Lesson 9. The Art Gallery Problem

I628E – Information Processing Theory

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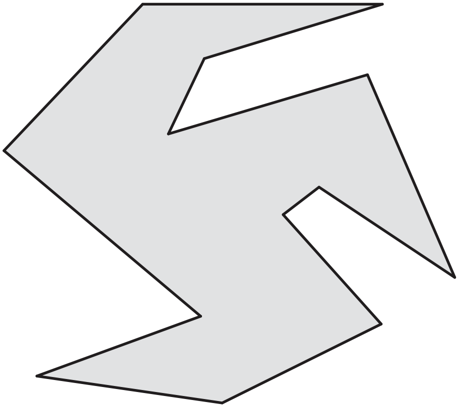
JAIST – January 20, 2020

Art Gallery Problem

Klee, 1973: Given a polygon, choose a minimum number of points (called "guards") that collectively see its whole interior.



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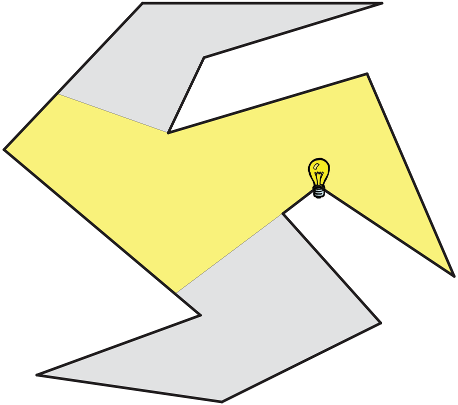

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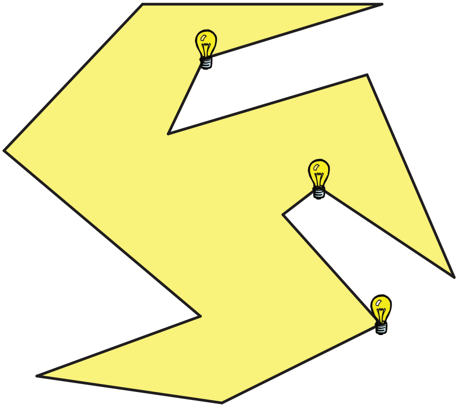
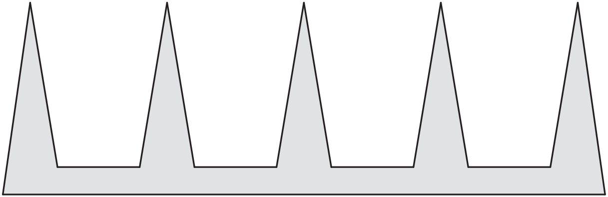


Chvátal's lower bound

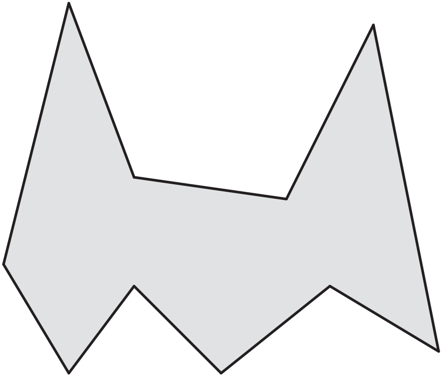
Chvátal, 1975: For every n, there are polygons with n vertices where $\left\lfloor \frac{n}{3} \right\rfloor$ guards are necessary:



No point in this polygon can see the tip of more than one "spike". Hence we need at least one guard per spike.

Fisk's solution

Fisk, 1978: For every polygon with n vertices, $\left\lfloor \frac{n}{3} \right\rfloor$ vertex guards are sufficient.



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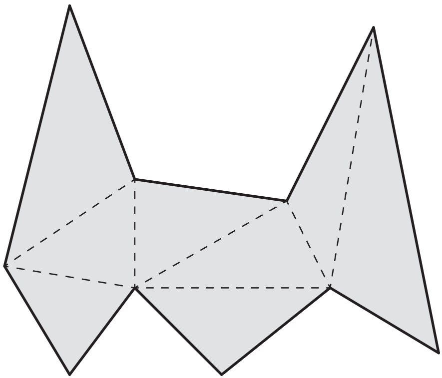

Step 1: triangulate the polygon.

Fisk's solution

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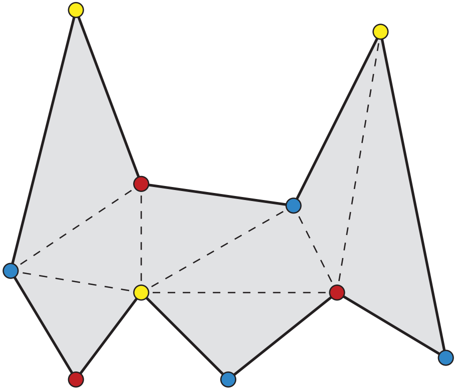

Step 2: since the *dual graph* of the triangulation is a tree, we can inductively 3-color the vertices.

