A Marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic

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Abstract

We investigate which part of Brouwer's Intuitionistic Mathematics is finitistically justifiable or guaranteed in Hilbert's Finitism, in the same way as similar investigations on Classical Mathematics (i.e., which part is equiconsistent with **PRA** or consistent provably in **PRA**) already done quite extensively in proof theory and reverse mathematics. While we already knew a contrast from the classical situation concerning the continuity principle, more contrasts turn out: we show that several principles are finitistically justifiable or guaranteed which are classically not. Among them are: (i) fan theorem for decidable fans but arbitrary bars; (ii) continuity principle and the axiom of choice both for arbitrary formulae; and (iii) Σ_2 induction and dependent choice. We also show that Markov's principle MP does not change this situation; that neither does lesser limited principle of omniscience LLPO (except the choice along functions); but that limited principle of omniscience LPO makes the situation completely classical.

1 Introduction

1.1 Brouwer's Intuitionism and Hilbert's Finitism

Brouwer's Intuitionism is considered as the precursor of many varieties of constructivism and finitism which reject the law of excluded middle (LEM) for statements concerning infinite objects. It is said that even Hilbert, a most severe opponent of Brouwer's, adopted a part of Brouwer's idea in his proposal for meta-mathematics or proof theory, and this partial adoption is now called *Hilbert's Finitism*. However, there are several essential differences between these two varieties of constructivism or finitism.

First, they are different in their original aims. Brouwer's Intuitionism is a claim how mathematics in its entirety should be, and the mathematics practiced according to this is called *Intuitionistic Mathematics* (INT). On the other hand, Hilbert's Finitism was intended to apply only to a particular part of mathematics, called *proof theory* or *meta-mathematics*. The aim of this part *was* "saving" the entirety of mathematics from the fear of inconsistency. The "entirety of mathematics" in Hilbert's idea is far beyond finitism and now called *Classical Mathematics* (CLASS) in the context of comparison among kinds of mathematics.

This difference might explain why Hilbert's Finitism is stricter than Brouwer's Intuitionism: for example, the induction schema on numbers is granted for free in the latter whereas it is allowed only if restricted to finitely checkable statements in the former. The acceptance of the schema for properties not finitely checkable (even though LEM for such properties is not allowed) seems to be reason enough not to call Brouwer's Intuitionism a finitism, and moreover it requires transcendental assumptions which basically assert that everything is to be constructed (cf. the notion of *choice sequence*) contradicting CLASS.

It is worth mentioning Bishop's constructivism, a third variety of constructivism. *Bishop-style constructive mathematics* (BISH) is considered to be completely constructive, in the sense that it does not assume any transcendental assertion. Thus all the theorems of BISH, as formal sentences, are contained in those of CLASS and INT. Nonetheless, it does not seem plausible to call it a finitism either, for it also accepts the induction schema applied to properties not finitely checkable. It accepts the axiom of choice applied to such properties as well, which is also beyond the finitistically justifiable part of CLASS.

Another contrast between Brouwer and Hilbert is in their attitudes towards formalization: while Brouwer did not formalize INT, Hilbert tried to formalize CLASS and since then his Finitism (now identified with what is formalizable in *Primitive Recursive Arithmetic* **PRA**; see [39]) has been established as the meta-theory of handling formalization, or, in which proof theory is practiced. This contrast has, however, been gradually losing significance: followers of Brouwer formalized INT, and now our interest is in how different it is, as a formal theory,[1] from CLASS and from BISH as well as from *Russian Recursive Mathematics* RUSS. The last requires a different transcendental assumption asserting that everything is computable.

Unfortunately a difference is also in popularity: CLASS has been investigated extensively, e.g., identifying the finitistically secured part, while there seems to have been no similar systematic investigation for INT.

Given these contrasts, the aim of the present series of papers, the identification of the part of INT and addable axioms that Hilbert would recognize as secured, has multi-fold motivations. To repeat: from the viewpoint that Brouwer's Intuitionism is the precursor of various kinds of constructivism and finitism; from the historical perspective that Brouwer and Hilbert were severe opponents of each other; and from the necessity of the identification as has been done for CLASS in order to develop INT in parallel to CLASS.

1.2 Reducibility and interpretability

By what criterion would Hilbert recognize a fragment of mathematics as secured according to his Finitism? We may distinguish two criteria: a fragment is said to be (i) *finitistically guaranteed* if it is consistent provably in **PRA**; and (ii) *finitistically justifiable* if it is consistent relative to **PRA** provably in **PRA**. It is likely that these were not distinguished in Hilbert's original intention prior to Gödel's incompleteness theorem.

Proof theory, to which Hilbert's Finitism was originally intended to apply, has refined (ii) above (see [30, Subsec. 2.5]): a formal theory T_1 is *(proof theoretically) reducible to* another T_2 over a class $\mathcal{C}$ of sentences if there is a primitive recursive function f such that provably within **PRA**, for any sentence A from $\mathcal{C}$, if x is a proof of A in T_1 then $f(x)$ is a proof of A in T_2. Usually $\mathcal{C}$ contains the absurdum $\perp$ and so this notion yields the comparison of *externally defined consistency strengths* (namely, the consistency of T_2 implies that of T_1 or consistency-wise implication) provably in **PRA**. In many interesting cases, the theories essentially contain a fragment of arithmetic and we can assume $\mathcal{C}$ includes Π^0_1 or Π^0_2 sentences. As the Gödel sentence (of a reasonable theory) is Π^0_1, it also yields the comparison of *internally defined consistency strength*: any reasonable formal theory consistent provably in T_1 is consistent provably also in T_2. Now $\mathbf{I\Sigma}_1$, $\mathbf{RCA}_0$ and $\mathbf{WKL}_0$ are parts of Classical Mathematics that are known to be proof theoretically reducible to **PRA**. As a subtheory is trivially reducible to a supertheory, these four theories are *proof theoretically equivalent*.

For our purposes, however, we can use a stronger notion, interpretability. We will prove reducibility results by giving concrete interpretations, among which are Gödel-Gentzen negative interpretation and realizability interpretation. Our notion of interpretability is slightly broader than that in some literature, in the sense that logical connectives can be interpreted non-trivially (as in the aforementioned examples).[2] An interpretation I is called $\mathcal{C}$*-preserving*, if any $\mathcal{C}$ sentence A is implied by its interpretation A^I in the interpreting theory T_2. All interpretations in the present paper are Π^0_1-preserving, and so imply reducibility with $\mathcal{C}=\Pi^0_1$. Whereas reducibility concerns only proofs ending with sentences in $\mathcal{C}$, interpretability means that all mathematical practice formalized in one theory can be simulated in another. As each step of proofs in T_1 is transformed into a uniformly bounded number of steps in T_2, the induced transformation f of proofs belongs to even lower complexity, and so the consistency-wise implication is proved in meta-theories weaker than **PRA**.

The difference between reducibility and interpretability becomes essential when we talk about the relations between finitistically guaranteed theories (hence weaker than **PRA**): while the reducibility is proved typically by cut elimination, which requires commitment to superexponential functions, such a commitment yields the consistency of $\mathbf{B\Sigma_1ex}$, $\mathbf{RCA}^*_0$ and $\mathbf{WKL}^*_0$, typical finitistically guaranteed theories, and so collapses the hierarchy of the externally defined consistency strengths of such weaker theories.

[1] "Intuitionism as an opponent of formalism" is also a quite interesting topic, which has not yet been investigated enough so far. For instance, in the authors' opinion, Brouwer's original proof of bar induction should be analyzed from this viewpoint.

[2] We *could* give a tentative definition: a map I from $\mathcal{L}_1$ to $\mathcal{L}_2$ is an *interpretation* of an $\mathcal{L}_1$-theory T_1 in an $\mathcal{L}_2$-theory T_2 iff (a) $T_2 \vdash \perp^I \to \perp$, (b) $T_2 \vdash A^I$ for any axiom A of T_1, and (c) there is a polynomial-time computable function p such that if C is from A, B by a *single* logical rule then $p(A,B,C)$ is a *derivation* of C^I from A^I, B^I in T_2. However, we will not need such a definition but only basic properties that the word suggests (and which follow from the definition above): (i) a composition of interpretations is an interpretation; (ii) all those we will define with name "interpretation" in this paper are interpretations and (iii) the existence of a $\mathcal{C}$-preserving interpretation implies both $\mathcal{C}$ conservation provable in $\mathbf{B\Sigma_1ex}$ and the reducibility over $\mathcal{C}$.

1.3 Characteristic axioms of Intuitionistic Mathematics

Up to the present, there seems to be a consensus on what characterizes INT. An informal explanation of such characterizing axioms is as follows, where the terminology might differ from Brouwer's original.

Intuitionistic logic neither the law of excluded middle ($A \vee \neg A$) nor double negation elimination ($\neg\neg A \to A$) is accepted unless A is finitely checkable (while the explosion axiom $\perp \to A$ is accepted);

Basic arithmetic basic properties which are finitely checkable and which govern the natural numbers and fundamental operations, are accepted;

Induction on natural numbers the induction schema on ω for all the legitimate properties[3] (not necessarily finitely checkable) is accepted;

Bar induction transfinite induction along the well-founded tree (coded by a *bar*, which intersects any infinite sequence) of finite sequences of numbers, with various restriction[4], is accepted;

Fan theorem classically equivalent to a form of König's lemma or **weak fan theorem**, restricted to binary trees but defined by any legitimate properties, is an important consequence of bar induction in many applications; either of them is taken as an axiom of INT instead of bar induction in some literature;

Axiom of choice for any legitimate property A of sorts i and j, if for any x of sort i there is y of j such that $A[x, y]$ holds then there exists a function f of sort $i \to j$ such that $A[x, f(x)]$ holds for any x of i;

Continuity principle a function on Baire space ω^ω defined by any legitimate property is locally continuous.

The last contradicts CLASS, and the others, except the first two and weak fan theorem, are classically beyond Finitism. Since *Heyting arithmetic*, consisting only of the first three, is mutually interpretable with Peano arithmetic, and hence already beyond Finitism, we need to restrict these axioms, as in CLASS.

The first half of the main purpose of the present series of papers is thus to clarify how large fragments of these axioms are *jointly* reducible to Hilbert's Finitism (i.e., finitistically justifiable) or jointly consistent provably in Finitism (i.e., finitistically guaranteed). This paper, the first in the series, addresses this question, in the language $\mathcal{L}_{\mathrm{F}}$ of function-based second order arithmetic (similar to that of **EL** from [40, Ch.3, 6.2]), where we need some twist to state the existence of choice functions on Baire space (see 2.5.5) or where we could say that the axiom of choice for such sorts is illegitimate at all (see f.n.11).

