1$$
$$\mathsf{s}_{\mathbb{N}}(X) = X + 1 \qquad +_{\mathbb{N}}(X, Y) = X + 2Y$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\bigwedge \left\{ \begin{array}{llll} [\alpha]_{\mathbb{N}}(x + 0) & = X + 2 & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = X + 2Y + 2 & > X + 2Y + 1 & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ [\alpha]_{\mathbb{N}}(x \times 0) & = 2X + 2 & > 1 & = [\alpha]_{\mathbb{N}}(0) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{s}(y)) & = \ldots & > \ldots & = [\alpha]_{\mathbb{N}}(x \times y + x) \end{array} \right\}$$

this holds, hence TRS is terminating

how to check constraint automatically?

How to Solve Polynomial Constraints?

in general undecidable but there is easy (incomplete) criterion

FACT

polynomial is non-negative if all coefficients of are non-negative

EXAMPLE

$$
\begin{aligned}
& 2XY + 2X + 2Y + 2 > X + Y \\
\Leftrightarrow\ & (2XY + 2X + 2Y + 2) - (X + Y) - 1 \geq 0 \\
\Leftrightarrow\ & 2XY + 1X + 1Y + 1 \geq 0 \\
\Leftarrow\ & \bigwedge \left\{ \begin{array}{ll} 2 & \geq 0 \\ 1 & \geq 0 \\ 1 & \geq 0 \\ 1 & \geq 0 \end{array} \right\}
\end{aligned}
$$

Example

TRS $\mathcal{R}$

$$\begin{array}{lll} x + 0 & \to & x \\ x \times 0 & \to & 0 \end{array} \qquad \begin{array}{lll} x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\ x \times \mathsf{s}(y) & \to & x \times y + x \end{array}$$

take well-founded strictly monotone interpretations

$$\begin{array}{rl} 0_{\mathbb{N}} = 1 & \qquad \times_{\mathbb{N}}(X, Y) = 2XY + Y + 1 \\ \mathsf{s}_{\mathbb{N}}(X) = X + 1 & \qquad +_{\mathbb{N}}(X, Y) = X + 2Y \end{array}$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\bigwedge \left\{ \begin{array}{llll} [\alpha]_{\mathbb{N}}(x + 0) & = X + 2 & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = X + 2Y + 2 & > X + 2Y + 1 & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ [\alpha]_{\mathbb{N}}(x \times 0) & = 2X + 2 & > 1 & = [\alpha]_{\mathbb{N}}(0) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{s}(y)) & = \ldots & > \ldots & = [\alpha]_{\mathbb{N}}(x \times y + x) \end{array} \right\}$$

this holds, hence TRS is terminating

how to find suitable interpretations?

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = a \qquad +_{\mathbb{N}}(X, Y) = dXY + eX + fY + g$$
$$\mathsf{s}_{\mathbb{N}}(X) = bX + c \qquad \times_{\mathbb{N}}(X, Y) = iXY + jX + kY + l$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\vDash \bigwedge \left\{ \begin{array}{lll} [\alpha]_{\mathbb{N}}(x + 0) & = bdX + eX + af + g & \\ & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = bXY + eX + bfY + cf + g & \\ & > bdXY + beX + bfY + bg + c & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ & \ldots & \end{array} \right\}$$

how to solve polynomial constraint problem?

Translating to Diophantine Constraints

in general undecidable but there is easy (incomplete) criterion

FACT

polynomial is non-negative if all coefficients are non-negative

EXAMPLE

$$
\begin{aligned}
& \models aXY + bX + cY + d > eX + fY + g \\
\Leftrightarrow \quad & \models aXY + (b-e)X + (c-f)Y + (d-g-1) \geq 0 \\
\Leftarrow \quad & \models \bigwedge \left\{ \begin{array}{rl} a & \geq 0 \\ b-e & \geq 0 \\ c-f & \geq 0 \\ d-g-1 & \geq 0 \end{array} \right\}
\end{aligned}
$$

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = a \qquad +_{\mathbb{N}}(X, Y) = dXY + eX + fY + g$$
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$$\models \bigwedge \left\{ \begin{array}{lll} [\alpha]_{\mathbb{N}}(x + 0) & = bdX + eX + af + g & \\ & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = bXY + eX + bfY + cf + g & \\ & > bdXY + beX + bfY + bg + c & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ & \ldots & \end{array} \right\}$$

$\wedge$?

