Stainless Formal Verification

on the interdependency between mathematical foundations, semantics of specifications, and proof score programming

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Outline

1. Introduction
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   - Signatures, algebras, sentences, satisfaction
   - Equational proof theory
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     - Natural numbers
     - Bubble sorting
   - Proof management
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Goal of lecture

Gain some understanding on what is a correct specification and verification process through clarification of

- mathematical foundations
- semantics
- specification
- proof score programming

and of the relationships between them.

Most of ‘formal verification’ practice, while assumed to be rigorous by definition, in reality still confused wrt some of above aspects, hence difficult to be trusted.

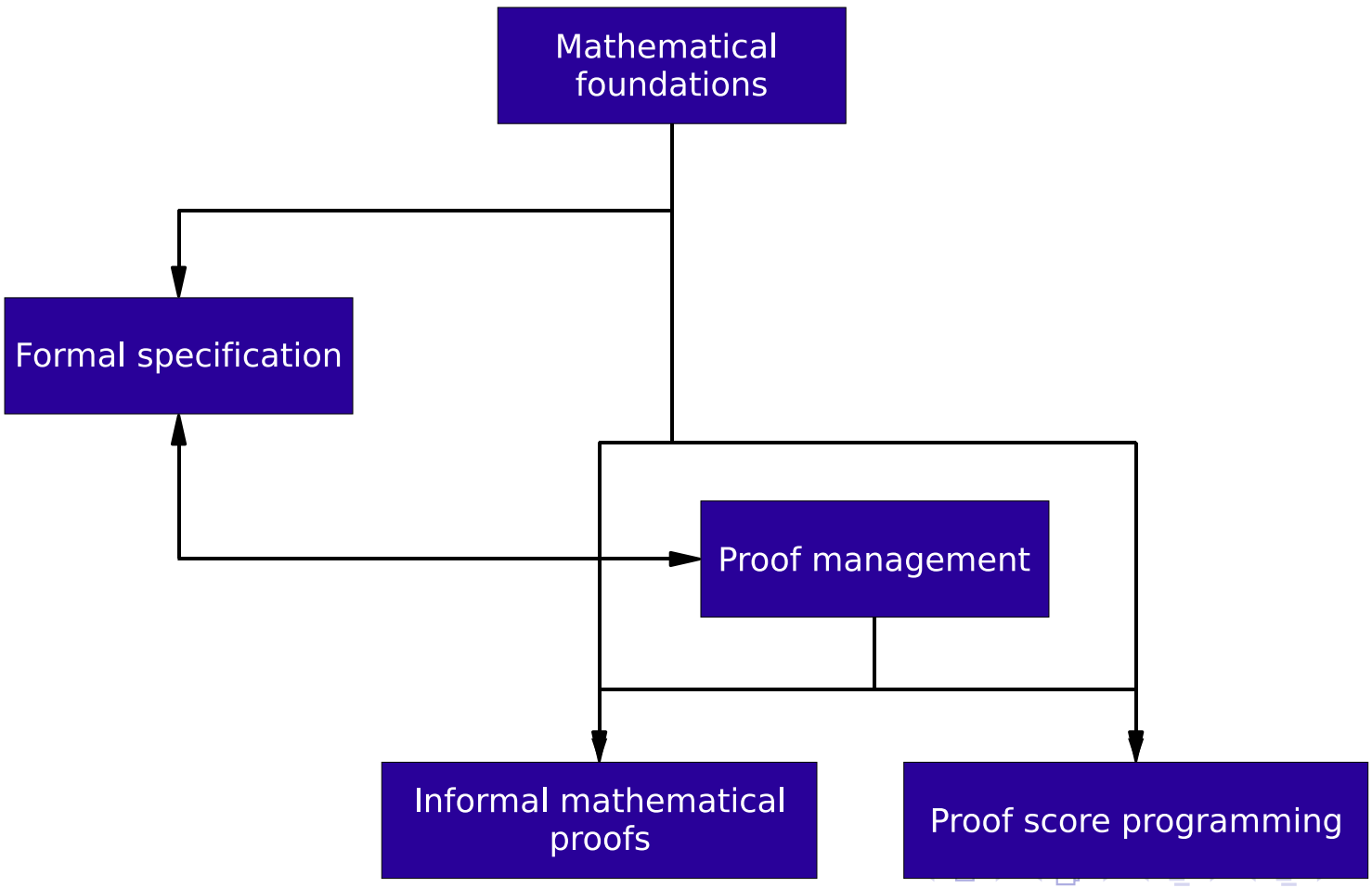
‘Stainless’ does not mean ‘perfectly formal’

But rather

- being clear about what are the formal and the informal proof parts,
- justification of informal parts by solid clear mathematical arguments,
- proof scores for the formal proof parts strictly based upon mathematical foundations, and
- clean(!) mathematical foundations.