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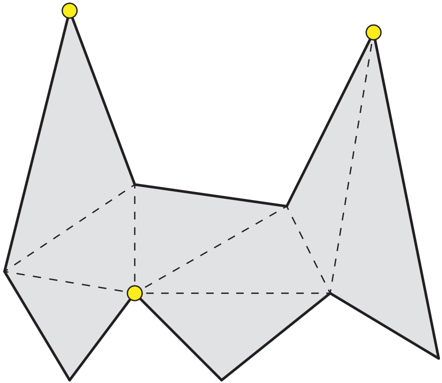


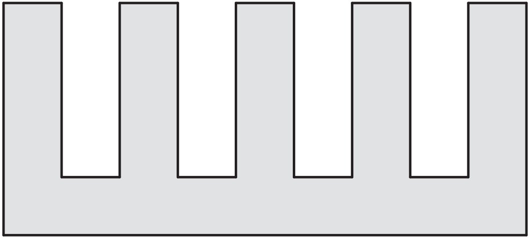
Step 3: choose the vertices of the less frequent color as guards.

Guarding orthogonal polygons

What if the polygon is *orthogonal* (i.e., its edges meet at right angles)? Can we guard it with fewer guards?

Chvátal's lower bound of $\left\lfloor \frac{n}{3} \right\rfloor$ guards no longer holds...

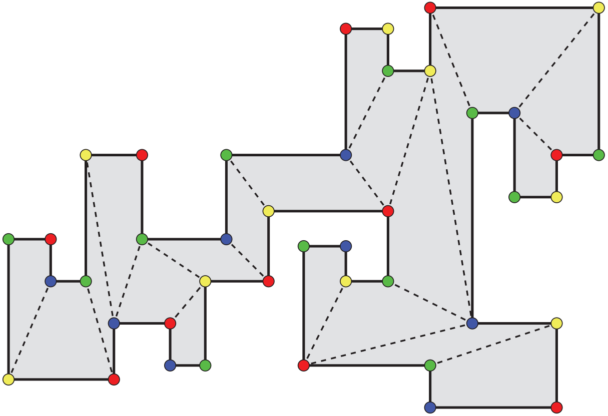
But the following example shows that there are orthogonal polygons with n vertices where $\left\lfloor \frac{n}{4} \right\rfloor$ guards are *necessary*:



Are $\left\lfloor \frac{n}{4} \right\rfloor$ guards *sufficient* for every orthogonal polygon?

Guarding orthogonal polygons

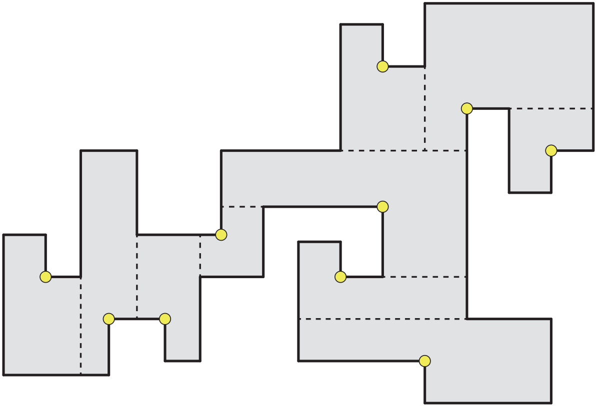
Kahn et al., 1983: For every orthogonal polygon with n vertices, $\left\lfloor \frac{n}{4} \right\rfloor$ vertex guards are sufficient.



Indeed, any orthogonal polygon can be decomposed into *convex quadrilaterals*, which induce a 4-coloring of the vertices.

Guarding orthogonal polygons

We give an alternative proof, essentially due to O'Rourke:



Any orthogonal polygon with r *reflex vertices* can be decomposed into $\left\lfloor \frac{r}{2} \right\rfloor + 1$ *L-shaped pieces*, each of which requires only 1 guard.

Are the two statements equivalent? Is $\left\lfloor \frac{r}{2} \right\rfloor + 1$ the same as $\left\lfloor \frac{n}{4} \right\rfloor$?

Observation: In any orthogonal polygon, $n = 2r + 4$.

Indeed, recall that the sum of the internal angles of a polygon with n vertices is $\pi(n-2)$.

- Each of the r reflex vertices gives a contribution of $3\pi/2$,
- Each of the $n-r$ convex vertices gives a contribution of $\pi/2$.

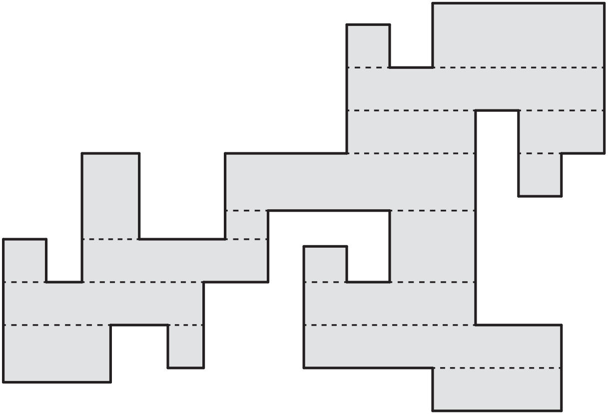
So, $\pi(n-2) \;=\; r \cdot 3\pi/2 + (n-r)\cdot \pi/2$.