The expositions of axioms here are informal or pre-formal, and it is quite delicate how to formalize them. We follow a standard way, but some discussions are unavoidable and will be addressed in Section 2.

We define fragments of the axioms basically by requiring the relevant properties to be in classes of formulae, e.g., Σ^0_n's and Π^0_n's (which however do not exhaust all arithmetical formulae because of the lack of prenex normal form theorem), and by controlling the sorts in the axiom of choice.

1.4 Finitistically justifiable and guaranteed parts of Intuitionistic Mathematics

We will see that the following with $\mathbf{EL}_0^-$ (i.e., the logic and basic arithmetic) are jointly reducible to **PRA**:

- induction on natural numbers restricted to Σ^0_2 properties (Σ^0_2-Ind);
- bar induction restricted to Π^0_1 properties (Π^0_1-BI, see the exact formulation in Def.2.25);
- fan theorem for fans (decidable by definition) and bars defined by any legitimate properties ($\mathcal{L}_{\mathrm{F}}$-FT);
- axiom of choice for all legitimate properties and dependent choice of numbers for Σ^0_2 ones (Σ^0_2-DC^0);
- continuity principle for functions defined by any legitimate properties ($\mathcal{L}_{\mathrm{F}}$-$\mathsf{WC!}^0$ and $\mathcal{L}_{\mathrm{F}}$-$\mathsf{WC!}^1$),

and that, with the following further restrictions, jointly consistent provably in **PRA**: induction on numbers to decidable properties; dependent choice and bar induction omitted; fans to be complete binary ($\mathcal{L}_{\mathrm{F}}$-WFT).

Besides the well known contrast with the classical situation concerning the continuity principle, we see further contrasts, as any of the following is, classically, beyond **PRA**: Σ^0_2-Ind; fan theorem restricted either to decidable bars Δ^0_0-FT or to complete binary fans and Π^0_1 bars Π^0_1-WFT; and Π^0_1 axiom of choice.

[3] It is debatable whether the properties involving third or higher-order quantifiers are legitimate in Brouwer's Intuitionism. If not, it is also a plausible not to call them properties. However, to emphasize the limitation on what we can consider, we call a property legitimate if we can consider it. This terminology is parallel to Feferman's (e.g. [15]) in the context of predicativity.

[4] There is a debate on the right formulation of Brouwer's intension. See 2.5.1.

Our method is *Kleene's functional realizability*, known to be able to interpret most part of INT in CLASS. We examine which fragments of INT are interpreted by this in $\mathbf{WKL}_0$ or $\mathbf{WKL}_0^*$. As it is based on a Π^0_2-definable application "$|$" for functions, unlike the number realizability, naïve attempts of proof easily rely on Π^0_2 or higher induction. The proof is, in general, not straightforward from previously known one.

As a byproduct, we can add *Markov's principle* MP (i.e., Σ^0_1-DNE double negation elimination restricted to Σ_1 assertions) to the combinations above. MP is accepted from some constructive views and called *semi-constructive*. While it seems agreed not to accept MP in Intuitionism, it is not agreed to accept its negation.[5] We need no interpretations that exclude MP, as the interpretability of T+MP trivially implies that of T.

Moreover, we will see that these fragments are optimal: none of Π^0_2-Ind, Σ^0_1-BI_D (restricted to decidable bars), Π^0_2-$\mathsf{DC!}^0$ (with uniqueness in the premise) and Π^0_1-$\mathsf{DC!}^1$ (dependent choice of functions) can *only* with $\mathbf{EL}_0^-$ be reducible to $\mathbf{PRA}$; none of Π^0_1-Ind, Σ^0_1-Ind, Δ^0_0-BI_D, Δ^0_0-$\mathsf{DC!}^0$ and Δ^0_0-FT only with $\mathbf{EL}_0^-$ is consistent provably in $\mathbf{PRA}$. For the former, we interpret $\mathbf{I\Sigma}_2$, which proves the consistency of $\mathbf{PRA}$, by generalizing Coquand and Hofmann's method [11]. For the latter, we interpret $\mathbf{I\Sigma}_1$ which is equiconsistent with $\mathbf{PRA}$.

Note that, by Gödel's second incompleteness, if a theory T_1 proves the consistency $\mathrm{Con}(T_2)$ of another T_2, then T_1 is not reducible to (nor interpretable in) T_2, since otherwise T_1 proves its own consistency.

1.5 Effects of semi-constructive or semi-classical principles

Hilbert's Finitism did not intend to restrict the mathematics, but to maximize the set of acceptable axioms that are directly beyond Finitism but that are secured on his Finitistic ground through meta-mathematics. So we should continue to clarify which axioms beyond Intuitionism can be added to the secure parts of INT without losing finitistic guaranteedness or justifiability. The aforementioned byproduct on MP is a part of answer, and it is natural to try to answer more generally: which part of classical logic, or even of CLASS, is finitistically guaranteed or justifiable jointly with major parts[6] of INT? As many classically valid principles are known not to imply full classical logic, the other half of our purpose is to ask: how does the secured part change from the intuitionistic situation to classical one, along the hierarchy of such *semi-classical* principles?[7]

Among famous ones are *limited principle of omniscience* LPO (i.e., Σ^0_1-LEM the law of excluded middle for Σ^0_1) and *lesser limited principle of omniscience* LLPO (i.e., $\Pi^0_1{\vee}\Pi^0_1$-DNE double negation elimination for $\Pi^0_1{\vee}\Pi^0_1$). LLPO is implied by LPO and, as shown in [1], independent of MP. In the presence of full induction, LLPO is equivalent to $\mathsf{B}\Sigma^0_2$-DNE and to Σ^0_1-GDM, *generalized De Morgan's law* $\neg(\forall x<y)A \to (\exists x<y)\neg A$ for Σ^0_1 properties. With restricted induction, however, all the implications we know among these are as follows.

$$\begin{tikzcd}
\cdots \arrow[dr] & \Pi^0_2\vee\Pi^0_2\text{-}\mathsf{DNE} \arrow[r] & \Sigma^0_1\text{-}\mathsf{LEM} \equiv \mathsf{LPO} \arrow[rr] \arrow[drr] & & \Pi^0_1\vee\Pi^0_1\text{-}\mathsf{DNE} \equiv \mathsf{LLPO} \arrow[r] & \Sigma^0_0\text{-}\mathsf{LEM} \arrow[r, leftrightarrow] & 0=0 \arrow[d, leftrightarrow] \\
\cdots \arrow[ur] & \Sigma^0_2\text{-}\mathsf{DNE} \arrow[r] \arrow[ur] & \Sigma^0_1\text{-}\mathsf{GDM} \arrow[r] & \mathsf{B}\Sigma^0_2\text{-}\mathsf{DNE} \arrow[ur] & \Sigma^0_1\text{-}\mathsf{DNE} \equiv \mathsf{MP} \arrow[r] \arrow[ur] & \Sigma^0_0\text{-}\mathsf{GDM} \arrow[r, leftrightarrow] & 0=0
\end{tikzcd}$$

Unlike MP, by *weak counterexample argument*[8] we can presume that Brouwer would reject the idea of LLPO (and hence all principles above it). Thus the status of LLPO in Intuitionism is as that of WKL in Finitism, since WKL is definitely *directly* unacceptable in Finitism, and actually they are equivalent in the presence of axiom of choice (cf. 3.9). Because accepting WKL *indirectly* by consistency proof was the core of Simpson's "partial realizations of Hilbert's Program" from [36], LLPO should be of particular interest in our context.

[5] Brouwer's *creative subject*, a method controversial even among Intuitionists, or its formalization *Kripke's schema* yields the negation of MP. However we confine ourselves to "objective Intuitionism" in Beeson's [5] term, excluding such "subjectivities".

[6] Since the entirety of INT is not finitistically justifiable, we do not need to stick to the consistency with full INT.

[7] We can ask the same for RUSS, characterized by MP, $\mathcal{L}_{\mathrm{F}}$-AC^{0i} and CT $\forall\alpha\exists e\forall x(\alpha(x)=\{e\}(x))$, where $\{\text{-}\}$ is Kleene bracket. We knew the inconsistency of $\mathbf{EL}_0^-$+CT+Δ^0_0-WFT (by the famous counterexample; cf. [42, §3]) and of $\mathbf{EL}_0^-$+CT+Π^0_2-WC^0 (as $\forall x(\alpha(x)=\{e\}(x))$ is Π^0_2). As the so-called KLS Theorem needs only decidable induction (cf. [40, Ch.6, 4.12, 5.5]), the combination in 1.4 with (W)FT replaced by CT, i.e., $\mathbf{EL}_0^-$+CT+$\mathcal{L}_{\mathrm{F}}\{\text{-}\mathsf{AC}^{0i}, \text{-}\mathsf{WC!}^i\}$ (or $+\Sigma^0_2\{\text{-}\mathsf{Ind}, \text{-}\mathsf{DC}^0\}+\Sigma^0_1\text{-}\mathsf{DC}^1+\Pi^0_1\text{-}\mathsf{BI}$) is interpreted by Kleene's number realizability (extended to $\mathcal{L}_{\mathrm{F}}$ trivially) in $\mathbf{B\Sigma_1ex}$ (or in $\mathbf{I\Sigma}_1$, as our argument will collaterally show; see Prop.3.46 and below it) and so finitistically guaranteed (or justifiable). Thus only CT+Π^0_1-WC^i remains to be asked.

Instead of keeping CT, our proof will also show that the combination in 1.4 with (W)FT replaced by a *semi-Russian* axiom NCT $\forall\alpha\neg\forall e\neg\forall x(\alpha(x)=\{e\}(x))$ (and so the formula-version, by $\mathcal{L}_{\mathrm{F}}$-AC^{00}) is finitistically justifiable or guaranteed, as shown in Thm.5.6. NCT seems to imply that there is no *lawless choice sequence*, which had been rejected in early stages of Intuitionism.

[8] Let $\alpha(n)\neq 0$ iff the first successive m occurrence of 9's in the decimal expansion of Napier's constant e starts at the n-th digit; then $\neg(\exists n\neg(\alpha(2n)=0)\wedge\exists n\neg(\alpha(2n+1)=0))$ and LLPO implies $\forall n(\alpha(2n)=0)\vee\forall n(\alpha(2n+1)=0)$; i.e., either the first successive m occurrence of 9's, if exists, starts at an odd digit or if it exists it starts at an even digit; however it is open for large enough m which disjunct holds. Recall that in Intuitionism to claim a disjunction, we need to know which disjunct is true.