Specification and verification schema

Each level is defined and makes sense wrt the upper levels.



Mathematical foundations

- There is a formal logical system, including both model theory (for semantics) and proof theory. Very desirable that these constitute an *institution*.
- The model theory is sufficiently developed in order to support necessary specification properties.
- The eventual operational level of the proof theory (e.g. rewriting) is rigorously supported by mathematics.

Formal specification

- There is a formal specification language such that the language constructs correspond exactly to mathematical entities in the underlying logic.
- A specification consists of
  - a set of *axioms* in the underlying logic (this includes the specification of a corresponding signature), and
  - eventually, structuring constructs.
- Each such specification defines the *class of models* satisfying its axioms.
  - In the structured case, this is also determined by the structuring constructs (requires a bit of mathematical sophistication).

The whole point of formal specification:

axiomatic definition of certain classes of models.

Role of semantics and foundations

Tendency to forget this sense of formal specification and verification by neglecting semantics.

No semantics = No meaning!

This implies incorrect non-sense specification and verification.

Proof management

- Introduce auxiliary entities (e.g. functions, predicates) necessary for the proof; this means extension of the original specification.
- Separate the formal from the informal parts of the proof.
  - Informal parts get mathematical proofs.
  - Formal parts are proved by proof score running. Much higher ‘horizontal’ complexity than the informal parts.

Proof score programming

- Specification of the proof structure, including lemmas, conditions, proof tasks to be executed by the system, etc.
- Should be rigorously, directly and *transparently* based upon mathematical results lying foundations to corresponding proof methodologies.
  - In particular, this means to avoid abuse or even any use of extra-logical features of the language (such as ==, etc.)

Many sorted algebra (MSA)

- It is the most classical logical system of algebraic specification.
  - Deeply rooted in conventional mathematics.
  - Originating from early mathematical foundations of semantics of programming languages and of abstract data types.
- Has very convenient model theoretic and computational properties, supporting high integration between the specification and the verification aspects.
- However, there are myriads of other logical systems used for formal specification, some of them just refinements of MSA, others quite different (at least through a gross perspective).

Signatures

S-sorted signature (S, F)

- S – set of sort symbols,
- $F = \{F_{w \to s} \mid w \in S^*,\ s \in S\}$ – indexed family of operation symbols.

Algebras

(S,F)-*algebra* A consists of

- a set A_s for each $s \in S$, and
- a function $A_{\sigma:w\to s} : A_w \to A_s$ for each $\sigma \in F_{w\to s}$ (where $A_w = A_{s_1} \times \cdots \times A_{s_n}$, for $w = s_1 \ldots s_n$).

(S,F)-*homomorphism* $h : A \to B$ consists of

- $h_s : A_s \to B_s$ for each $s \in S$,

such that the following *homomorphism condition*

$$h_s(A_\sigma(a)) = B_\sigma(h_w(a)).$$

for each $a \in A_w$.

Sentences

The set of (S, F)-*sentences* is the least set such that:

- Each (S, F)-equation $t = t'$ is an (S, F)-sentence.
- If ρ_1 and ρ_2 are (S, F)-sentences then
  - $\rho_1 \wedge \rho_2$ (*conjunction*),
  - $\rho_1 \vee \rho_2$ (*disjunction*),
  - $\rho_1 \Rightarrow \rho_2$ (*implication*) and
  - $\neg \rho_1$ (*negation*)

  are also (S, F)-sentences.
- If X is a set of variables for (S, F), then $(\forall X)\rho$ and $(\exists X)\rho$ are (S, F)-sentences whenever ρ is an $(S, F \cup X)$-sentence.

Satisfaction

Defined recursively on the structure of the sentences.

- $A \models t = t'$ if and only if $A_t = A_{t'}$ (where $A_{\sigma(t_1,\ldots,t_n)} = A_\sigma(A_{t_1},\ldots,A_{t_n})$),
- $A \models \rho_1 \wedge \rho_2$ if and only if $A \models \rho_1$ and $A \models \rho_2$,
- $A \models \rho_1 \vee \rho_2$ if and only if $A \models \rho_1$ or $A \models \rho_2$,
- $A \models \rho_1 \Rightarrow \rho_2$ if and only if $A \not\models \rho_1$ or $A \models \rho_2$,
- $A \models \neg\rho_1$ if and only if $A \not\models \rho_1$,
- $A \models_{(S,F)} (\forall X)\rho$ if and only if $A' \models_{(S,F\cup X)} \rho$ for each $(S, F \cup X)$-expansion A' of A, and
- $A \models (\exists X)\rho$ if and only if $A \not\models (\forall X)\neg\rho$.

Initial semantics

Initial algebra A in class $\mathscr{C}$ of algebras:
for each $B \in \mathscr{C}$ there exists an unique $h : A \to B$.