Solving for n, we get $n = 2r + 4$.

Now, $\left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{2r+4}{4} \right\rfloor = \left\lfloor \frac{r}{2} \right\rfloor + 1$.

Guarding orthogonal polygons

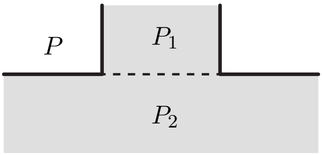
Here goes O'Rourke's proof:



We draw horizontal *cutlines* at reflex vertices, subdividing the polygon into rectangles. We reason by induction on r...

Guarding orthogonal polygons

If two reflex vertices are endpoints of the same cutline, we can cut the polygon and work on the two resulting subpolygons separately:

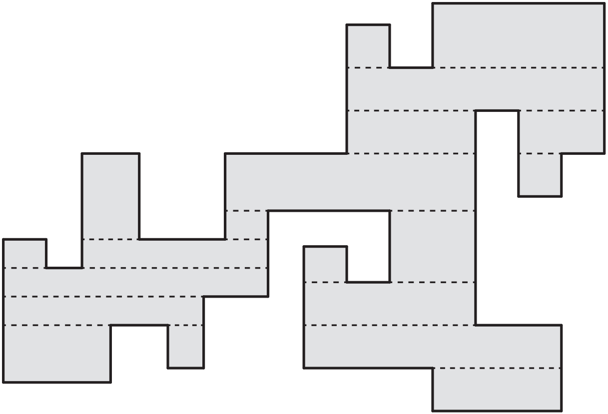


Indeed, if P, P_1, P_2 have r, r_1, r_2 reflex vertices respectively, then $r = r_1 + r_2 + 2$ (two reflex vertices are resolved by the cutline).

By induction, P_1 and P_2 can be guarded by $\left\lfloor \frac{r_1}{2} \right\rfloor + 1$ and $\left\lfloor \frac{r_2}{2} \right\rfloor + 1$ guards, respectively. In total, this is $\left\lfloor \frac{r_1+r_2+2}{2} \right\rfloor + 1 = \left\lfloor \frac{r}{2} \right\rfloor + 1$.

So, P can be guarded by $\left\lfloor \frac{r}{2} \right\rfloor + 1$ guards.

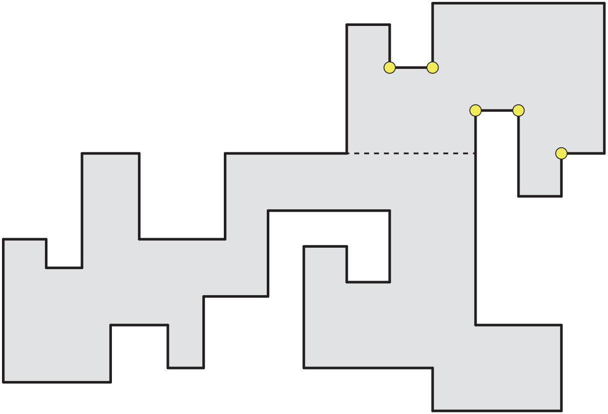
Therefore, we may assume that each cutline contains only one reflex vertex:



Note that each rectangle has at most 2 neighboring rectangles above and at most 2 below. (Why?)

Guarding orthogonal polygons

A cutline is *odd* if it splits the polygon in two parts, one of which contains an odd number of reflex vertices:



Odd cutlines turn out to be very desirable!

Guarding orthogonal polygons

Why are odd cutlines desirable?

Because, if we cut a polygon P along an odd cutline, we can work on the two resulting subpolyongs P_1 and P_2 separately.

Indeed, if P, P_1, P_2 have r, $r_1 = 2k+1$, r_2 reflex vertices, then $r = 2k + r_2 + 2$ (one reflex vertex is resolved by the cutline).

By induction, P_1 and P_2 can be guarded by $\left\lfloor \frac{r_1}{2} \right\rfloor + 1 = k + 1$ and $\left\lfloor \frac{r_2}{2} \right\rfloor + 1$ guards. In total, this is $\left\lfloor \frac{2k+r_2+2}{2} \right\rfloor + 1 = \left\lfloor \frac{r}{2} \right\rfloor + 1$.