We show that adding Σ^0_1-GDM (and so LLPO), even jointly with MP, does not change the intuitionistic situation described in 1.4, except the axiom of function-number and function-function choice. Though these choices cannot be formalized in $\mathcal{L}_{\mathrm{F}}$, continuous choice (CC), whose Π^0_1 fragment contradicts LLPO, could be seen as conjunctions of them and continuity principle (cf. 2.5.5). Our main tool is van Oosten's *Lifschitz-style functional realizability* from [29], in the definition of which, a bounded Σ^0_2 property plays a central role. Thus the arguments on the finitistic ground is much more delicate than in van Oosten's original context.

On the other hand, we will see that LPO already makes the situation completely classical, that is, any of the following *separately*, but together with $\mathbf{EL}^-_0$+LPO, is already non-reducible to **PRA**:

- Σ^0_2 induction on numbers (Σ^0_2-Ind);
- fan theorem restricted to Δ^0_0 bars but without the binary constraint (Δ^0_0-FT);
- weak fan theorem restricted to (complete binary fans and) Π^0_1-bars (Π^0_1-WFT); and
- Π^0_1 axiom of choice even with the uniqueness assumption in the premise (Π^0_1-AC!^{00}).

For the second and fourth we will show the interpretability of $\mathbf{ACA}_0$ with Gödel-Gentzen negative interpretation. For the others, we need the combination with intuitionistic forcing to interpret $\mathbf{I\Sigma}_2$ or $\mathbf{ACA}_0$.

1.6 Constructive reverse mathematics on consistency strength

Our study also contributes to the research field, called *constructive reverse mathematics* (cf. e.g., [18, 20]). There implications, on a constructive ground, between (fragments of) axioms from CLASS, INT and RUSS, are investigated and, for the unprovability of these implications, questions of the following type are of interest:

which combination of axioms (from different kinds of mathematics) is consistent and which is not?

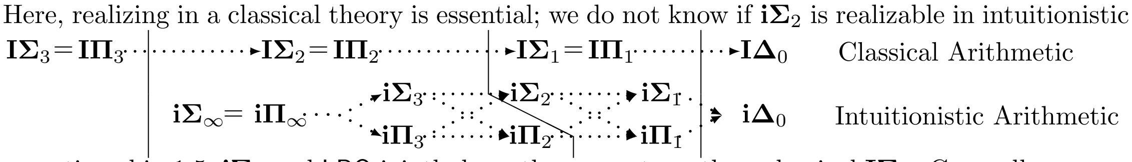
Namely, it has been asked only whether a combination is consistent or inconsistent.

Now our investigation is on the proof theoretic or consistency strengths of combinations. In other words, we ask how consistent (or to which extent consistent) the combination is. Thus the question becomes refined:

which combination of axioms (from different kinds of mathematics) is how much consistent?

The proof theoretic investigation of intuitionistic theories seems much less developed than classical ones.

Even the consistency strengths of Σ_n or Π_n induction schemata, the most basic targets of the study, were identified only in 1990s. Then Visser (in his unpublished note, see [45]) pointed out that $\mathbf{i\Sigma}_\infty = \mathbf{i\Pi}_\infty$, Heyting arithmetic with induction restricted to prenex formulae, is mutually Π_2-preservingly interpretable with $\mathbf{i\Pi}_2$, and so with classical $\mathbf{I\Sigma}_2$. This shows the drastic contrast with the classical situation, as classical $\mathbf{I\Sigma}_n$'s form a strict hierarchy exhausting Peano arithmetic **PA**. $\mathbf{i\Sigma}_1$ and $\mathbf{I\Sigma}_1$ are mutually Π_2-preservingly interpretable (see [11, 3]), and so are $\mathbf{i\Pi}_1$ and $\mathbf{I\Pi}_1 = \mathbf{I\Sigma}_1$ as shown easily by Gödel-Gentzen negative interpretation (but only Π_1-preserving, as shown in [45]). Thus any of $\mathbf{i\Sigma}_n$ ($n \geq 3$) and $\mathbf{i\Pi}_n$ ($n \geq 2$) has the same strength as classical $\mathbf{I\Sigma}_2$, and both $\mathbf{i\Sigma}_1$ and $\mathbf{i\Pi}_1$ as classical $\mathbf{I\Sigma}_1$ (and so **PRA**). What remains is $\mathbf{i\Sigma}_2$, which [10, Corollary 2.27] interpreted in a fragment of Gödel's **T** of the same proof theoretic strength as $\mathbf{I\Sigma}_1$ by Dialectica interpretation. We will show these results by realizability but also that these strengths are not affected by adding the fragments of Brouwerian axioms. While for this goal we need functional realizability, our proof also shows that Kleene's number realizability, used in [45], interprets intuitionistic $\mathbf{i\Sigma}_2$ in classical $\mathbf{I\Sigma}_1$. Here, realizing in a classical theory is essential; we do not know if $\mathbf{i\Sigma}_2$ is realizable in intuitionistic $\mathbf{i\Sigma}_1$.



As mentioned in 1.5, $\mathbf{i\Sigma}_2$ and LPO jointly have the same strength as classical $\mathbf{I\Sigma}_2$. Generally, our method shows that $\mathbf{i\Sigma}_{n+1}$+Σ_n-LEM is mutually Π_{n+2}-preservingly interpretable with $\mathbf{I\Sigma}_{n+1}$, whereas Gödel-Gentzen negative interpretation needs stronger $\mathbf{i\Sigma}_{n+1}$+Σ_{n+1}-DNE to interpret $\mathbf{I\Sigma}_{n+1}$.

Besides induction, there seem to have been no proof theoretic studies (in the sense of 1.2) on intuitionistic theories of the strength below **HA**.[9] The present paper leads to this large field of proof theoretic study.

[9] Those above **HA**, e.g., many variants of **CZF**, have been investigated. Some works of proof mining (e.g., [24]) are related but not exactly: e.g., induction for all negative formulae has no strength in their sense, although it interprets full induction.

1.7 Conclusions

Although bar induction (BI) was accepted in Brouwer's original idea, the accumulation of studies has shown that weak fan theorem (WFT), a consequence of BI, and continuous choice (CC) suffice in most cases. These two have been perceived even to characterize Intuitionistic Mathematics (INT) in constructive reverse mathematics (see [9, Ch.5], [20, p.44, l.-7] or [12, §4] where WFT is called fan theorem). If we agree with this perception,[10] we could conclude that *Brouwer's Intuitionism is compatible with Hilbert's Finitism*, for WFT and CC both for arbitrary formulae are jointly reducible to, and, even provably consistent in **PRA**.

Moreover, some semi-classical principles, e.g., Markov's principle MP and lesser limited principle of omniscience LLPO, do not destroy the compatibility and are hence consistent with Intuitionism and Finitism[11] (Fig.1) even though Brouwer did not accept them. Thus MP *and* LLPO *are acceptable in the same (indirect) sense as WKL is acceptable in Hilbert's Finitism.* On the other hand, *limited principle of omniscience* LPO *is, by no means, consistent with Intuitionism and Finitism*: it is finitistically consistent only with those fragments of Brouwerian axioms with which the entire classical logic is finitistically consistent (Fig.2).

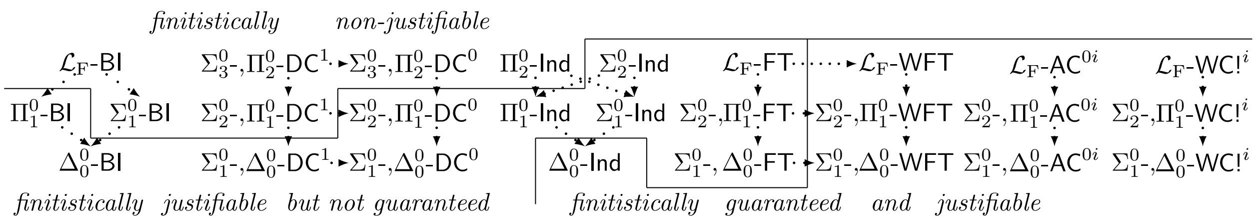


Figure 1: "Intuitionisitic Situation" – over any base theory between $\mathbf{EL}_0^-$ and $\mathbf{EL}_0^-$+MP+Σ^0_1-GDM

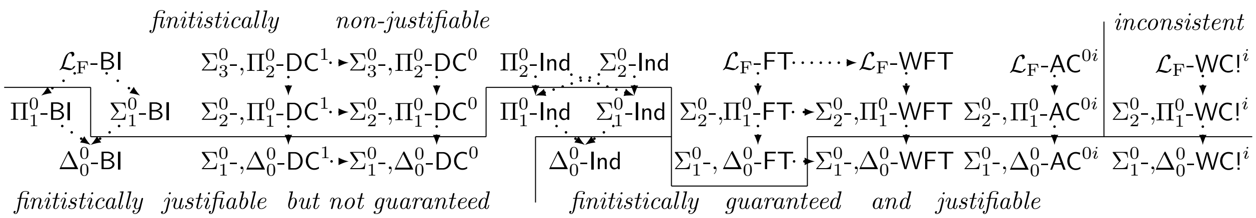


Figure 2: "Classical Situation" – over any base theory between $\mathbf{EL}_0^-$+LPO and $\mathbf{EL}_0^-$+$\mathcal{L}_F$-LEM

1.8 A marriage of Brouwer's Intuitionism and an ultrafinitism

After Hilbert's Finitism in the early 20th century, ultrafinitisms, stricter kinds of finitism than Hilbert's, have been proposed. Some are motivated by the development of computational complexity theory in the latter half of the century: only functions of a certain complexity are admitted, in the same sense as Hilbert's (formalized as **PRA**) admits only primitive recursive ones. Which part of INT, with which semi-classical principle, is justifiable or guaranteed with respect to them? An abundance of complexity classes (not yet proved to be identical), and hence of ultrafinitisms, makes this question too big to answer in one paper.

Here we consider only the easiest kind, which admits only Kalmár's elementary functions.[12] This could be formalized as $\mathbf{B\Sigma_1ex}$. All our finitistic guaranteedness results yield justifiability with respect to this kind of ultrafinitism, as they are proved via interpretability in $\mathbf{B\Sigma_1ex}$. Recall that the notion of proof theoretic reducibility collapses the consistency of such weak theories but that of interpretability does not.