Fact

Initial algebras are unique up to isomorphisms.

Theorem

Each set of conditional equations, i.e. sentences of the form $(\forall X)(t_1 = t_1') \wedge \cdots \wedge (t_n = t_n') \Rightarrow (t = t')$, *has an initial algebra.*

Initial algebras are the models of tight denotation specifications (`mod!` in CafeOBJ, `fmod` in Maude)

Entailment systems

Entailment relation (for a signature Σ) consists of a binary relation $\vdash_\Sigma$ between sets of Σ-sentences such that:

1. *union:* if $\Gamma \vdash_\Sigma \Gamma_1$ and $\Gamma \vdash_\Sigma \Gamma_2$ then $\Gamma \vdash_\Sigma \Gamma_1 \cup \Gamma_2$,
2. *monotonicity:* if $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash_\Sigma \Gamma$, and
3. *transitivity:* if $\Gamma \vdash_\Sigma \Gamma_1$ and $\Gamma_1 \vdash_\Sigma \Gamma_2$ then $\Gamma \vdash_\Sigma \Gamma_2$.

An *entailment system* $\vdash$ consists of an entailment relation $\vdash_\Sigma$ for each signature Σ.

Example: semantic entailment

$\Gamma \models_{(S,F)} \Gamma'$ if and only if
$A \models_{(S,F)} \Gamma$ implies $A \models_{(S,F)} \Gamma'$ for each (S,F)-algebra A.

Modus Ponens (meta-rule)

Meta-rules are properties of the entailment systems.

$$\Gamma \vdash_{(S,F)} (H \Rightarrow C) \text{ if and only if } \Gamma \cup H \vdash_{(S,F)} C.$$

The semantic entailment $\models$ has Modus-Ponens.

Universal Quantification (meta-rule)

$$\Gamma \vdash_{(S,F)} (\forall X)\rho \text{ if and only if } \Gamma \vdash_{(S,F\cup X)} \rho.$$

The semantic entailment $\models$ has Universal Quantification.

Equational proof rules

Reflexivity: $\dfrac{\emptyset}{t = t}$

Symmetry: $\dfrac{\{t = t'\}}{t' = t}$

Transitivity: $\dfrac{\{t = t',\ t' = t''\}}{t = t''}$

Congruence: $\dfrac{\{t_i = t'_i \mid 1 \leq i \leq n\}}{\sigma(t_1, \ldots, t_n) = \sigma(t'_1, \ldots, t'_n)}$

Substitutivity: $\dfrac{\{(\forall X)H \Rightarrow C\}}{\{\theta(H \Rightarrow C)\}} \quad \theta : X \to T_{(S,F)}.$

The equational entailment system

Definition (*entailment system for conditional equations*, $\vdash^{e}$)

The least entailment system, containing the 5 proof rules and satisfying the 2 meta-rules above.

Unlike $\models$, $\vdash^{e}$ is finitely defined.
This makes $\vdash^{e}$ usable for formal proofs.

Soundness

An absolutely necessary property for any logical system, constitutes the basis for the correctness of formal verification.

Not very hard to establish.

Theorem (*Soundness of equational deduction*)

$\Gamma \vdash^{\mathrm{e}}_{(S,F)} \rho$ *implies* $\Gamma \models_{(S,F)} \rho$.

Completeness

A very desirable property of logical systems, but not absolutely necessary.

In general, much harder to establish than soundness.

Theorem (*Completeness of equational deduction*)

$\Gamma \models_{(S,F)} \rho$ *implies* $\Gamma \vdash^{\mathrm{e}}_{(S,F)} \rho$.

Soundness and completeness means that $\models$ (for conditional equations) and $\vdash^{\mathrm{e}}$ are the same, hence, very importantly, we have a finitary definition of $\models$ (for conditional equations).

Rewriting

The standard way to mechanize equational deduction, to have it as a computation process.

Abandons *Symmetry* and replaces *Substitutivity* and *Congruence* by the following rule:

$$\textit{Rewriting:} \quad \frac{\{(\forall X)H \Rightarrow (t = t')\} \cup \theta(H)}{c[\theta(t)] = c[\theta(t')]}$$

In general completeness is lost, however under conditions such as *confluence* and *termination*, completeness may be retained.

Inductive properties

These are the properties satisfied by the initial algebra of a given set Γ of conditional equations, i.e. $0_\Gamma \models \rho$.

For example, for the specification Γ as follows:

```
0 : -> Nat
s : Nat -> Nat
+ : Nat Nat -> Nat
(∀ X) X + 0 = X
(∀ X,Y) X + s(Y) = s(X + Y)
```

we have that $0_\Gamma \models (\forall \mathtt{X})\mathtt{0 + X} = \mathtt{X}$ but $\Gamma \not\models (\forall \mathtt{X})\mathtt{0 + X} = \mathtt{X}$. Hence, direct ordinary (equational) deduction not enough for proving inductive properties.