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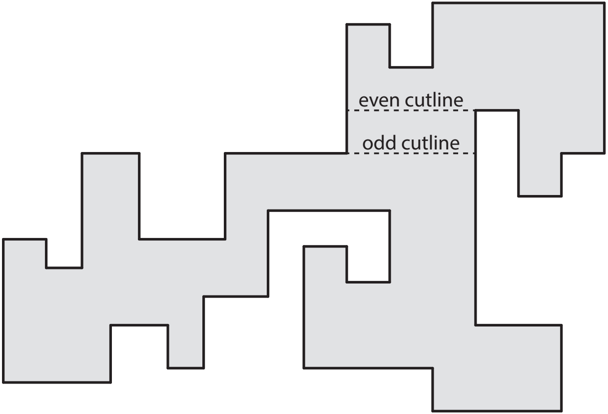
So, P can be guarded by $\left\lfloor \frac{r}{2} \right\rfloor + 1$ guards.

Observation: If r is even, then every cutline of P is odd.

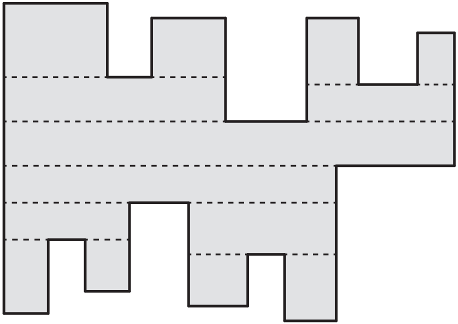
Indeed, a cutline resolves one reflex vertex, so $r_1 + r_2$ must be odd, implying that either r_1 or r_2 is odd.

Guarding orthogonal polygons

Observation: If a rectangle has exactly one neighboring rectangle above and exactly one below, then there is an odd cutline.



The only case left to consider is the one where there is an odd total number of reflex vertices, and no rectangle has exactly one neighboring rectangle above and exactly one below:



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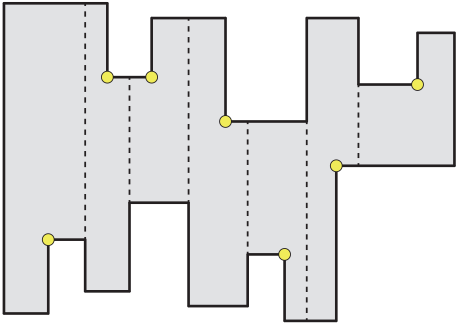


To guard this type of polygon, we sort the reflex vertices horizontally, and we place guards on the odd ones.

NP-hardness

So far we have given optimal bounds on the number of guards that hold for every (orthogonal) polygon. But certainly, for most polygons, much fewer than $\left\lfloor \frac{n}{3} \right\rfloor$ guards are enough!

What if we wanted to find an optimal positioning of guards for every given polygon?

Lee and Lin, 1986: The problem of computing the minimum number of guards for any given polygon is NP-hard.

The original proof reduces from 3-SAT. We will give a reduction from Vertex Cover, essentially due to Katz and Roisman, which has the advantage of producing orthogonal polygons.

Art Gallery Problem (decision version):

Input: An orthogonal polygon P and an integer k.

Output: YES if P can be guarded by at most k guards placed on its boundary. NO otherwise.

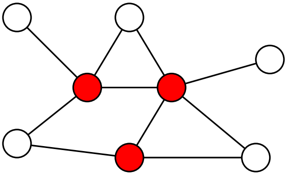
NP-hardness: Vertex Cover

The reduction is from the NP-complete problem Vertex Cover:

Vertex Cover (decision version):

Input: a graph $G = (V, E)$ and an integer k.

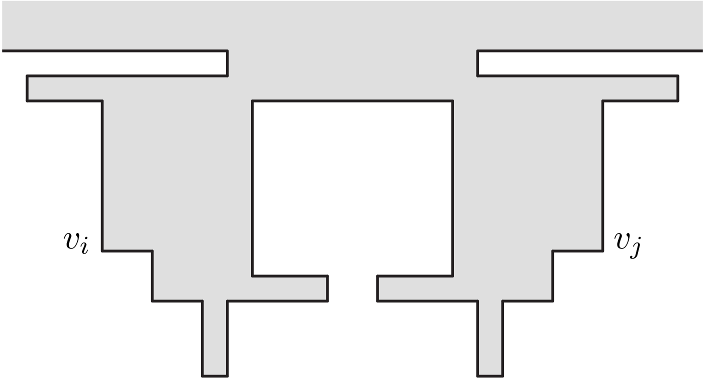
Output: YES if there is a subset $U \subseteq V$ of exactly k vertices such that each edge in E has at least one endpoint in U. NO otherwise.



$k = 3$

NP-hardness: edge gadget

We represent an edge (v_i, v_j) by this gadget:



At least 3 guards need to be placed within the gadget.
But placing 3 guards leaves a small triangle uncovered, which requires an "external contribution" from a 4th guard.