Theories for even stricter kinds of ultrafinitism require the distinction between large and small numbers (i.e., x's and $|x|$'s), and therefore, in such a context, the natural formulations of some axioms, e.g., fan theorem, are not clear. The authors hope that they could treat these topics somewhere in the near future.

[10] This seems plausible as far as the "antique" fields of mathematics (established until ca.1900) are concerned. Other fields may go beyond this perception (e.g., [43, 41] used BI not only FT in combinatorics), needless to say that of CLASS beyond $\mathcal{L}_F$.

[11] For this claim on LLPO, we need to keep continuity principle without replacing it by CC (so the axiom of function-number and function-function choice are excluded) as an axiom of INT. This might be supported by the fact that Brouwer talked about "assignments" rather than left-total binary relations and by the argument triggered by creative subject as will be in f.n.13.

[12] Such functions form the third level $\mathcal{E}^3$ of *Grzegorczyk hierarchy.* We can replace it by $\mathcal{E}^n$ for any $n \geq 3$ without changing the result, as ultrafinitistic non-justifiability is by the interpretability of $\mathbf{I\Sigma_1}$ which proves the consistency of the theory for $\mathcal{E}^n$.

1.9 Outline and prerequisites

§2 introduces our base theory $\mathbf{EL}_0^-$ and some variants, as well as semi-classical principles and Brouwerian axioms whose strengths we will investigate, with basic properties. §3 gives upper bounds of the strengths of combinations of them, with Kleene's functional realizability and van Oosten's variant for Lifschitz-style, whose characterization by axioms will be generalized extensively. Folklore results from classical arithmetic, refined in Subsection 3.1, plays vital roles. §4 gives lower bounds, with Gödel-Gentzen negative interpretation and by generalizing Coquand-Hofmann forcing interpretation. §5 will present the results in final forms, with supplementary results, further problems and related works.

While §2 summarizes basic definitions and results on function-based second order arithmetic, the readers are assumed to be familiar with set-based counterpart from, e.g., [37]. They are supposed to know the systems $\mathbf{RCA}_0^*$, $\mathbf{WKL}_0^*$, $\mathbf{RCA}_0$, $\mathbf{WKL}_0$ and $\mathbf{ACA}_0$ as well as the axiom schema Π^1_m-TI, which is known to be equivalent over $\mathbf{ACA}_0$ to the transfinite induction along well-founded trees represented by sets. Comprehension axioms below are central in defining theories. By convention, *we always assume that there are no collisions of free variables with bound ones.* Thus below we implicitly assume that X is not free in $A[x]$.

($\mathcal{C}$-CA) $\exists X \forall x(x\in X \leftrightarrow A[x])$ for A from $\mathcal{C}$.

2 Preliminaries

2.1 The system $\mathbf{EL}_0^-$ of basic arithmetic

Definition 2.1 (languages $\mathcal{L}_1$ and $\mathcal{L}_\mathrm{F}$)**.** (1) The language $\mathcal{L}_1$ is a one-sorted first order language with equality $=$ consisting of constants 0 and 1, binary function symbols $+$, $\cdot$ and exp and a binary predicate $<$. (2) The language $\mathcal{L}_\mathrm{F}$ of elementary analysis is the two-sorted first order language, whose sorts are called *number* and *function*, which includes $\mathcal{L}_1$ as the part of the number sort, and which, additionally, has two function symbols Ev and Rest of arity one function and one number and of value number.

Notice that $\mathcal{L}_\mathrm{F}$ does not have the equality for the function sort. We call the systems on this language *function-based second order arithmetic*, in order to distinguish them from *set-based second order arithmetic*, systems on the language $\mathcal{L}_\mathrm{S}$ (called L_2 in [37]), which has been common in classical reverse mathematics.

Notation 2.2. Variables of the number sort are denoted by lower-case Latin letters x, y, z, u, v, etc., and those of the function are by Greek ones α, β, etc.. $\alpha(x) := \mathrm{Ev}(\alpha, x)$ and $\alpha{\restriction}x := \mathrm{Rest}(\alpha, x)$. $(\exists x<t)A$ stands for $\exists x(x<t \wedge A)$, $(\forall x<t)A$ for $\forall x(x<t \to A)$, $\alpha<\beta$ for $\forall x(\alpha(x)<\beta(x))$ and $\alpha=\beta$ for $\forall x(\alpha(x)=\beta(x))$.

Let $\exists! x A[x] :\equiv \exists x A[x] \wedge \forall y, z(A[y] \wedge A[z] \to y=z)$ and $\exists! \alpha A[\alpha] :\equiv \exists \alpha A[\alpha] \wedge \forall \beta, \gamma(A[\beta] \wedge A[\gamma] \to \beta=\gamma)$.

Definition 2.3 ($\mathcal{C}{\wedge}\mathcal{D}$, $\mathcal{C}{\vee}\mathcal{D}$, $\mathcal{C}{\to}\mathcal{D}$, $\neg\mathcal{C}$, $\mathsf{B}\forall^i\mathcal{C}$, $\mathsf{B}\exists^i\mathcal{C}$, $\forall^i\mathcal{C}$ and $\exists^i\mathcal{C}$)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be classes of formulae.

$\mathcal{C} \,\square\, \mathcal{D}$ consists of all formulae of the form $A \,\square\, B$ with A and B from $\mathcal{C}$ and $\mathcal{D}$ respectively, for $\square \equiv \wedge, \to, \vee$.

Moreover $\neg\mathcal{C}$, $\mathsf{B}\forall^0\mathcal{C}$, $\mathsf{B}\exists^0\mathcal{C}$, $\forall^0\mathcal{C}$, $\exists^0\mathcal{C}$, $\mathsf{B}\forall^1\mathcal{C}$, $\mathsf{B}\exists^1\mathcal{C}$, $\forall^1\mathcal{C}$ and $\exists^1\mathcal{C}$ consist of all those formulae of the forms $\neg A, (\forall x<t)A, (\exists x<t)A, \forall x A, \exists x A, (\forall \xi<\alpha)A, (\exists \xi<\alpha)A, \forall \xi A$ and $\exists \xi A$ respectively with A from $\mathcal{C}$.

Definition 2.4 (Δ^0_0, $\mathsf{B}\Pi^0_{n+1}$, $\mathsf{B}\Sigma^0_{n+1}$, Π^0_n, Σ^0_n, Π^0_∞, Σ^0_∞, Δ^1_0)**.** A formula of $\mathcal{L}_\mathrm{F}$ is called Δ^0_0 (as well as Σ^0_0 and Π^0_0) if all the quantifiers in it are number and bounded, i.e., only in the forms $\forall x<t$ and $\exists x<t$. Let $\mathsf{B}\Pi^0_{n+1} :\equiv \mathsf{B}\forall^0\Sigma^0_n$; $\mathsf{B}\Sigma^0_{n+1} :\equiv \mathsf{B}\exists^0\Pi^0_n$; $\Pi^0_{n+1} :\equiv \forall^0\Sigma^0_n$; and $\Sigma^0_{n+1} :\equiv \exists^0\Pi^0_n$. A formula is called *arithmetically prenex* (Π^0_∞ and Σ^0_∞) if it is Π^0_n or Σ^0_n for some n; and called Δ^1_0 if it contains no function quantifiers.

Definition 2.5 (**iQex**)**.** The intuitionistic $\mathcal{L}_1$-theory **iQex** is generated by the equality axioms, $x+0=x$; $x+(y+1)=(x+y)+1$; $x\cdot 0=0$; $x\cdot(y+1)=(x\cdot y)+x$; $\exp(x,0)=1$; and $\exp(x,y+1)=\exp(x,y)\cdot x$, and

(ir) $\neg(x<x)$; (tr) $x<y \wedge y<z \to x<z$; (s0) $x<x+1$; (s1) $x<y \to (x+1<y) \vee (x+1=y)$.

Definition 2.6 ($\mathcal{C}$-Ind, $\mathcal{C}$-Bdg, $\mathcal{C}$-LNP, $\mathcal{C}$-LEM and $\mathcal{C}$-DNE)**.** For a class $\mathcal{C}$ of $\mathcal{L}_1$ or $\mathcal{L}_\mathrm{F}$ formulae, define:

($\mathcal{C}$-Ind) $A[0] \wedge (\forall x<n)(A[x] \to A[x{+}1]) \to A[n]$ for any formula A from $\mathcal{C}$;

($\mathcal{C}$-Bdg) $(\forall x<n)\exists y A[x,y,n] \to \exists u(\forall x<n)(\exists y<u)A[x,y,n]$ for any formula A from $\mathcal{C}$;

($\mathcal{C}$-LNP) $A[x] \to (\exists y \le x)(A[y] \wedge (\forall z<y)\neg A[z])$, where $y\le x$ stands for $y<x{+}1$, for any formula A from $\mathcal{C}$;

($\mathcal{C}$-LEM) $A \vee \neg A$ for any formula A from $\mathcal{C}$; ($\mathcal{C}$-DNE) $\neg\neg A \to A$ for any formula A from $\mathcal{C}$.

Definition 2.7 ($\mathbf{i\Pi}_{n+1}$, $\mathbf{i\Sigma}_{n+1}$, $\mathbf{B\Sigma}_1\mathbf{ex}$, $\mathbf{I\Sigma}_{n+1}$)**.** Let $\Delta_0 \equiv \Delta^0_0 \cap \mathcal{L}_1$, $\Sigma_n \equiv \Sigma^0_n \cap \mathcal{L}_1$ and $\Pi_n \equiv \Pi^0_n \cap \mathcal{L}_1$. Define

$\mathbf{i\Pi}_{n+1} :\equiv \mathbf{iQex}+\Pi_{n+1}\text{-}\mathsf{Ind}$; $\quad \mathbf{i\Sigma}_{n+1} :\equiv \mathbf{iQex}+\Sigma_{n+1}\text{-}\mathsf{Ind}$;

$\mathbf{I\Delta}_0\mathbf{ex} :\equiv \mathbf{iQex}+\mathcal{L}_1\text{-}\mathsf{LEM}+\Delta_0\text{-}\mathsf{Ind}$; $\quad \mathbf{B\Sigma}_1\mathbf{ex} :\equiv \mathbf{I\Delta}_0\mathbf{ex}+\Sigma_1\text{-}\mathsf{Bdg}$; $\quad \mathbf{I\Sigma}_{n+1} :\equiv \mathbf{i\Sigma}_{n+1}+\mathcal{L}_1\text{-}\mathsf{LEM}$.