Constructors

A great device for reducing the complexity of proofs of inductive properties.

Sub-signature of constructors

Signature (S, F), Γ a set of conditional equations for (S, F). (S, F^c) is a *sub-signature of constructors for* Γ when

- $F^c_{w \to s} \subseteq F_{w \to s}$, and
- $0_{(S,F^c)} \to 0_\Gamma\restriction_{(S,F^c)}$ surjective.

Example: `0` and `s` form a sub-signature of constructors for the specification above.

Sufficient completeness

The following equivalent characterization for sub-signature of constructors has two advantages:

1. does not depend on existence of initial algebras for Γ, hence applicable within more general situations, and
2. it gives a method to actually prove the constructors property.

Proposition

(S, F^c) is a sub-signature of constructors for Γ if and only if for each (S, F)-term t there exists an (S, F^c)-term t' such that $\Gamma \models_{(S,F)} t = t'$.

Proving inductive properties

The following gives a sufficient condition for proving inductive properties by an *infinite* set of ordinary proofs.

Theorem

1. Γ *set of conditional equations for a signature* (S,F),
2. (S,F^c) *sub-signature of constructors for* Γ, *and*
3. *('lemmas':)* E *set of any sentences such that* $0_\Gamma \models E$.

Then for any $(S,F \cup X)$*-sentence* ρ

$$0_\Gamma \models (\forall X)\rho \;\; \textit{if} \;\; \Gamma \cup E \models \theta(\rho) \;\; \textit{for all substitutions} \;\; \theta : \; X \to T_{(S,F^c)}.$$

Structural induction

Theorem (*sufficient **finitary proof method for inductive properties*)**

1. (S, F^c) *sub-signature of constructors for* Γ ***any*** *set of* (S,F)*-sentences,*
2. X *finite set of variables for* (S,F)*,*
3. ρ ***any*** $(S, F \cup X)$*-sentence.*

If for any sort preserving mapping $Q: \ X \to F^c$

$$\Gamma \cup \{\psi(\rho) \mid \psi: \ X \to Z = \cup_{x \in X} Z_x \ \text{ with } \ \psi(x) \in Z_x\} \models_{(S, F \cup Z)} Q^\sharp(\rho)$$

where

- Z_x *are strings of variables for the arguments of* Q_x *such that* $Z_{x1} \cap Z_{x2} = \emptyset$ *for* $x1 \neq x2 \in X$*, and*
- $Q^\sharp$ *is the substitution* $X \to T_{(S, F^c \cup Z)}$ *defined by* $Q^\sharp(x) = Q_x(Z_x)$*,*

then $\Gamma \models_{(S,F)} \theta(\rho)$ *for all substitutions* $\theta: \ X \to T_{(S,F^c)}$.

Generation of proof goals by Structural Induction

The Structural Induction Theorem together with the predecesor Theorem lead to the following proof goal generation rule.

Structural Induction

$$\frac{\{\Gamma \cup \{\psi(\rho) \mid \psi : X \to Z \ldots\} \models_{(S,F \cup Z)} Q^\sharp(\rho) \mid Q : X \to F^c\}}{0_\Gamma \models_{(S,F)} (\forall X)\rho}$$

The finitary character of structural induction

Finite number of proof goals always involving finite conditions because:

- for any Q, finite Z because finite X and finite arities of operations, hence finite $\{\psi \mid \psi : \ X \to Z\}$, and
- if finite F^c, then finite $\{Q \mid Q : \ X \to F^c\}$ since finite X .
  - moreover smaller F^c implies fewer proof goals.

Generality of foundations

Many of MSA mathematical concepts and results above can be interpreted in the same form in other logical systems (such as *preordered algebra* for specification with transitions).

For example, the structural induction method above has such general character.

This means foundations can be mathematically developed at the level of abstract *institutions*.

Example of ‘stainless’ formal verification

We illustrate

- formal specification based rigorously upon mathematical foundations,
- proof management,
- proof score building based rigorously upon equational proof theory and the structural induction theorem above,
- the inter-dependency between semantic-oriented specification and proof score programming.

Specification of natural numbers

The natural numbers with the usual zero and succesor operations constitute the initial algebra of PNAT=.