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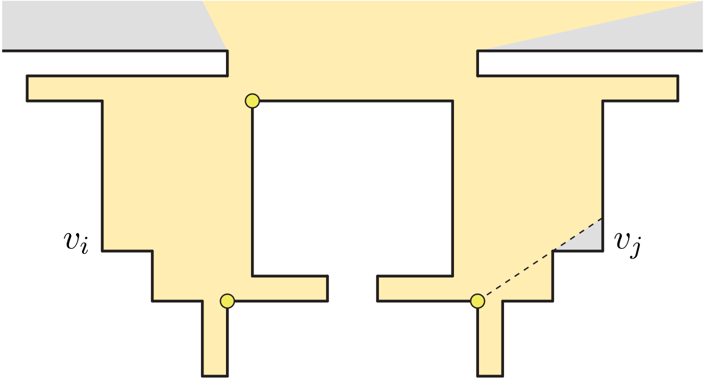

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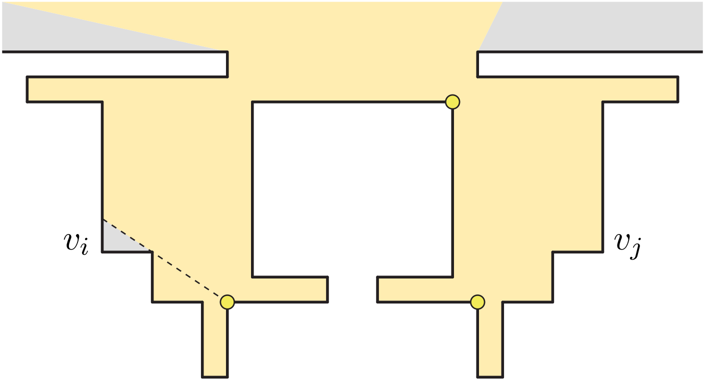

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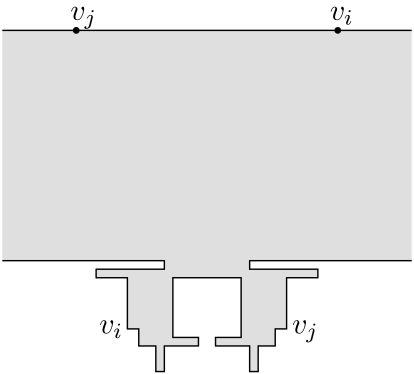
NP-hardness: representing vertices

This “contribution” must come from an external guard placed at a specific location, which represents an endpoint of the edge.



To minimize guards, we would like the same external guard to contribute to many edge gadgets...

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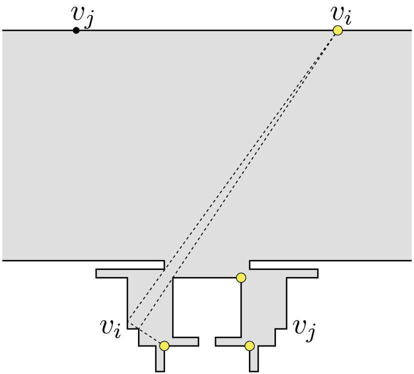

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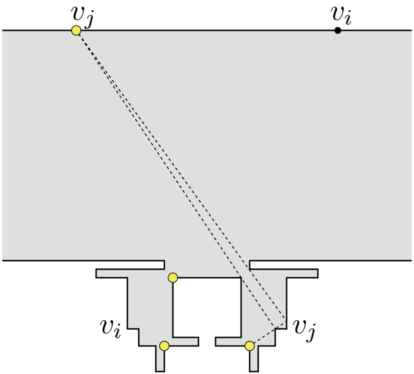
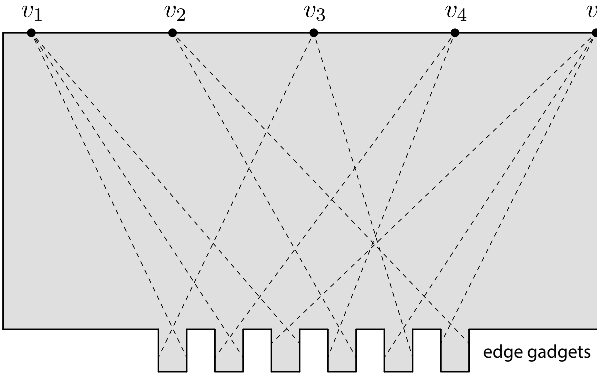


To minimize guards, we would like the same external guard to contribute to many edge gadgets...

NP-hardness: full construction

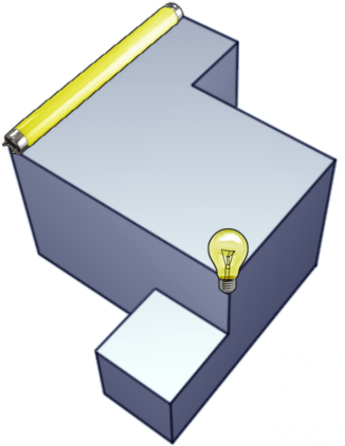
We tweak each edge gadget so that it can receive contributions only from the two locations corresponding to its endpoints in G.



This polygon can be guarded by $3|E| + k$ guards
if and only if the edges of G can be covered by k vertices.

3-dimensional art galleries

What if our art gallery is not a 2D polygon but a 3D polyhedron?
Do the previous results generalize to 3D shapes?