Proposition 2.8. (1) $\mathbf{iQex} + \Delta_0\text{-}\mathsf{Ind}$ proves (i) $0 < x+1$; (ii) $x < y \vee x = y \vee y < x$; and (iii) $\Delta_0\text{-}\mathsf{LEM}$.
(2) (i) $\mathbf{iQex} + \mathsf{B}\forall^0\neg\mathcal{C}\text{-}\mathsf{Ind} + \mathsf{B}\exists^0(\mathcal{C} \wedge \mathsf{B}\forall^0\neg\mathcal{C})\text{-}\mathsf{DNE} \vdash \mathcal{C}\text{-}\mathsf{LNP}$. In particular, (ii) $\mathbf{iQex} + \Delta_0\text{-}\mathsf{Ind} \vdash \Delta_0\text{-}\mathsf{LNP}$.
(3) (i) $\mathsf{B}\forall^0\Sigma_n \subseteq \Sigma_n$ up to equivalence over $\mathbf{iQex}+\Sigma_n\text{-}\mathsf{Bdg}$; (ii) $\mathbf{iQex}+\Sigma_n\text{-}\mathsf{Ind} \vdash \Sigma_n\text{-}\mathsf{Bdg}$.

Proof. (1) (i) is by $\Delta_0\text{-}\mathsf{Ind}$, (s0) and (tr). For (ii), let $A[x, y] :\equiv (x < y \vee x = y \vee y < x)$. Now $A[0, 0]$ and, by (i), $A[0, y] \to A[0, y+1]$. Thus $\Delta_0\text{-}\mathsf{Ind}$ yields $\forall y A[0, y]$. Because of $\Delta_0\text{-}\mathsf{Ind}$ it remains to show $A[x, y] \to A[x+1, y]$. $x < y \to A[x+1, y]$ is by (s1), $x = y \to A[x+1, y]$ by (s0) and $y < x \to A[x+1, y]$ by (s0) and (tr).

We see (iii) by induction on A. The atomic cases are by (ii), where (ir) implies $x < y \vee y < x \to \neg(x = y)$ and (ir) and (tr) imply $x < y \vee x = y \to \neg(y < x)$. The cases of $\wedge$ and $\to$ logically follow from the induction hypothesis. For $Q \equiv \exists, \forall$, let $B[n] :\equiv (Qx < n)A[x] \vee \neg(Qx < n)A[x]$. By (s1), (i) and (tr), if $x < 0$ then $x+1 < 0 \vee x+1 = 0$ and $x+1 < x+1$ contradicting (ir). Thus $\neg(x < 0)$ and $B[0]$. Now $x < n+1 \to x < n \vee x = n$ by (s1) and (ii). $B[n] \wedge (A[n] \vee \neg A[n]) \to B[n+1]$ and $B[n] \to B[n+1]$ by the hypothesis for A. Apply $\Delta_0\text{-}\mathsf{Ind}$.
(2) Let A be $\mathcal{C}$ and $B[y] :\equiv (\forall z \le y)\neg A[z]$. $\neg(\exists y \le x)(A[y] \wedge (\forall z < y)\neg A[z])$, i.e., $(\forall y \le x)((\forall z < y)\neg A[z] \to \neg A[y])$ implies $B[0] \wedge (\forall y < x)(B[y] \to B[y+1])$ and $B[x]$ by $\mathsf{B}\forall^0\neg\mathcal{C}\text{-}\mathsf{Ind}$. So $A[x] \to \neg\neg(\exists y \le x)(A[y] \wedge (\forall z < y)\neg A[z])$.
(3) (ii) Let A be Π_{n-1}. If $(\forall x < m)\exists y, z A[x, y, z]$, by $\Sigma_n\text{-}\mathsf{Ind}$ on $k \le m$, $\exists u(\forall x < k)(\exists y, z < u)A[x, y, z]$. ☐

Notation 2.9. (1) While $\mathcal{L}_{\mathrm{F}}$ has no function symbols besides $+$, $\cdot$ and exp, we can treat a *bounded* Δ^0_0 *definable function* f (i.e., defined by $A[\vec{x}, \vec{\alpha}, y]$ from Δ^0_0 and bounded by a term $t[\vec{x}, \vec{\alpha}]$) as follows: for a formula $B[y]$, by $B[f(\vec{x}, \vec{\alpha})]$ we mean $(\exists y < t[\vec{x}, \vec{\alpha}])(A[\vec{x}, \vec{\alpha}, y] \wedge B[y])$. If $B[y]$ is Δ^0_0, so is $B[f(\vec{x}, \vec{\alpha})]$. In this way, we can introduce fundamental operations on pairing and sequences of numbers without affecting the complexity: we fix, for each standard n, a bounded Δ^0_0 definable bijection $(\text{-}, ..., \text{-}) : \mathbb{N}^n \to \mathbb{N}$ and the associated projections $(\text{-})^n_i$ satisfying $(x)^n_i \le x$; and also a bijection $\mathbb{N}^{<\omega} \to \mathbb{N}$ so that basic operations are bounded Δ^0_0 definable up to the identification, e.g., evaluation $[u, x] \mapsto u(x)$; concatenation $[u, v] \mapsto u{*}v$ and $[u, x, \alpha] \mapsto (u{*}\alpha)(x)$; length-1 sequence $x \mapsto \langle x\rangle$; length $u \mapsto |u|$; and restriction $[u, n] \mapsto u{\restriction}n$. Assume $\max(u(x), |u|, u{\restriction}n) \le u$.
(2) Define $(\beta)^n_i = \lambda x.(\beta(x))^n_i$, $(\beta, \gamma) = \lambda x.(\beta(x), \gamma(x))$, $(\beta)_y = \lambda x.\beta((y, x))$, $\beta \ominus y = \lambda x.\beta(y+x)$ and $\underline{z} = \lambda x.z$, which are all bounded Δ^0_0 definable. Alternatively, for example, $A[(\beta)^n_i]$ is the result of replacing all the occurrences of $\alpha(t)$ in $A[\alpha]$ by $(\beta(t))^n_i$ and those of $\alpha{\restriction}t$ by corresponding bounded Δ^0_0 definable terms.
(3) We assume that classes of formulae are closed under (i) conjunctions and disjunctions with Δ^0_0, and (ii) substitutions of the expressions from (1) and (2). The operations in Def.2.3 preserve these closure properties.

Definition 2.10 ($\mathbf{EL}^-_0$)**.** The $\mathcal{L}_{\mathrm{F}}$-theory $\mathbf{EL}^-_0$ is generated over intuitionistic logic with equality for numbers, by (a) $\mathbf{iQex}$, (b) $\Delta^0_0\text{-}\mathsf{Ind}$, (c) $\alpha{\restriction}0 = \langle\,\rangle$, $\alpha{\restriction}(x+1) = (\alpha{\restriction}x){*}\langle\alpha(x)\rangle$; and (d) Δ^0_0 bounded search defined below:
($\mathcal{C}$ bounded search**)** $\exists\beta\forall x((\exists y < t[x])A[x, y] \to \beta(x) < t[x] \wedge A[x, \beta(x)])$ for A from $\mathcal{C}$ and a term $t[x]$.

$\mathbf{EL}^-_0$ is almost equivalent to $\mathbf{EL}_{\mathrm{ELEM}}$ from [18], which however has terms for all elementary functions by the help of functionals. Our $\mathbf{EL}^-_0$ proves the existence of those functions by the axiom (d) but shares the important feature with $\mathcal{L}_{\mathrm{S}}$ from classical reverse mathematics that second order terms are only variables.

Since Σ^0_n is Σ_n with $\mathcal{L}_F$-terms substituted for x's, 2.8 holds with Δ_0 and Σ_n replaced by Δ^0_0 and Σ^0_n.

Lemma 2.11. In $\mathbf{iQex}$ or $\mathbf{EL}^-_0$, $A \vee B$ is equivalent to $(\exists i < 2)((i = 0 \to A) \wedge (i = 1 \to B))$ for any A and B.

A key fact in second order arithmetic is a formal version of famous Kleene's normal form theorem. While in references (e.g., [37, Theorem II.2.7]) the proof is omitted or very sketchy, we give a little details.

Definition 2.12 (D_C, B_C)**.** For a Δ^0_0 formula $C[\vec{x}, \vec{\alpha}]$, we define D_C and B_C as follows.
(1) $\mathrm{D}_C[\vec{x}, \vec{u}]$ is the result of replacing $\alpha_i(s)$ and $\alpha_i{\restriction}s$ by $u_i(s)$ and $u_i{\restriction}s$, respectively, in C.
(2) (i) For atomic C, $\mathrm{B}_C[\vec{x}, v, \vec{\alpha}] :\equiv \bigwedge_i (v > t_i[\vec{x}, \vec{\alpha}])$ where $t_i[\vec{x}, \vec{\alpha}]$'s are all subterms in C; (ii) for $\Box \equiv \wedge, \to, \vee$, $\mathrm{B}_{C_1 \Box C_2}[\vec{x}, v, \vec{\alpha}] :\equiv \bigwedge_{i=1,2} \mathrm{B}_{C_i}[\vec{x}, v, \vec{\alpha}]$; (iii) $\mathrm{B}_{(Qz<t)C}[\vec{x}, v, \vec{\alpha}] :\equiv (\forall z < t[\vec{x}, \vec{\alpha}])\mathrm{B}_C[z, \vec{x}, v, \vec{\alpha}] \wedge \mathrm{B}_{0<t[\vec{x},\vec{\alpha}]}[\vec{x}, v, \vec{\alpha}]$.

$\mathrm{B}_C[\vec{x}, v, \vec{\alpha}]$ means "$C[\vec{x}, \vec{\alpha}]$ refers α only below v". So we take '$\bigwedge_{i=1,2}$' even for $\to, \vee$ and '$(\forall z < t[\vec{x}, \vec{\alpha}])$' for $\exists$. In $\mathcal{L}_{\mathrm{S}}$, $v > t_C[\vec{x}]$ can play the role of B_C for a suitable t_C (cf. [25, Lemma 2.13], where $t(i, \vec{k})$ on p.162, l.9 is a typo of $t''(i, \vec{k})$). Below $\beta(x_0, ..., x_n)$ stands for $\beta((x_0, ..., x_n))$ where the inner $(...)$ is from 2.9(1).