FACT

- polynomial interpretation is weakly monotone if all coefficients are non-negative
- polynomial interpretation is strictly monotone if all coefficients are positive
- polynomial interpretation is non-negative if it is weakly monotone and constant is non-negative

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = a \qquad +_{\mathbb{N}}(X, Y) = dXY + eX + fY + g$$
$$\mathsf{s}_{\mathbb{N}}(X) = bX + c \qquad \times_{\mathbb{N}}(X, Y) = iXY + jX + kY + l$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\vDash \bigwedge \left\{ \begin{array}{lll} [\alpha]_{\mathbb{N}}(x + 0) & = bdX + eX + af + g & \\ & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = bXY + eX + bfY + cf + g & \\ & > bdXY + beX + bfY + bg + c & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ & \ldots & \end{array} \right\}$$

$$\wedge \boxed{\bigwedge \{b, d, e, f, i, j, k > 0, \quad a, c, g, l \geq 0\}}$$

Dependency Pair Method

Dependency Pair Method

Arts, Giesl TCS 2000

NOTATION

for $t = f(t_1, \ldots, t_n)$, $t^\sharp$ denotes $f^\sharp(t_1, \ldots, t_n)$ where $f^\sharp$ is fresh symbol

DEFINITION

$l^\sharp \to t^\sharp$ is dependency pair of TRS $\mathcal{R}$ if $\exists l \to r \in \mathcal{R}$:
$r \trianglerighteq t$, $l \not\triangleright t$, $\mathrm{root}(t)$ is defined symbol

DEFINITION

dependency graph of $\mathcal{R}$

- nodes are dependency pairs of $\mathcal{R}$
- draw arrow from $s^\sharp \to t^\sharp$ to $u^\sharp \to v^\sharp$
  if $t\sigma \to_{\mathcal{R}}^* u\tau$ for some substitution σ and τ

dependency graph $\approx$ recursive call graph

Dependency Pair Methods

dependency pair method (informally):

- termination is shown by termination of recursive functions
  Giesl, Arts, Ohlebusch JSC 2002

powerful refinements

- argument filterings Arts, Giesl TCS 2000

- usable rules Hirokawa, Middeldorp IC 2007
  Giesl, Thiemann, Schneider-Kamp, Falke JAR 2007

Argument Filterings

- argument filtering $\pi(f^{(n)})$ is of form i or $[i_1, \ldots, i_m]$
  $(0 \leq i \leq n,\ 1 \leqslant i_1 < \cdots < i_m \leqslant n)$
- extension to term

$$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \ldots, t_n),\ \pi(f) = i \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n),\ \pi(f) = [i_1, \ldots, i_m] \end{cases}$$

EXAMPLE

for $\pi(\mathsf{f}) = [2]$, $\pi(\mathsf{g}) = 1$

$$\pi(\mathsf{f}(\mathsf{g}(x), \mathsf{g}(\mathsf{g}(x)))) = \mathsf{f}(\mathsf{g}(x), \mathsf{g}(\mathsf{g}(x))) = \mathsf{f}(x)$$

Eratosthenes' sieve

TRS $\mathcal{R}$

$$
\begin{array}{rlrl}
1\colon & x - 0 \to x & 10\colon & \mathsf{filter}(x, \mathsf{nil}) \to \mathsf{nil} \\
2\colon & \mathsf{s}(x) - \mathsf{s}(y) \to x - y & 11\colon & \mathsf{filter}(x, y : ys) \to \mathsf{if\_filter}(x \mid y, x, y : ys) \\
3\colon & 0 \leqslant y \to \mathsf{true} & 12\colon & \mathsf{if\_filter}(\mathsf{true}, x, y : ys) \to \mathsf{filter}(x, ys) \\
4\colon & \mathsf{s}(x) \leqslant 0 \to \mathsf{false} & 13\colon & \mathsf{if\_filter}(\mathsf{false}, x, y : ys) \to y : \mathsf{filter}(x, ys) \\
5\colon & \mathsf{s}(x) \leqslant \mathsf{s}(y) \to x \leqslant y & 14\colon & \mathsf{sieve}(\mathsf{nil}) \to \mathsf{nil} \\
6\colon & \mathsf{if}(\mathsf{true}, x, y) \to x & 15\colon & \mathsf{sieve}(x : xs) \to x : \mathsf{sieve}(\mathsf{filter}(x, xs)) \\
7\colon & \mathsf{if}(\mathsf{false}, x, y) \to y & & \\
8\colon & x \mid 0 \to \mathsf{true} & & \\
9\colon & \mathsf{s}(x) \mid \mathsf{s}(y) \to \mathsf{if}(x \leqslant y, \mathsf{s}(x) \mid (y - x), \mathsf{false}) & &
\end{array}
$$