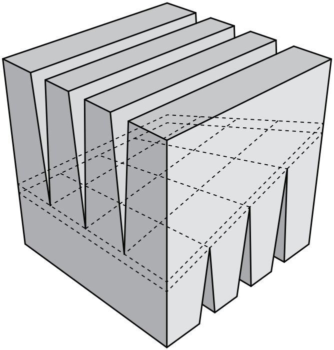
In this setting, we consider both *vertex guards* and *edge guards*.

Convex partitions of polyhedra

Most 2D techniques involve decomposing polygons into a linear number of convex parts, and then placing a guard in each part.

Unfortunately, no such partitions are possible in 3D.

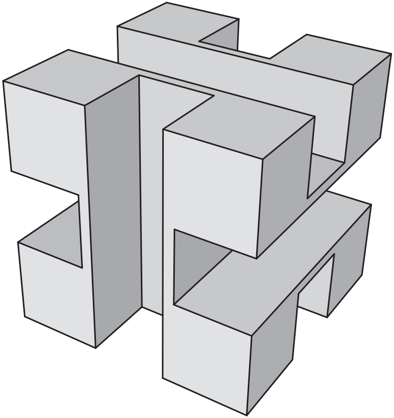
Chazelle, 1984: There are polyhedra with n vertices that can only be partitioned into $\Omega(n^2)$ convex parts:



Although in this polyhedron all edges are straight line segments, the central warped region has the shape of a hyperbolic paraboloid. A convex set can only have a small part of its volume inside this region.

Vertex-guarding orthogonal polyhedra

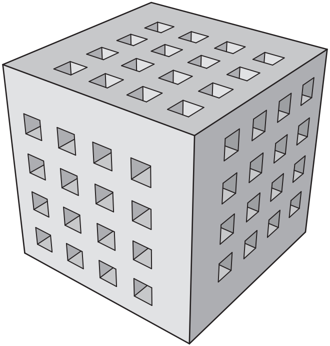
The Art Gallery Problem for *vertex guards* may be unsolvable, even in some orthogonal polyhedra:



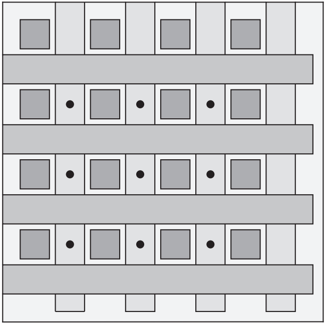
Some points in the central region are invisible to all vertices!
(Hence this polyhedron is not even decomposable into *tetrahedra*.)

Point-guarding orthogonal polyhedra

So, we must consider *point guards* that do not lie on vertices.
But there are orthogonal polyhedra that require $\Omega(n\sqrt{n})$ guards!



outer view



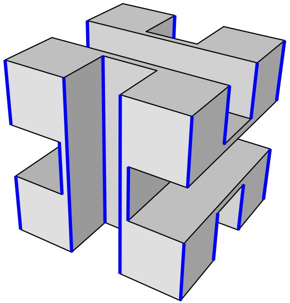
cross section

Paterson and Yao, 1992: Every orthogonal polyhedron yields a BSP tree of size $O(n\sqrt{n})$, hence this many guards suffice.

Edge-guarding polyhedra

Edge guards are more effective than vertex guards:
placing an edge guard on every (reflex) edge is sufficient. (Why?)

Can we do better? What about orthogonal polyhedra?



Placing edge guards only on the edges oriented in one of the 3 directions is sufficient (indeed, the cross sections orthogonal to the chosen direction are collections of polygons whose vertices hold guards).

Choosing the direction with fewest edges yields an upper bound of $\left\lfloor \frac{m}{3} \right\rfloor$ edge guards, where m is the total number of edges.

Can we refine this strategy?

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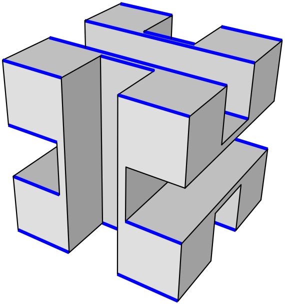

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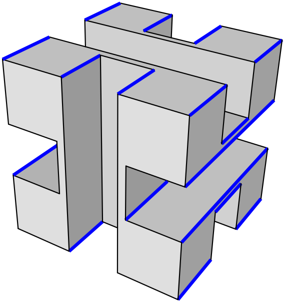

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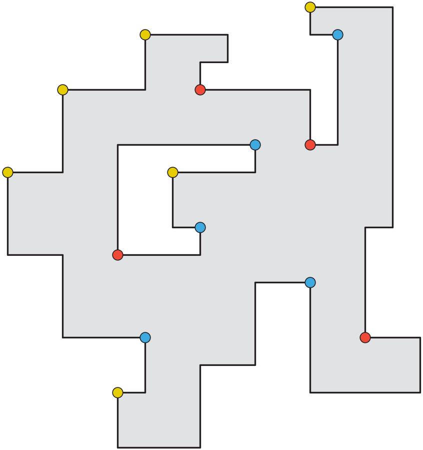
Can we refine this strategy?