Lemma 2.13. For a Δ^0_0 formula $C[\vec{x},\vec{\alpha}]$, (i) $\mathbf{EL}_0^- \vdash \mathrm{B}_C[\vec{x},u,\vec{\alpha}] \wedge u \le v \to \mathrm{B}_C[\vec{x},v,\vec{\alpha}]$ (upward closure) and (ii) $\mathbf{EL}_0^- \vdash \exists\beta\forall\vec{x}\,\mathrm{B}_C[\vec{x},\beta(\vec{x}),\vec{\alpha}] \wedge \forall u,\vec{x}(\mathrm{B}_C[\vec{x},u,\vec{\alpha}] \to (C[\vec{x},\vec{\alpha}] \leftrightarrow \mathrm{D}_C[\vec{x},\vec{\alpha}{\restriction}u]))$.

Proof. By induction on C. (ii) First let C be atomic, whose all subterms are $t_i[\vec{x},\vec{\alpha}]$'s. Take $\beta(\vec{x}) = 1+\sum_i t_i[\vec{x},\vec{\alpha}]$ by Axiom (d). Then $\mathrm{B}_C[\vec{x},\beta(\vec{x}),\vec{\alpha}]$. For the latter conjunct, assume $\mathrm{B}_C[\vec{x},u,\vec{\alpha}]$. Now $t_i[\vec{x},\vec{\alpha}] < u$ and so $\alpha_j(t_i[\vec{x},\vec{\alpha}]) = (\alpha_j{\restriction}u)(t_i[\vec{x},\vec{\alpha}])$. Thus $t_i[\vec{x},\vec{\alpha}] = t_i[\vec{x},\vec{\alpha}{\restriction}u]$ by induction on t_i and hence $C[\vec{x},\vec{\alpha}] \leftrightarrow \mathrm{D}_C[\vec{x},\vec{\alpha}{\restriction}u]$. In the quantifier case, the induction hypotheses yield γ,δ with $\forall\vec{x},z\,\mathrm{B}_C[z,\vec{x},\gamma(z,\vec{x}),\vec{\alpha}]$ and $\forall\vec{x}\mathrm{B}_{0<t[\vec{x},\vec{\alpha}]}[\vec{x},\delta(\vec{x}),\vec{\alpha}]$. So $\forall\vec{x}\,\mathrm{B}_{(Qz<t)C}[\vec{x},\beta(\vec{x}),\vec{\alpha}]$ with $\beta(\vec{x}) := \gamma{\restriction}(t[\vec{x},\vec{\alpha}],\vec{x})+\delta(\vec{x})$ yielded by Axiom (c). If $\mathrm{B}_{(Qz<t)C}[\vec{x},u,\vec{\alpha}]$, then $(\forall z < t[\vec{x},\vec{\alpha}])\mathrm{B}_C[z,\vec{x},u,\alpha]$ and, since $C[z,\vec{x},\vec{\alpha}] \leftrightarrow \mathrm{D}_C[z,\vec{x},\vec{\alpha}{\restriction}u]$ for each $z < t[\vec{x},\vec{\alpha}]$ by the induction hypothesis, we have $(Qz < t[\vec{x},\vec{\alpha}])C[z,\vec{x},\vec{\alpha}] \leftrightarrow \mathrm{D}_{(Qz<t)C}[z,\vec{x},\vec{\alpha}{\restriction}u]$. The other cases are proved similarly. ☐

Theorem 2.14. For any $A[\vec{\alpha}]$ from Σ^0_1 there is $D[\vec{u}]$ from Δ^0_0 without $\vec{\alpha}$ with $\mathbf{EL}_0^- \vdash \forall\vec{\alpha}(A[\vec{\alpha}] \leftrightarrow \exists nD[\vec{\alpha}{\restriction}n])$.

Proof. For simplicity, let $\vec{\alpha} = \alpha$. Define $D[u] :\equiv (\exists x < |u|)(\mathrm{B}_C[x,|u|,u{*}\underline{0}] \wedge \mathrm{D}_C[x,u])$ for $A[\alpha] \equiv \exists xC[x,\alpha]$. Note $\mathrm{B}_C[x,n,\alpha] \to \forall\beta\,\mathrm{B}_C[x,n,(\alpha{\restriction}n){*}\beta]$. If $\exists nD[\alpha{\restriction}n]$, say $x < n \wedge \mathrm{B}_C[x,n,(\alpha{\restriction}n){*}\underline{0}] \wedge \mathrm{D}_C[x,\alpha{\restriction}n]$, then, by 2.13, $C[x,\alpha]$. Conversely, if $C[x,\alpha]$, 2.13 yields $n > x$ with $\mathrm{B}_C[x,n,\alpha]$ and so $\mathrm{D}_C[x,\alpha{\restriction}n]$. ☐

2.2 Choice axioms along numbers

Besides the existence of some specific functions and the closure conditions 2.10(d), $\mathbf{EL}_0^-$ has no constraints on the second order domain. It seems common to use choice axioms to govern the domain in the function-based setting, while in the set-based one comprehension axioms are more common.

Among variants of dependent choice, we set the premise to be $\mathrm{Ran}(R) \subseteq \mathrm{Dom}(R)$ for the relation R.

Definition 2.15 (choice schema)**.** For a class $\mathcal{C}$ of formulae, define the following axiom schemata. Moreover $\mathcal{C}$-$\mathsf{AC!}^{0i}$ and $\mathcal{C}$-$\mathsf{DC!}^{i}$ for $i = 0,1$ are defined with $\exists$ replaced by $\exists!$ in the premises.

($\mathcal{C}$-AC^{00}) $\forall x\exists yA[x,y] \to \exists\alpha\forall xA[x,\alpha(x)]$, **($\mathcal{C}$-$\mathsf{AC}^{01}$)** $\forall x\exists\beta A[x,\beta] \to \exists\alpha\forall xA[x,(\alpha)_x]$ both for any A from $\mathcal{C}$;
($\mathcal{C}$-DC^{0}) $\forall x,y(A[x,y] \to \exists zA[y,z]) \to \forall x,y(A[x,y] \to \exists\alpha(\alpha(0) = x \wedge \forall zA[\alpha(z),\alpha(z{+}1)]))$ for any A from $\mathcal{C}$;
($\mathcal{C}$-DC^{1}) $\forall\beta,\gamma(A[\beta,\gamma] \to \exists\delta A[\gamma,\delta]) \to \forall\beta,\gamma(A[\beta,\gamma] \to \exists\alpha((\alpha)_0 = \beta \wedge \forall zA[(\alpha)_z,(\alpha)_{z+1}]))$ for any A from $\mathcal{C}$.

Lemma 2.16. (1) Over $\mathbf{EL}_0^- + \mathcal{C}$-$\mathsf{LNP}$, $(\mathcal{C} \wedge \mathsf{B}\forall^0\neg\mathcal{C})$-$\mathsf{AC!}^{00}$ implies $\mathcal{C}$-AC^{00}; $(\mathcal{C} \wedge \mathsf{B}\forall^0\neg\mathcal{C})$-$\mathsf{DC!}^{0}$ implies $\mathcal{C}$-DC^{0}.
(2) Over $\mathbf{EL}_0^-$, (i) $\mathcal{C}$-DC^{i} yields $\exists^i\mathcal{C}$-DC^{j}; (ii) $\mathcal{C}$-DC^{i} yields $\mathcal{C}$-AC^{0j}; (iii) $\mathcal{C}$-AC^{0i} yields $\exists^i\mathcal{C}$-AC^{0j}, for $j \le i \in \{0,1\}$; (iv) $\mathcal{C}$-$\mathsf{DC!}^{i}$ yields $\mathcal{C}$-$\mathsf{AC!}^{0i}$; (v) $\mathcal{C}{\wedge}\Pi^0_1$-$\mathsf{DC!}^{1}$ yields $\mathcal{C}$-$\mathsf{DC!}^{0}$; (vi) $\mathcal{C}{\wedge}\Pi^0_1$-$\mathsf{AC!}^{01}$ yields $\mathcal{C}$-$\mathsf{AC!}^{00}$.
(3) (i) $\mathbf{EL}_0^-{+}\mathcal{C}$-$\mathsf{DC!}^{0}{\vdash}\,\mathcal{C}$-$\mathsf{Ind}$; (ii) $\mathbf{EL}_0^-{+}\mathcal{C}$-$\mathsf{AC}^{00}{\vdash}\,\mathcal{C}$-$\mathsf{Bdg}$. (4) $\mathbf{EL}_0^- + \mathsf{B}\forall^0\mathcal{C}$-$\mathsf{AC!}^{00} + \exists^0(\mathsf{B}\forall^0\mathcal{C})$-$\mathsf{Ind} \vdash \mathcal{C}$-$\mathsf{DC!}^{0}$.
(5) $\mathbf{EL}_0^-{+}\forall^0(\mathcal{C}{\wedge}\neg\mathcal{C})$-$\mathsf{DC!}^{1}{+}\mathcal{C}$-$\mathsf{LNP} \vdash \forall^0\exists^0\mathcal{C}$-$\mathsf{DC!}^{1}$.

Proof. Let A be $\mathcal{C}$. (2)(i) ($i{=}0$) If $\forall x,y(\exists uA[x,y,u] \to \exists z,vA[y,z,v]))$ then $\forall x,y(B[x,y] \to \exists zB[y,z])$ where $B[x,y] \equiv A[(x)^2_0,(y)^2_0,(y)^2_1]$. For any x,y with $\exists uA[x,y,u]$, since $\exists y,uB[(x,0),(y,u)]$, $\mathcal{C}$-DC^{0} yields β with $\beta(0) = (x,0)$ and $\forall zB[\beta(z),\beta(z{+}1)]$. Define α by $\alpha(x) = (\beta(x))^2_0$. ($i{=}1$) is similarly proved.
(ii) ($i{=}0$) If $\forall x\exists yA[x,y]$ then $\forall u\exists v((v)^2_0 = (u)^2_0{+}1 \wedge A[(u)^2_0,(v)^2_1])$, and $\mathcal{C}$-DC^{0} yields α with $\alpha(0) = (0,0)$ and $\forall x((\alpha(x{+}1))^2_0 = (\alpha(x))^2_0{+}1 \wedge A[(\alpha(x))^2_0,(\alpha(x{+}1))^2_1])$. Δ^0_0-Ind shows $(\alpha(x))^2_0 = x$ and so $\forall xA[x,(\alpha(x{+}1))^2_1]$.
($i{=}1$) If $\forall x\exists\gamma A[x,\gamma]$, $\mathcal{C}$-DC^{1} yields α with $\forall x((\alpha)_{x+1}(0) = (\alpha)_x(0){+}1 \wedge A[(\alpha)_x(0),(\alpha)_{x+1}{\ominus}1])$ and $(\alpha)_0 = \underline{0}$.
(v)(vi) If $\exists!zA[z]$ then $\exists!\gamma(A[\gamma(0)] \wedge \gamma{\ominus}1 = \underline{0})$ and vice versa, where $\gamma{\ominus}1 = \underline{0}$ is Π^0_1.
(3)(i) Let $B[x,y] :\equiv y = x+1{\wedge}(y \le n \to A[y])$. If $A[0]{\wedge}(\forall x < n)(A[x] \to A[x{+}1])$, as $\forall x,y(B[x,y] \to \exists!zB[y,z])$ and $B[0,1]$, $\mathcal{C}$-$\mathsf{DC!}^{0}$ yields α with $\forall xB[\alpha(x),\alpha(x{+}1)]$ and $\alpha(0) = 0$. By Δ^0_0-Ind $(\forall x \le n)(x = \alpha(x))$ and $A[n]$.
(4) Let $\forall x,y(A[x,y] \to \exists!zA[y,z])$ and $A[x,y]$. $\exists^0(\mathsf{B}\forall^0\mathcal{C})$-$\mathsf{Ind}$ shows $\forall n\exists!uC[n,u]$ where

$$C[n,u] :\equiv |u| = n{+}2 \wedge u(0) = x \wedge u(1) = y \wedge (\forall k < n{+}1)A[u(k),u(k{+}1)].$$