Eratosthenes' sieve

$$
\begin{array}{rl}
1: & x - 0 \to x \\
2: & \mathsf{s}(x) - \mathsf{s}(y) \to x - y \\
3: & 0 \leqslant y \to \mathsf{true} \\
4: & \mathsf{s}(x) \leqslant 0 \to \mathsf{false} \\
5: & \mathsf{s}(x) \leqslant \mathsf{s}(y) \to x \leqslant y \\
6: & \mathsf{if}(\mathsf{true}, x, y) \to x \\
7: & \mathsf{if}(\mathsf{false}, x, y) \to y \\
8: & x \mid 0 \to \mathsf{true} \\
9: & \mathsf{s}(x) \mid \mathsf{s}(y) \to \mathsf{if}(x \leqslant y, \mathsf{s}(x) \mid (y - x), \mathsf{false})
\end{array}
$$

$$
\begin{array}{rl}
10: & \mathsf{filter}(x, \mathsf{nil}) \to \mathsf{nil} \\
11: & \mathsf{filter}(x, y : ys) \to \mathsf{if\_filter}(x \mid y, x, y : ys) \\
12: & \mathsf{if\_filter}(\mathsf{true}, x, y : ys) \to \mathsf{filter}(x, ys) \\
13: & \mathsf{if\_filter}(\mathsf{false}, x, y : ys) \to y : \mathsf{filter}(x, ys) \\
14: & \mathsf{sieve}(\mathsf{nil}) \to \mathsf{nil} \\
15: & \mathsf{sieve}(x : xs) \to x : \mathsf{sieve}(\mathsf{filter}(x, xs))
\end{array}
$$

termination of $\mathcal{R}$ is established by termination of

- recursion of $-$-rules
- recursion of $\leq$-rules
- recursion of $\mid$-rules
- mutual recursion of filter- and if_filter-rules, and
- recursion of sieve-rules (hardest)

Termination of sieve

dependency pair $\mathcal{C} \subseteq >$

$$\mathsf{sieve}^\sharp(x : xs) > \mathsf{sieve}^\sharp(\mathsf{filter}(x, xs))$$

usable rules $\mathcal{U}(\mathcal{C}) \subseteq \gtrsim$

$$\boxed{1}: \quad x - 0 \gtrsim x$$
$$\boxed{2}: \quad \mathsf{s}(x) - \mathsf{s}(y) \gtrsim x - y$$
$$\boxed{3}: \quad 0 \leqslant y \gtrsim \mathsf{true}$$
$$\boxed{4}: \quad \mathsf{s}(x) \leqslant 0 \gtrsim \mathsf{false}$$
$$\boxed{5}: \quad \mathsf{s}(x) \leqslant \mathsf{s}(y) \gtrsim x \leqslant y$$
$$\boxed{6}: \quad \mathsf{if}(\mathsf{true}, x, y) \gtrsim x$$
$$\boxed{7}: \quad \mathsf{if}(\mathsf{false}, x, y) \gtrsim y$$
$$\boxed{8}: \quad x \mid 0 \gtrsim \mathsf{true}$$
$$\boxed{9}: \quad \mathsf{s}(x) \mid \mathsf{s}(y) \gtrsim \mathsf{if}(x \leqslant y, \mathsf{s}(x) \mid (y - x), \mathsf{false})$$
$$\boxed{10}: \quad \mathsf{filter}(x, \mathsf{nil}) \gtrsim \mathsf{nil}$$
$$\boxed{11}: \quad \mathsf{filter}(x, y : ys) \gtrsim \mathsf{if\_filter}(x \mid y, x, y : ys)$$
$$\boxed{12}: \quad \mathsf{if\_filter}(\mathsf{true}, x, y : ys) \gtrsim \mathsf{filter}(x, ys)$$
$$\boxed{13}: \quad \mathsf{if\_filter}(\mathsf{false}, x, y : ys) \gtrsim y : \mathsf{filter}(x, ys)$$
$$14: \quad \mathsf{sieve}(\mathsf{nil}) \gtrsim \mathsf{nil}$$
$$15: \quad \mathsf{sieve}(x : xs) \gtrsim x : \mathsf{sieve}(\mathsf{filter}(x, xs))$$