Moreover we have a specification of the equality of numbers

```
mod! PNAT=
  [ Nat ]
  op 0 :  -> Nat
  op s_ : Nat -> Nat
  op _=_ : Nat Nat -> Bool {comm}
  vars M N : Nat
  eq ((s M) = 0) = false .
  eq (0 = 0) = true .
  eq [succ=] : (s M = s N) = (M = N) .
}
```

Specification of $<$ on natural numbers

The following adds a specification of the ‘strictly less than’ relation on the naturals numbers.

```
mod! PNAT< {
  protecting(PNAT=)
  op _<_ : Nat Nat -> Bool
  vars M N : Nat
  eq [succ<] : (s M) < (s N) = M < N .
  eq 0 < (s M) = true .
  eq M < 0 = false .
}
```

Strings of natural numbers

The strings of natural numbers with concatenation and empty string constitute the initial algebra of `STRG-PNAT`.

```
mod! STRG-PNAT {
  [ Nat < Strg ]
  op nil :  -> Strg
  op _;_ : Strg Strg -> Strg {assoc id: nil}
}
```

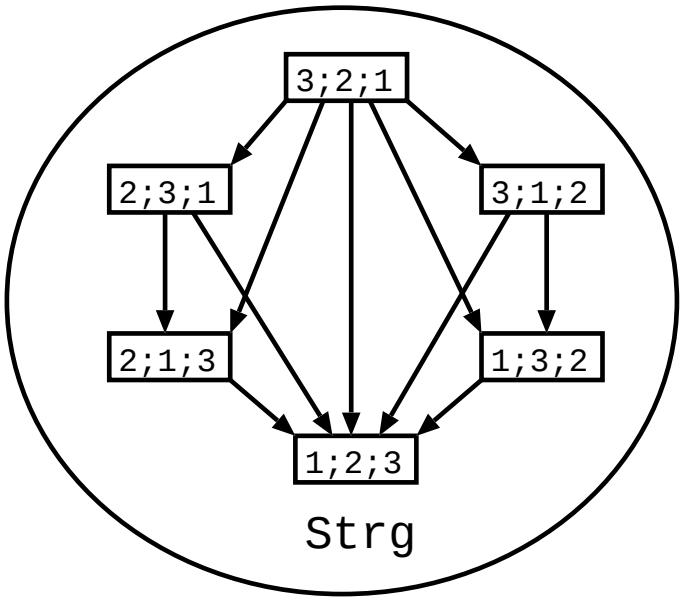
Specification of bubble sorting of strings of natural numbers

Bubble sorting algorithm appears as an instance of rewriting modulo associativity (of concatenation).

```
mod! SORTING-STRG-PNAT {
  protecting(STRG-PNAT))
  vars E E' : Nat
  ctrans E ; E' => E' ; E  if (E' < E) .
}
```

The semantics of SORTING-STRG-PNAT

The initial model is the *preordered algebra* that has strings of naturals as elements and the transitions given by the sorting algorithm as the preorder relation.



Sorting strings of natural numbers

We may use this specification as an actual sorting (very high level) program by executing it by rewriting modulo associativity.

```
exec s s s 0 ; s 0 ; 0 ; s s 0 .
```

Topological sorting

However, if we look more carefully into the specification of bubble sorting, we note that it essentially requires only a binary relation $<$, no commitement to any property of the naturals, not even to the naturals as elements of the strings.

This means bubble sorting is very general, has a *generic* nature.

Such kind of sorting over binary relations is sometimes known as *topological* sorting.

Specification of generic strings

The following parameterized module specifies strings over *any* set (`Elt`) of elements.

```
mod! STRG (X :: TRIV) {
  [ Elt < Strg ]
  op nil :  -> Strg
  op _;_ : Strg Strg -> Strg {assoc id: nil}
}
```

Specification of generic pseudo-order

The following specifies a generic binary relation < on the elements, to be used for the sorting.
We also specify a loose negation of <, namely `not<`, mainly for operational reasons.

```
mod* PSEUDO-ORDER {
  [ Elt ]
   op _<_ : Elt Elt -> Bool
   op _not<_ : Elt Elt -> Bool
   vars E1 E2 E3 : Elt
   cq (E1 not< E2) = true
       if E2 < E1 or not(E1 < E2) .
}
```

Note on specification of generic pseudo-order

Note that the sentence specifying `not<`

```
cq (E1 not< E2) = true
      if E2 < E1 or not(E1 < E2) .
```

is *not* a conditional equation (although CafeOBJ notation refers to it as conditional equation).

However this is OK since this is loose semantics specification, which does not require existence of initial algebras.

Recovering the ordering of the naturals

The following instantiate the above specified pseudo-orders to the standard ordering of the natural numbers.

```
view PNAT<asPO from PSEUDO-ORDER to PNAT<
     {op (E:Elt not< E':Elt) ->
          ((E:Nat = E':Nat) or (E' < E))} .
```

It uses default mapping mechanism, hence only the mapping of `not<` needs to be specified explicitly.

View definitions require proofs

The view specification requires a proof that the initial algebra of PNAT< satisfies the axiom of PSEUDO-ORDER through the translation given by the view.