$\mathsf{B}\forall^0\mathcal{C}$-$\mathsf{AC!}^{00}$ yields β with $\forall nC[n,\beta(n)]$. $\beta(n) \subset \beta(n{+}1)$ by Δ^0_0-Ind. $\forall kA[\alpha(k),\alpha(k{+}1)]$ for $\alpha(k) = \beta(k)(k)$.
(5) Since $\Pi^0_1{\wedge}\forall^0(\mathcal{C}{\wedge}\neg\mathcal{C}) \subseteq \forall^0(\Delta^0_0{\wedge}\mathcal{C}{\wedge}\neg\mathcal{C}) \subseteq \forall^0(\mathcal{C}{\wedge}\neg\mathcal{C})$, (2)(iv)(v) and $\mathcal{C}$-LNP yield

$$\exists!\eta\forall x\exists yA[\xi,\eta,x,y] \leftrightarrow \exists!\eta B[\xi,\eta] \text{ where } B[\xi,\eta] :\equiv \forall x(\forall y < (\eta)^2_1(x))(A[\xi,(\eta)^2_0,x,(\eta)^2_1(x))]{\wedge}\neg A[\xi,(\eta)^2_0,x,y]).$$

If $\forall\beta,\gamma(\forall x\exists yA[\beta,\gamma,x,y] \to \exists!\delta\forall x\exists yA[\gamma,\delta,x,y])$, then $\forall\beta,\gamma(B[(\beta)^2_0,\gamma] \to \exists!\delta B[(\gamma)^2_0,\delta])$ by the equivalence, and $\forall^0(\mathcal{C}{\wedge}\neg\mathcal{C})$-$\mathsf{DC!}^{1}$ yields γ with $\forall zB[((\gamma)_z)^2_0,(\gamma)_{z+1}]$ which implies $\forall z,x\exists yA[((\gamma)_z)^2_0,((\gamma)_{z+1})^2_0,x,y]$. ☐

Definition 2.17 ($\mathbf{EL}_0^*$, $\mathbf{EL}_0$ and $\mathbf{EL}$)**.** $\mathbf{EL}_0^*$ is $\mathbf{EL}_0^-{+}\Delta^0_0$-$\mathsf{AC}^{00}$; $\mathbf{EL}_0$ is $\mathbf{EL}_0^*{+}\Sigma^0_1$-$\mathsf{Ind}$; and $\mathbf{EL}$ is $\mathbf{EL}_0{+}\mathcal{L}_{\mathrm{F}}$-$\mathsf{Ind}$.

By 2.8(2)(ii) and 2.16(1), $\mathbf{EL}_0^- \vdash \Delta^0_0$-$\mathsf{DC}^{0} \leftrightarrow \Delta^0_0$-$\mathsf{DC!}^{0}$. By 2.16(2)(i)(ii)(3)(i)(4), $\mathbf{EL}_0 = \mathbf{EL}_0^-{+}\Delta^0_0$-$\mathsf{DC}^{0}$.

2.3 Relation to set-based systems

One might consider that the study of our function-based second order arithmetic is equivalent to that of the famous set-based one (extensively done, e.g., in [37]), since functions are coded by sets as graphs and sets are coded by functions as characteristic functions. This expectation is true if we consider only classical systems not sensitive to arithmetical complexity. Otherwise there are several delicate differences. We first clarify the correspondence between the two settings along which we consider similarity and dissimilarity.

Definition 2.18 (characteristic function interpretation $\mathfrak{ch}$)**.** Assign injectively function variables α_X of $\mathcal{L}_\mathrm{F}$ to set variables X of $\mathcal{L}_\mathrm{S}$. For an $\mathcal{L}_\mathrm{S}$ formula A, define an $\mathcal{L}_\mathrm{F}$ formula $A^{\mathfrak{ch}}$ by

$\perp^{\mathfrak{ch}} :\equiv \perp$; $\quad (t \in X)^{\mathfrak{ch}} :\equiv \alpha_X(t) = 0$; $\quad (s \,\mathrm{R}\, t)^{\mathfrak{ch}} :\equiv s \,\mathrm{R}\, t$ for $\mathrm{R} \equiv =, <$; $\quad (A \Box B)^{\mathfrak{ch}} :\equiv A^{\mathfrak{ch}} \Box B^{\mathfrak{ch}}$ for $\Box \equiv \wedge, \to, \vee$;
$(QxA)^{\mathfrak{ch}} :\equiv QxA^{\mathfrak{ch}}$ for $Q \equiv \forall, \exists$; $\quad (\forall XA)^{\mathfrak{ch}} :\equiv \forall \alpha_X(\alpha_X < \underline{2} \to A^{\mathfrak{ch}})$; $\quad (\exists XA)^{\mathfrak{ch}} :\equiv \exists \alpha_X(\alpha_X < \underline{2} \wedge A^{\mathfrak{ch}})$.

Definition 2.19 (graph interpretation $\mathfrak{g}$)**.** Assign injectively variables X_α of $\mathcal{L}_\mathrm{S}$ to variables α of $\mathcal{L}_\mathrm{F}$. For an $\mathcal{L}_\mathrm{F}$-term t, define $[\![t]\!](x)$: $[\![x]\!](y) :\equiv x = y$; $[\![c]\!](y) :\equiv c = y$ for $c = 0, 1$; $[\![t \circ t']\!](y) :\equiv \exists x, x'([\![t]\!](x) \wedge [\![t']\!](x') \wedge y = x \circ x')$ for $\circ \equiv +, \cdot, \exp$; $[\![\alpha(t)]\!](y) :\equiv \exists z([\![t]\!](z) \wedge (z, y) \in X_\alpha)$; and $[\![\alpha \restriction t]\!](u) :\equiv [\![t]\!](|u|) \wedge (\forall x < |u|)((x, u(x)) \in X_\alpha))$. For A in $\mathcal{L}_\mathrm{F}$, define $A^{\mathfrak{g}}$ in $\mathcal{L}_\mathrm{S}$ as follows, where $\mathrm{Func}[X] :\equiv \forall x \exists! y((x, y) \in X)$:

$\perp^{\mathfrak{g}} :\equiv \perp$; $\quad (s \,\mathrm{R}\, t)^{\mathfrak{g}} :\equiv \exists x, y([\![s]\!](x) \wedge [\![t]\!](y) \wedge x \mathrm{R} y)$ for $\mathrm{R} \equiv =, <$; $(A \Box B)^{\mathfrak{g}} :\equiv A^{\mathfrak{g}} \Box B^{\mathfrak{g}}$ for $\Box \equiv \wedge, \to, \vee$;
$(QxA)^{\mathfrak{g}} :\equiv QxA^{\mathfrak{g}}$ for $Q \equiv \forall, \exists$; $\quad (\forall \alpha A)^{\mathfrak{g}} :\equiv \forall X_\alpha(\mathrm{Func}[X_\alpha] \to A^{\mathfrak{g}})$; $\quad (\exists \alpha A)^{\mathfrak{g}} :\equiv \exists X_\alpha(\mathrm{Func}[X_\alpha] \wedge A^{\mathfrak{g}})$.

Lemma 2.20. The graph interpretation $\mathfrak{g}$ interprets $\mathbf{EL}_0^- + \Sigma^0_1\text{-}\mathsf{Bdg} + \Delta^0_0\text{-}\mathsf{AC}^{00} + \Sigma^0_n\text{-}\mathsf{Ind}$ in $\mathbf{RCA}_0^* + \Sigma^0_n\text{-}\mathsf{Ind}$.

Proof. As $\mathbf{RCA}_0^*$ proves Σ^0_1-Bdg, we have $\exists x [\![t]\!](x) \to \forall X_\alpha(\forall x \exists! y((x, y) \in X_\alpha \to \exists v [\![\alpha \restriction t]\!](v))$ for any term t. Thus $\exists! x [\![t]\!](x)$ by induction on t, and hence $(s \,\mathrm{R}\, t)^{\mathfrak{g}}$ is equivalent to $\forall x, y([\![s]\!](x) \wedge [\![t]\!](y) \to x \mathrm{R} y)$. Thus, if A is Δ^0_0, then $A^{\mathfrak{g}}$ is Δ^0_1 and $\mathbf{RCA}_0^*$ yields $X_\alpha = \{(x, y) : A[x, y]^{\mathfrak{g}} \wedge (\forall z < y) \neg A[x, y]^{\mathfrak{g}}\}$. If $(\forall x \exists y A[x, y])^{\mathfrak{g}}$, then $\forall x \exists! y((x, y) \in X_\alpha)$ by Δ^0_0-LNP, which is provable in $\mathbf{RCA}_0^*$. Now $\forall x \exists y((x, y) \in X_\alpha \wedge A[x, y]^{\mathfrak{g}})$ i.e., $(\forall x A[x, \alpha(x)])^{\mathfrak{g}}$. Thus $(\Delta^0_0\text{-}\mathsf{AC}^{00})^{\mathfrak{g}}$. The interpretability of the remaining axioms by $\mathfrak{g}$ is obvious. $\square$

Thus $\mathfrak{g}$ seems to require Δ^0_1-CA in $\mathcal{L}_\mathrm{S}$. To interpret it, $\mathfrak{ch}$ seems to require Δ^0_0-AC^{00} and hence $\mathbf{EL}_0^*$.