Guarding with parallel edges

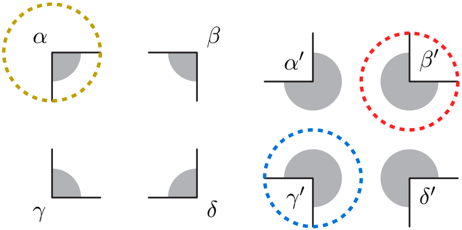
V., 2011: In the cross sections, instead of placing guards on all vertices, it is sufficient to choose vertices of only 3 “types”:



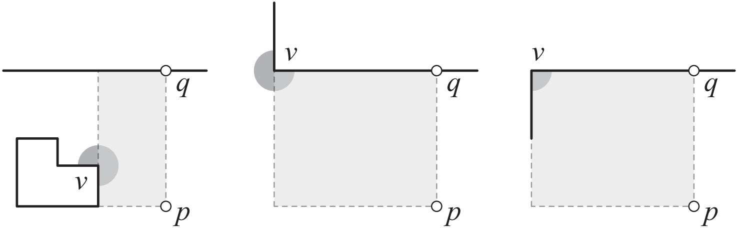
This selection of guards remains consistent through cross sections.

Guarding with parallel edges

There are 8 types of vertices in total, and we picked only 3 of them:



This selection works because, for every point p, we can shoot a ray upward and then shift it to the left until we hit a vertex v: this vertex is of one of the 3 chosen types, and therefore guards p:



Guarding with parallel edges

Let m_x be the total number of edges oriented in direction x. Each of these edges is of one of the 8 types, so we have

$$\alpha + \beta + \gamma + \delta + \alpha' + \beta' + \gamma' + \delta' \quad = \quad m_x.$$

There are 4 symmetric ways of choosing 3 edge types to obtain a valid guard placement:

$$\alpha + \beta' + \gamma',$$
$$\delta + \beta' + \gamma',$$
$$\beta + \alpha' + \delta',$$
$$\gamma + \alpha' + \delta'.$$

The sum is

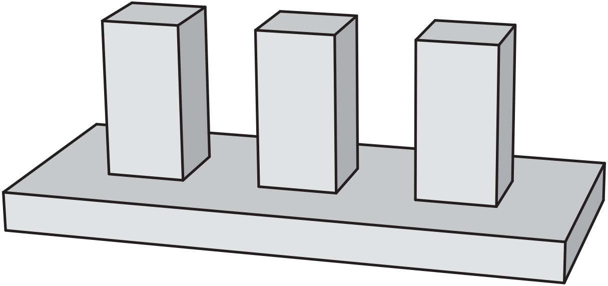
$$\alpha + \beta + \gamma + \delta + 2\alpha' + 2\beta' + 2\gamma' + 2\delta' \quad \leq \quad 2m_x.$$

Hence, one of the 4 choices yields at most $\left\lfloor \frac{m_x}{2} \right\rfloor$ edges.

By selecting the direction x that minimizes m_x, we place at most $\left\lfloor \frac{m}{6} \right\rfloor$ parallel edge guards, where m is the total number of edges.

Edge guards: lower bound for orthogonal polyhedra

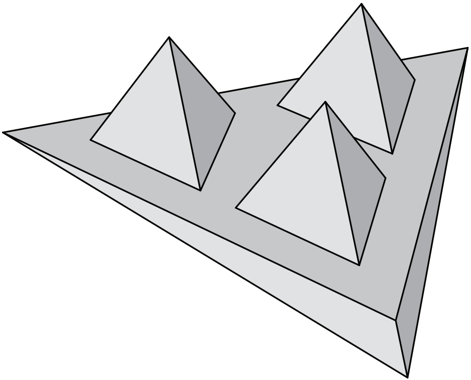
So, $\left\lfloor \frac{m}{6} \right\rfloor$ edge guards are always sufficient, and there are orthogonal polyhedra where $\left\lfloor \frac{m}{12} \right\rfloor$ edge guards are necessary:



Open problem: Are $\left\lfloor \frac{m}{12} \right\rfloor$ edge guards always sufficient?

Edge-guarding general polyhedra

For *general polyhedra*, there are examples where $\left\lfloor \frac{m}{6} \right\rfloor$ edge guards are necessary:

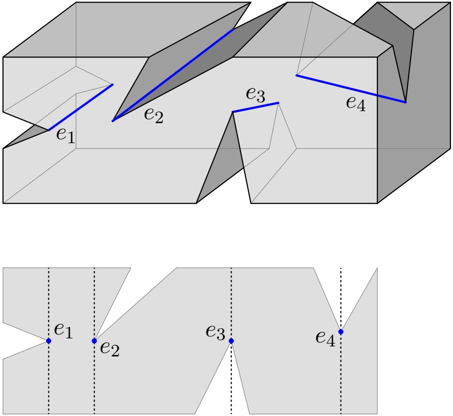


Is there a non-trivial upper bound?