This means an inductive proof, that can be done by proof score programming based upon the Structural Induction Theorem; we skip this here.

Note that in this case *proof score programming is involved at the stage of specification writing*.

Specification of the generic bubble sorting algorithm

```
mod! SORTING-STRG(Y :: PSEUDO-ORDER) {
  protecting(STRG(Y))
  vars E E' : Elt
  ctrans E ; E' => E' ; E  if (E' < E) .
}
```

`SORTING-STRG-PNAT` can be obtained as an instance of `SORTING-STRG` by the above defined view `PNAT<asPO`.

```
select SORTING-STRG(PNAT<asPO) .
```

Specification of the generic bubble sorting algorithm

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```
select SORTING-STRG(PNAT<asPO) .
```

Properties of topological sorting

While for the bubble sorting of strings of naturals termination and confluence are quite obvious, in the generic case these are rather unclear.

In fact, in general they may not hold.

In the following we focus on (proving) termination.

Proof of termination of topological (generic) bubble sorting

For this we go back to the specification level and define a function `disorder : Strg -> Nat` such that

$(\forall S, S')[S \texttt{=>} S' \text{ implies } \texttt{disorder}(S') < \texttt{disorder}(S)]$.

(note this is a Horn clause in the *POA*, the logical system of preordered algebra).

Then the mathematical argument of well-foundness of the natural numbers leads to the (informal) mathematical proof of termination.

Specification of auxiliary functions

The definition of function `disorder` requires specification of other auxiliary functions too:

```
mod! PNAT+ {
  protecting(PNAT=)
  op _+_ : Nat Nat -> Nat
  vars M N : Nat
  eq [succ+] : M + (s N) = s(M + N) .
  eq M + 0 = M .
}
```

Specification of auxiliary functions

- $E \gg S$ computes how many elements of S are less than E.
- `disorder`(S) computes how many steps of the sorting algorithm are needed for the sorting of S.

```
mod! SORTING-DISORDER (Y :: PSEUDO-ORDER) {
    protecting(SORTING-STRG(Y) + PNAT+)
    op _>>_ : Elt Strg -> Nat
    op disorder : Strg -> Nat
    eq E >> nil = 0 .
    cq E >> E' =  s 0 if (E' < E) .
    cq E >> E' =  0   if (E' not< E) .
    eq E >> (S ; S') = (E >> S) + (E >> S') .
    eq disorder(E) = 0 .
    eq disorder(E ; S) = disorder(S) + (E >> S) .
}
```

Specification of auxiliary functions

The following specifies an equivalence relation on strings defined by

$$\text{S} <> \text{S'} \quad \text{if and only if} \quad (\forall E) E\text{»}S = E\text{»}S'.$$

Although this is not required by the specification of `disorder`, it is used in the formal proofs below.

As this is beyond CafeOBJ logic, at this level we under-specify it, however we will use its complete definition in the proof scores.