no suitable LPO, so apply argument filtering

Termination of sieve

dependency pair $\pi(\mathcal{C}) \subseteq >$

$$\mathsf{sieve}^\sharp(x : xs) > \mathsf{sieve}^\sharp(\mathsf{filter}(x, xs))$$

usable rules $\pi(\mathcal{U}(\mathcal{C}, \pi)) \subseteq \gtrsim$

$$1: \quad x - 0 \gtrsim x$$
$$2: \quad \mathsf{s}(x) - \mathsf{s}(y) \gtrsim x - y$$
$$3: \quad 0 \leqslant y \gtrsim \mathsf{true}$$
$$4: \quad \mathsf{s}(x) \leqslant 0 \gtrsim \mathsf{false}$$
$$5: \quad \mathsf{s}(x) \leqslant \mathsf{s}(y) \gtrsim x \leqslant y$$
$$6: \quad \mathsf{if}(\mathsf{true}, x, y) \gtrsim x$$
$$7: \quad \mathsf{if}(\mathsf{false}, x, y) \gtrsim y$$
$$8: \quad x \shortmid 0 \gtrsim \mathsf{true}$$
$$9: \quad \mathsf{s}(x) \shortmid \mathsf{s}(y) \gtrsim \mathsf{if}(x \leqslant y, \mathsf{s}(x) \shortmid (y - x), \mathsf{false})$$
$$10: \quad \mathsf{filter}(x, \mathsf{nil}) \gtrsim \mathsf{nil}$$
$$11: \quad \mathsf{filter}(x, y : ys) \gtrsim \mathsf{if\_filter}(x \shortmid y, x, y : ys)$$
$$12: \quad \mathsf{if\_filter}(\mathsf{true}, x, y : ys) \gtrsim \mathsf{filter}(x, ys)$$
$$13: \quad \mathsf{if\_filter}(\mathsf{false}, x, y : ys) \gtrsim y : \mathsf{filter}(x, ys)$$
$$14: \quad \mathsf{sieve}(\mathsf{nil}) \gtrsim \mathsf{nil}$$
$$15: \quad \mathsf{sieve}(x : xs) \gtrsim x : \mathsf{sieve}(\mathsf{filter}(x, xs))$$

use LPO with empty precedence

how to find suitable argument filtering?

Finding Suitable Argument Filtering

argument filtering and usable rules are powerful

$$\pi(\mathcal{C}) \cap > \neq \varnothing \quad \wedge \quad \pi(\mathcal{U}(\mathcal{C}, \pi) \cup \mathcal{C}) \subseteq \gtrsim$$

PROBLEM

- difficult to find suitable argument filtering:
  $4,234,032$ possible argument filterings for TRS of sieve

- simultaneous constraint between usable rules and argument filtering is problematic

SOLUTION

encode simultaneous constraints into propositional formulas

Extended Diophantine Constraints

SYNTAX

$$c ::= b \mid r \mid \neg c \mid c_1 \vee c_2 \mid c_1 \wedge c_2$$
$$r ::= p_1 \geq p_2 \mid p_1 \leq p_2 \mid p_1 > p_2 \mid p_1 < p_2 \mid p_1 = p_2 \mid p_1 \neq p_2$$
$$p ::= x \mid n \mid p_1 + p_1 \mid p_1 \times p_2$$

where $b \in \mathcal{B}$, $x \in \mathcal{V}$ and $n \in \mathbb{Z}$

SEMANTICS

- $\alpha, \beta \vDash b$ if $\beta(b) = \text{true}$
- $\alpha, \beta \vDash p_1 \geq p_2$ if $[\alpha]_{\mathbb{N}}(p_1) \geq [\alpha]_{\mathbb{N}}(p_2)$
- $\alpha, \beta \vDash \neg c$ if $\alpha, \beta \vDash c$ does not hold
- $\alpha, \beta \vDash c_1 \vee c_2$ if $\alpha, \beta \vDash c_1$ or $\alpha \vDash c_2$
- $\alpha, \beta \vDash c_1 \wedge c_2$ if $\alpha, \beta \vDash c_1$ and $\alpha, \beta \vDash c_2$

- $\vDash c$ if $\exists \alpha : \mathcal{V} \to \mathbb{Z}, \beta : \mathcal{B} \to \{\text{true}, \text{false}\}.\ \alpha, \beta \vDash c$