Urrutia et al., 2012: For every polhedron with m edges, $\left\lfloor \frac{27}{32} m \right\rfloor$ edge guards are sufficient.

Edge-guarding general polyhedra

Each edge e may be of one of 4 classes based on the position of its two incident faces with respect to the vertical plane α through e:



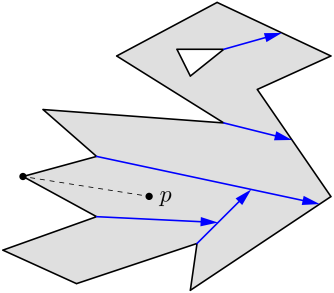
1. Both incident faces are on the left side of α.
2. Both incident faces are on the right side of α.
3. The incident faces are on opposite sides of α, and the polyhedron lies above e.
4. The incident faces are on opposite sides of α, and the polyhedron lies below e.

Edge-guarding general polyhedra

Lemma: If a point p in the polyhedron does not see any vertex, then p sees edges of at least 2 different classes.

Proof: By contradiction.

Case 1: p sees only edges of class 1.

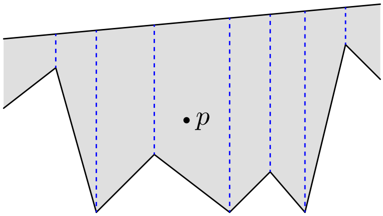


Consider a vertical cross section through p, and partition it into subpolygons by shooting rays along the angle bisectors of the reflex vertices corresponding to edges of class 1.

p lies in a convex subpolygon (why?), hence it sees its leftmost vertex, which must correspond to an edge of class 2. *Contradiction!*

Edge-guarding general polyhedra

Case 2: p sees only edges of class 3 and no vertices.



Partition the polyhedron into *cells* by drawing a vertical "wall" through each edge. Note that each cell has exactly two non-vertical faces: "floor" and "ceiling".

If the cell of p is convex, its floor must be a face of the polyhedron, and so p sees all its vertices. If the cell of p is non-convex, then p sees one of its vertical reflex edges, whose bottom vertex must be a vertex of the polyhedron. *Contradiction!*

Edge-guarding general polyhedra

Strategy: Pick an edge set that covers all vertices and, among the remaining edges, pick the ones in the 3 smallest classes.

Lemma: The vertex set is covered by $\left\lfloor \frac{3}{8}m \right\rfloor$ edges.
Proof: By classical matching theory (details omitted).

So, we pick $\left\lfloor \frac{3}{8}m \right\rfloor$ edges to cover all vertices,

plus at most $\frac{3}{4}$ of the remaining edges.

In total, we picked $\left\lfloor \frac{3}{8}m \right\rfloor + \left\lfloor \frac{3}{4}\left(m - \left\lfloor \frac{3}{8}m \right\rfloor\right) \right\rfloor \leq \left\lfloor \frac{27}{32}m \right\rfloor$ edges.

Open problem: Are $\left\lfloor \frac{m}{6} \right\rfloor$ edge guards always sufficient?

Summary

We established some general bounds on the number of guards for several 2D and 3D Art Gallery Problems:

General 2D polygons:	$\left\lfloor \frac{n}{3} \right\rfloor$ vertex guards
Orthogonal 2D polygons:	$\left\lfloor \frac{n}{4} \right\rfloor$ vertex guards
Orthogonal 3D polyhedra:	$\Theta(n\sqrt{n})$ point guards
	$\left\lfloor \frac{m}{12} \right\rfloor \sim \left\lfloor \frac{m}{6} \right\rfloor$ edge guards
General 3D polyhedra:	$\left\lfloor \frac{m}{6} \right\rfloor \sim \left\lfloor \frac{27}{32}m \right\rfloor$ edge guards

We also showed that computing the minimum number of guards is NP-hard, even for orthogonal polygons.

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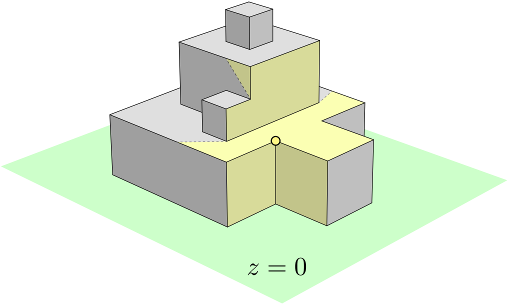
Assignment 1

Let the *ground* be the horizontal plane $z = 0$ in $\mathbb{R}^3$.
A *tower* is defined as an orthogonal polyhedron T whose vertices lie on the ground ($z = 0$) or above it ($z > 0$), such that the intersection between T and any vertical line is either empty or a segment with one endpoint on the ground.



We want to guard the external surface of a tower (above ground, i.e., excluding its "base") by placing point guards at some of its vertices. Give a good asymptotic *lower bound* on the number of vertex guards required for a tower with n vertices.