```
mod* SORTING<> (Y :: PSEUDO-ORDER) {
  protecting(SORTING-DISORDER(Y))
  op _<>_ : Strg Strg -> Bool
}
```

Proof management

The proof management of this problem means the following:

1. By an mathematical argument we have reduced the task of proving termination to proving a Horn clause sentence as inductive property in *POA*.
2. Extension of the original specification with new functions.
3. Mathematical proof of the fact that

   $(\forall S, S', E1, E2) E1 < E2$ implies $\texttt{disorder}(S; E1; E2; S') < \texttt{disorder}(S; E2; E1; S')$

   implies

   $(\forall S, S')$ [S=>S′ implies $\texttt{disorder}(S') < \texttt{disorder}(S)$].

   (This mathematical argument is related to the theory of rewriting modulo axioms.)

Proof management

4. Formal proof (by proof score programming and running) of

   $(\forall S, S', E1, E2)\ [\texttt{disorder}(S; E1; E2; S') < \texttt{disorder}(S; E2; E1; S')$

   This requires several lemmas and tranformation of the proof goals by meta-rules such as *Universal Quantification* or *Modus-Ponens*.
5. Mathematical proofs for sub-signatures of constructors.

Lemmas for the formal proof of
$(\forall S, S', E1, E2)\,\texttt{disorder}(S;E1;E2;S') < \texttt{disorder}(S;E2;E1;S')$ if $E1 < E2$

Lemma-1

$(\forall S1, S2)\ (S1 <> S2) \wedge (\texttt{disorder}(S1) < \texttt{disorder}(S2))$
implies $\texttt{disorder}(S;S1) < \texttt{disorder}(S;S2)$.

Lemma-2

$(\forall E, E', S)\ (E;E';S) <> (E';E;S)$.

Lemma-3

Commutativity and associativity of _+_.

Lemma-4

$(\forall M : \texttt{Nat})\ M < \texttt{s}M$.

How do we find lemmas?

In general lemmas have various different natures, no general method for finding lemmas.

However we may distinguish two main situations:

1. Lemmas that are very meaningful properties of the specified system (e.g. `Lemma-1` and `Lemma-2`). These are often hard to find and only on the basis of a deep understanding of the problem. Their formulation may even require definition of new functions!
2. Lemmas that have mainly a role to get the deduction (by rewriting) flow (e.g. `Lemma-3` and `Lemma-4`).. These are sometimes easier to find since they may ‘pop-up’ when the deduction gets blocked. (Unfortunately not the case here!)

Proof score of
$(\forall S, S', E1, E2)\texttt{disorder}(S; E1; E2; S') < \texttt{disorder}(S; E2; E1; S')$ if $E1 < E2$

By two steps transformation of the original proof goal:

by *Universal Quantification* into

$\Gamma \vdash_{\Sigma \cup \{E1, E2, S, S'\}} E1 < E2$ implies
$\texttt{disorder}(S; E1; E2; S') < \texttt{disorder}(S; E2; E1; S')$

and further by *Modus Ponens* into

$\Gamma \cup \{E1 < E2\} \vdash_{\Sigma \cup \{E1, E2, S, S'\}}$
$\texttt{disorder}(S; E1; E2; S') < \texttt{disorder}(S; E2; E1; S')$

Finally, we give this proof goal to the system.

Formal proof of `Lemma-1`:

$(\forall S, S1, S2)\texttt{disorder}(S;S1) < \texttt{disorder}(S;S2)$
if $S1$ `<>` $S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$.

By application of *Structural Induction* where

- The sub-signature of constructors consists of `nil`, all `e:Elt`, and `_;_`, this being established by mathematical proof.
- $X = \{S\}$, and
- ρ is
  $(\forall S1, S2)$ $S1$ `<>` $S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
  implies $\texttt{disorder}(S;S1) < \texttt{disorder}(S;S2)$.

the original proof goal gets transformed to