Encoding Argument Filtering Constraints

for n-ary function f introduce propositional variables

$$f^{\mathsf{list}}, f^1, f^2, \ldots, f^n$$

- replace $\pi(f) = i$ by

$$\neg f^{\mathsf{list}} \wedge f^i \wedge \bigwedge_{k \neq i} \neg f^k$$

- replace $\pi(f) = [i_1, \ldots, i_m]$ by

$$f^{\mathsf{list}} \wedge \bigwedge_{k \in \{i_1, \ldots, i_m\}} f^k \wedge \bigwedge_{k \notin \{i_1, \ldots, i_m\}} \neg f^k$$

- add valid form condition (e.g. $\neg f^{\mathsf{list}} \wedge f^1 \wedge f^2$ is invalid)

$$f^{\mathsf{list}} \vee \bigvee_{i \in \{1, \ldots, n\}} \left(f^i \wedge \bigwedge_{k \neq i} \neg f^i \right)$$

OBSERVE

$\pi(s) >_{\mathsf{lpo}} \pi(t)$ is extended Diophantine constraint

EXAMPLE

$$
\begin{aligned}
& \pi(\mathsf{f}(\mathsf{a}, x)) >_{\mathsf{lpo}} \pi(\mathsf{c}) \ \wedge \ \text{(valid form condition)} \\
\Leftrightarrow \ & (\mathsf{f}^{\mathsf{list}} \ \wedge \ \mathsf{c}^{\mathsf{list}} \ \wedge \ \mathsf{f} > \mathsf{c}) \vee \\
& (\neg \mathsf{f}^{\mathsf{list}} \ \wedge \ ((\mathsf{a}^1 \ \wedge \ \mathsf{c}^{\mathsf{list}} \ \wedge \ \mathsf{a}^{\mathsf{list}} \ \wedge \ \mathsf{a} > \mathsf{c}) \vee (\mathsf{a}^2 \ \wedge \ \mathsf{c}^{\mathsf{list}} \ \wedge \ \bot))) \wedge \\
& (\mathsf{f}^{\mathsf{list}} \ \vee \ (\mathsf{f}^1 \ \wedge \ \neg \mathsf{f}^2) \ \vee \ (\neg \mathsf{f}^1 \wedge \mathsf{f}^2)) \ \wedge \ \mathsf{a}^{\mathsf{list}} \ \wedge \ \mathsf{b}^{\mathsf{list}})
\end{aligned}
$$

solutions of this constraint

- $\mathsf{f}^{\mathsf{list}} = \mathsf{f}^1 = \mathsf{f}^2 = \mathsf{a}^{\mathsf{list}} = \mathsf{c}^{\mathsf{list}} = \mathsf{true}, \mathsf{f} = 1, \mathsf{a} = \mathsf{c} = 0$
  $\pi(f) = [1; 2], \pi(\mathsf{a}) = \pi(\mathsf{c}) = [], \mathsf{f} > \mathsf{a}, \mathsf{f} > \mathsf{c}$
- $\mathsf{f}^{\mathsf{list}} = \mathsf{f}^2 = \mathsf{false}, \mathsf{f}^1 = \mathsf{a}^{\mathsf{list}} = \mathsf{c}^{\mathsf{list}} = \mathsf{true}, \mathsf{a} = 1, \mathsf{f} = \mathsf{c} = 0$
  $\pi(f) = 1, \pi(\mathsf{a}) = \pi(\mathsf{c}) = [], \mathsf{a} > \mathsf{f}, \mathsf{a} > \mathsf{c}$
- ...

Encoding Usable Rule Criterion

for function f, introduce propositional variable

$$U_f$$

constraint

$$\pi(\mathcal{U}(\mathcal{C},\pi)\cup\mathcal{C})\subseteq\gtrsim$$

is encoded to

$$\bigwedge_{l\to r\in\mathcal{C}}\big(U_{\mathrm{root}(l)}\wedge(\pi(l)\gtrsim\pi(r))\big)\ \wedge\ \bigwedge_{l\to r\in\mathcal{R}}\big(U_{\mathrm{root}(l)}\to(\pi(l)\gtrsim\pi(r))\big)\ \wedge$$

$$\bigwedge_{l\to r\in\mathcal{R}\cup\mathcal{C}}\left(U_{\mathrm{root}(l)}\to\bigwedge_{\substack{p\in\mathcal{P}\mathsf{os}_{\mathcal{F}}(r)\\ \mathrm{root}(r|_p)\text{ is defined}}}\left(\bigwedge_{q,\,i\colon qi\leqslant p}\mathrm{root}(r|_q)^i\to U_{\mathrm{root}(r|_p)}\right)\right)$$