G8: $(Q_S = \texttt{nil}, Z = \emptyset)$

$\Gamma \models_\Sigma (\forall S1, S2)\ S1 \texttt{ <> } S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
implies $\texttt{disorder}(\texttt{nil}; S1) < \texttt{disorder}(\texttt{nil}; S2)$.

G9: $(Q_S = \texttt{e:Elt}, Z = \emptyset)$

$\Gamma \models_{\Sigma \cup \{\texttt{e}\}} (\forall S1, S2)\ S1 \texttt{ <> } S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
implies $\texttt{disorder}(\texttt{e}; S1) < \texttt{disorder}(\texttt{e}; S2)$.

G10: $(Q_S = \_;\_, Z = \{\texttt{x}, \texttt{y}\})$

$\Gamma \cup \{$ $(\forall S1, S2)\ S1 \texttt{ <> } S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
implies $\texttt{disorder}(\texttt{x}; S1) < \texttt{disorder}(\texttt{x}; S2)$,
$(\forall S1, S2)\ S1 \texttt{ <> } S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
implies $\texttt{disorder}(\texttt{y}; S1) < \texttt{disorder}(\texttt{y}; S2)$ $\}$
$\models_{\Sigma \cup \{\texttt{x},\texttt{y}\}}$ $(\forall S1, S2)\ S1 \texttt{ <> } S2$ and $\texttt{disorder}(S1) < \texttt{disorder}(S2)$
implies $\texttt{disorder}(\texttt{x}; \texttt{y}; S1) < \texttt{disorder}(\texttt{x}; \texttt{y}; S2)$.

Proof of **G8**, **G9**, **G10**

Each of the three proof goals is then manipulated by *Universal Quantification* and *Modus Ponens* and given to the reduction engine of the system. However the proof (by reduction) of **G9** and **G10**, resp., require the following lemmas, resp.:

Lemma-5

$(\forall M, N, P : \texttt{Nat})$ $M < P$ implies $M + N < P + N$.

Lemma-6

$(\forall S, S1, S2 : \texttt{Strg})$ $S1$<>$S2$ implies $S; S1$<>$S; S2$.

When $s1$<>$s2$ is used as condition through goal transformation by *Modus Ponens* it is given in its explicit format:

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eq E:Elt » s1 = E:Elt » s2 .
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On the different nature of `Lemma-5` and `Lemma-6`

`Lemma-5` is ‘easy’, not directly related to our problem, and moreover ‘pops-up’ in the reduction.
Without it the system gives the result:

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((disorder(s1) + (e » s2)) <
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Formal proof of `Lemma-2`:

$$(\forall E, E', S)(E; E'; S) <> (E'; E; S)$$

We have to do it in its explicit form: which means

$$(\forall E1, E, E', S)\ E1 \gg (E; E'; S) = E1 \gg (E'; E; S)$$

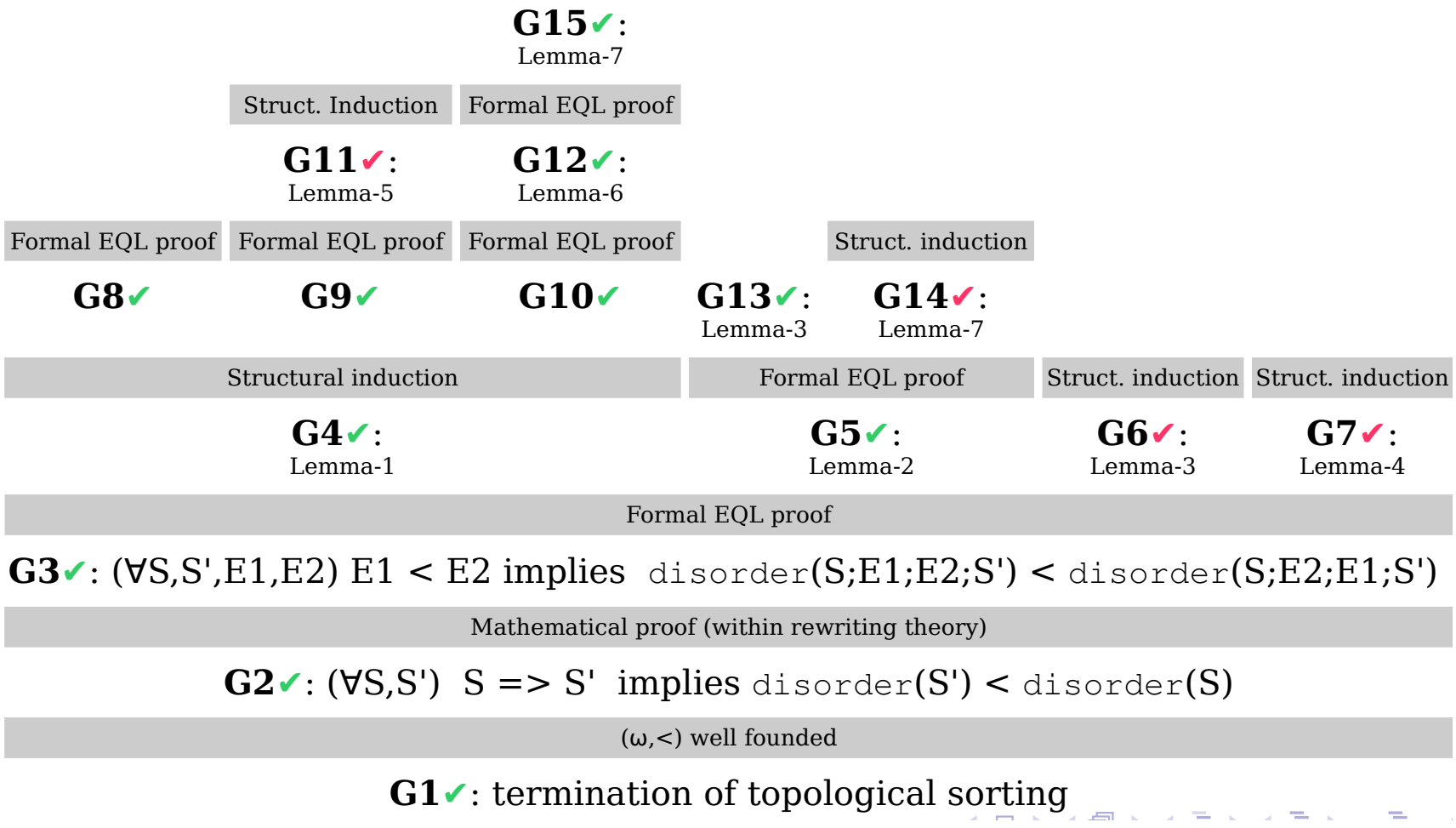
The proof by goal transformation through *Universal Quantification* and *Modus Ponens* requires two lemmas, `Lemma-3` (re-used) and

`Lemma-7`

$(\forall M : \texttt{Nat})\ M = M.$

Both are 'easy' lemmas, being discovered easily from the reduction process.

The structure of the verification process and of goal generation



Final conclusions

1. It is possible to have ‘stainless’ formal specification and verification
   - that is based rigorously and transparently on solid and clean mathematical foundations, and
   - that is also based on the realization of the inter-dependecies between mathematical foundations, semantics, specification and proof score programming methodologies, with semantics playing the central role.
2. This gives clarity and simplicity to the specification/verification process, and consequently *pragmatic* and a deep sense of trust.
3. However this may require some commitement to intelectual and scientific quality.