Experimental Results for LPO

dependency pair method with LPO

	IC 2005	Dio/SAT solving	
	$\mathcal{U}(\mathcal{C})$	$\mathcal{U}(\mathcal{C})$	$\mathcal{U}(\mathcal{C},\pi)$
# of success	188	188	247
# of timeout (60 s)	5	0	0
total time (s)	338	61	120

termination tool: TTT with Minisat problem set: 773 TRSs from Termination Problem Database
PC: Intel Xeon 2.40 GHz 512 MB memory

Experimental Results for Polynomial Orders

dependency pair method with linear form polynomial interpretation orders

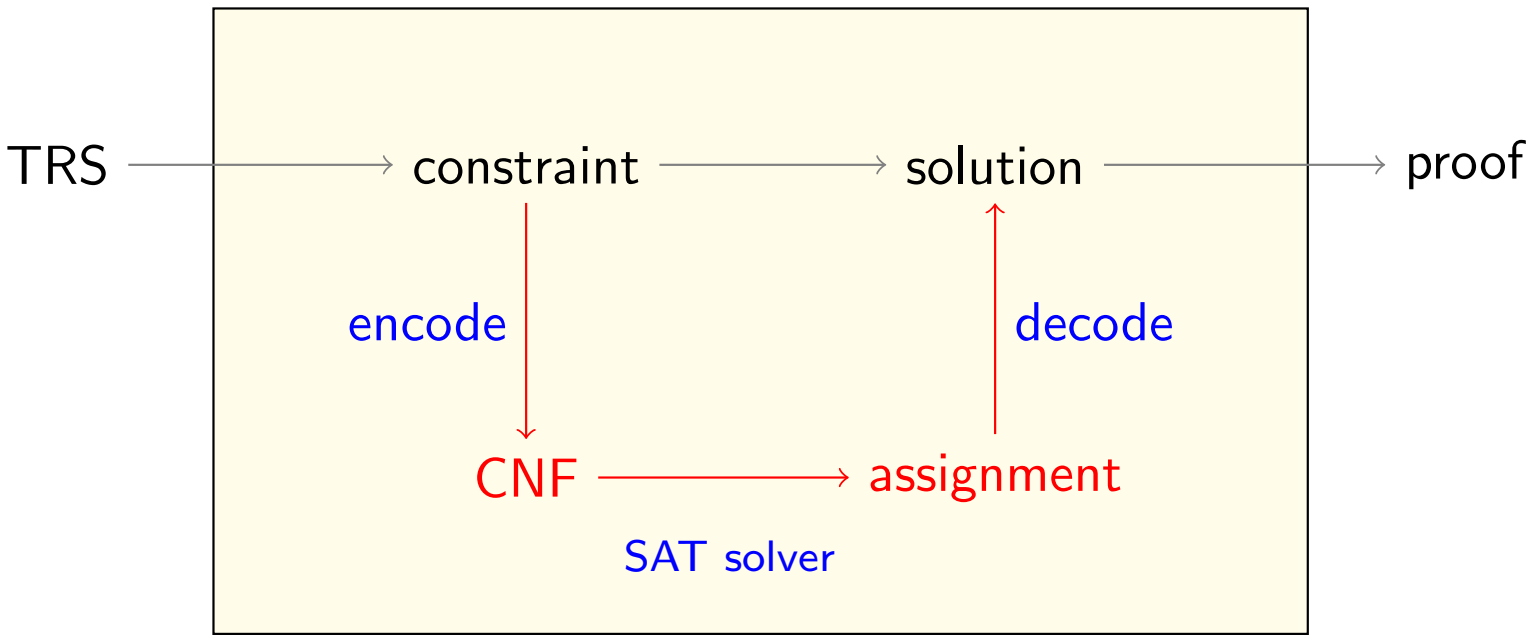
$$f_{\mathbb{N}}(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n + b \qquad a_1, \ldots, a_n, b \in \{0, 1, 2\}$$

	Gen & Test	Dio/SAT solving
	$\mathcal{U}(\mathcal{C})$	$\mathcal{U}(\mathcal{C}, \pi)$
# of success	247	431
# of timeout (60 s)	32	0
total time (s)	2568	91

termination tools: AProVE and TTT with Minisat problem set: 865 TRSs from Termination Problem Database
PC: AMD Athlon 64 2.20 GHz 512 MB memory

Conclusion

currently all termination tools use SAT solver or Diophantine constraint solver



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Kurihara, Kondo	IEA/AIE 2004
Zankl, Middeldorp, Hirokawa	SOFSEM 2007