The delicate differences are mainly caused by the clauses of the totality $\forall x \exists! y((x, y) \in X_\alpha)$ (which is known to be Π^0_2 complete in recursion theory) and of $\forall x(\alpha_X(x) < 2)$. For example, the premise $\forall x \exists \alpha A[x, \alpha]$ of the number-function choice $\mathcal{C}$-AC^{01} is interpreted by $\mathfrak{g}$ as $\forall x \exists X_\alpha(\mathrm{Func}[X_\alpha] \wedge A[x, \alpha]^{\mathfrak{g}})$ and so we cannot apply the number-set choice, unless the class is closed under conjunctions with Π^0_2 formulae. Conversely, $(\forall x \exists X A[x, X])^{\mathfrak{ch}}$ is $\forall x(\exists \alpha_X < \underline{2}) A[x, X]^{\mathfrak{ch}}$ and therefore we could say that the number-set choice is only a fragment of number-function choice, or bounded version of the latter. This motivates the following.

Definition 2.21 (bounded choice schema)**.** For a class $\mathcal{C}$ of formulae, define the following axiom schemata:

($\mathcal{C}$-BAC^{01}) $\forall x(\exists \beta < (\gamma)_x) A[x, \beta] \to (\exists \alpha < \gamma) \forall x A[x, (\alpha)_x]$ for any A from $\mathcal{C}$;
($\mathcal{C}$-BAC^{00}) $\forall x(\exists y < \beta(x)) A[x, y] \to (\exists \alpha < \beta) \forall x A[x, \alpha(x)]$ for any A from $\mathcal{C}$;
($\mathcal{C}$-$\mathsf{2AC}^{01}$) $\forall x(\exists \beta < \underline{2}) A[x, y] \to (\exists \alpha < \underline{2}) \forall x A[x, (\alpha)_x]$ for any A from $\mathcal{C}$;
($\mathcal{C}$-$\mathsf{2AC}^{00}$) $\forall x(\exists y < 2) A[x, y] \to (\exists \alpha < \underline{2}) \forall x A[x, \alpha(x)]$ for any A from $\mathcal{C}$.

2.4 Semi-classical or semi-constructive principles

Definition 2.22 (MP, LPO, $\mathcal{C}$-DM, $\mathcal{C}$-GDM and LLPO)**.** MP and LPO denote Σ^0_1-DNE and Σ^0_1-LEM both from 2.6 respectively. LLPO denotes Σ^0_1-DM, where for a class $\mathcal{C}$ of formulae, define the schemata:

($\mathcal{C}$-DM) $\neg(A \wedge B) \to \neg A \vee \neg B$ for A, B from $\mathcal{C}$; $\qquad$ ($\mathcal{C}$-GDM) $\neg(\forall x < y) A \to (\exists x < y) \neg A$ for A from $\mathcal{C}$.

Lemma 2.23. (1) $\mathcal{C}$-LEM yields $A \vee \neg A$ and $\neg\neg A \to A$ for any A built from $\mathcal{C}$ formulae by $\wedge$, $\vee$, $\to$ and $\neg$. (2) (i) $\mathsf{B}\exists^0(\forall^0 \neg \mathcal{C}) \subseteq \neg \exists^0 \mathsf{B} \forall^0 \mathsf{B} \exists^0 \mathcal{C}$ over $\mathbf{EL}_0^- + \exists^0\mathcal{C}$-$\mathsf{GDM}$+ $\mathcal{C}$-Bdg; (ii) $\mathbf{EL}_0^- \vdash (\neg \mathcal{C} \vee \neg \mathcal{C})$-$\mathsf{DNE} \leftrightarrow \mathcal{C}$-$\mathsf{DM}$. (3) $\mathbf{EL}_0^- + \mathcal{C}$-$\mathsf{GDM} \vdash \mathsf{B}\exists^0(\neg \mathcal{C})$-$\mathsf{DNE}$ and $\mathbf{EL}_0^- + \mathcal{C}$-$\mathsf{DNE}$+ $\mathsf{B}\exists^0(\neg \mathcal{C})$-$\mathsf{DNE} \vdash \mathcal{C}$-$\mathsf{GDM}$. (4) $\mathbf{EL}_0^- + \mathcal{C}$-$\mathsf{GDM} \vdash \mathcal{C}$-$\mathsf{DM}$.

Proof. Let A and B be $\mathcal{C}$. (2) (i) $\neg \exists u(\forall x < t)(\exists y < u) A[x, y]$ is equivalent to $\neg(\forall x < t) \exists y A[x, y]$ by $\mathcal{C}$-Bdg and to $(\exists x < t) \forall y \neg A[x, y]$ by $\exists^0\mathcal{C}$-GDM. (ii) $\neg\neg(\neg A \vee \neg B)$, $(\neg\neg A \wedge \neg\neg B) \to \perp$ and $\neg(A \wedge B)$ are equivalent. (3) $\neg\neg(\exists x < y) \neg A$ is equivalent to $\neg(\forall x < y) \neg\neg A$ and implies $\neg(\forall x < y) A$. $\mathcal{C}$-DNE yields the converse. $\square$

Lemma 2.24. (1) Over $\mathbf{EL}_0^-+\Sigma_n^0$-DNE, (i) $\neg\Pi_n^0=\Sigma_n^0$; (ii) $\neg\Sigma_{n+1}^0=\Pi_{n+1}^0$; and so (iii) Π_{n+1}^0-DNE holds. (2) Over $\mathbf{EL}_0^-$, (i) Σ_{n+1}^0-DNE yields Σ_n^0-GDM; (ii) Σ_n^0-GDM $\wedge\ \Sigma_{n-1}^0$-DNE yields $\mathsf{B}\Sigma_{n+1}^0$-DNE; (iii) $\mathsf{B}\Sigma_{n+1}^0$-DNE yields $\Pi_n^0\vee\Pi_n^0$-DNE; (iv) both Σ_{n+1}^0-DNE and $\Pi_{n+1}^0\vee\Pi_{n+1}^0$-DNE yield $\Sigma_n^0\vee\Pi_n^0$-DNE; (v) $\Sigma_n^0\vee\Pi_n^0$-DNE is equivalent to Σ_n^0-LEM; (vi) Σ_n^0-LEM yields Σ_n^0-DNE and $\Pi_n^0\vee\Pi_n^0$-DNE.

Proof. (1) By induction, $\neg\Pi_n^0=\neg\forall^0\neg\neg\Sigma_{n-1}^0=\neg\neg\exists^0\neg\Sigma_{n-1}^0=\neg\neg\exists^0\Pi_{n-1}^0=\Sigma_n^0$, and $\neg\Sigma_{n+1}^0=\forall^0\neg\Pi_n^0=\forall^0\Sigma_n^0$. (2) (i) and (ii) are by 2.23(3), since Σ_{n-1}^0-DNE implies $\neg\Sigma_n^0=\Pi_n^0$ and $\mathsf{B}\exists^0(\neg\Sigma_n^0)=\mathsf{B}\Sigma_{n+1}^0\subseteq\Sigma_{n+1}^0$. (iii) and (iv) are by 2.11. For (v), for A from Σ_{n-1}^0, Σ_{n-1}^0-DNE applied to $\neg((\forall xA)\wedge\neg\forall xA)$ yields $\neg((\forall x\neg\neg A)\wedge\neg\forall xA)$ and $\neg\neg(\exists x\neg A\vee\forall xA)$ where $\neg\Sigma_{n-1}^0=\Pi_{n-1}^0$. The rest of (v) and (vi) are by 2.23(1). ☐

We thus obtain the diagram in Subsection 1.5. [1] showed the independence of $\Pi_n^0\vee\Pi_n^0$-DNE and Σ_n^0-DNE and the non-reversibility of (2) (i), (iv) and (vi) for $n>0$. While (ii) and (iii) are reversible with Δ_0^1-Ind, we do not know over $\mathbf{EL}_0^-$ if they are nor if $\Pi_{n+1}^0\vee\Pi_{n+1}^0$-DNE or Σ_n^0-LEM implies $\mathsf{B}\Sigma_{n+1}^0$-DNE or Σ_n^0-GDM.

2.5 Brouwerian axioms

2.5.1 bar induction

Definition 2.25 (Bar and $\mathcal{C}$-BI$_D$, $(\mathcal{C},\mathcal{D})$-BI$_M$, $\mathcal{C}$-BI)**.** $\mathsf{Bar}[\gamma,\{u\colon B[u]\}]:\equiv\forall\alpha(\forall k(\gamma(\alpha\restriction k)=0)\to\exists nB[\alpha\restriction n])$.
($\mathcal{C}$-BI$_D$) $\mathsf{Bar}[\underline{0},\{u\colon\alpha(u)=0\}]\wedge\forall u(\forall xA[u{*}\langle x\rangle]\to A[u])\wedge\forall u(\alpha(u)=0\to A[u])\to A[\langle\,\rangle]$;
($(\mathcal{C},\mathcal{D})$-BI$_M$) $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]\wedge\forall u,v(B[u]\to B[u{*}v])\wedge\forall u(\forall xA[u{*}\langle x\rangle]\to A[u])\wedge\forall u(B[u]\to A[u])\to A[\langle\,\rangle]$;
($\mathcal{C}$-BI) $\mathsf{Bar}[\underline{0},\{u\colon A[u]\}]\wedge\forall u(\forall xA[u{*}\langle x\rangle]\to A[u])\to A[\langle\,\rangle]$, where, in all three, A is from $\mathcal{C}$ and B is from $\mathcal{D}$.

In $\mathcal{C}$-BI we do not distinguish B from A, since $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]$ and $\forall u(B[u]\to A[u])$ imply $\mathsf{Bar}[\underline{0},\{u\colon A[u]\}]$.

As LPO is absolutely against Brouwer's philosophy, 2.26 below shows that $\mathcal{L}_{\mathrm{F}}$-BI cannot be a Brouwerian axiom though Brouwer's original texts look to accept it. Whereas Kleene presumed that Brouwer had meant $\mathcal{L}_{\mathrm{F}}$-BI$_D$, it seems more common to consider $(\mathcal{L}_{\mathrm{F}},\mathcal{L}_{\mathrm{F}})$-BI$_M$ (see, e.g., [42]), which are, as will be shown in 2.27(1)(iii) and 2.37(4), equivalent to $\mathcal{L}_{\mathrm{F}}$-BI$_D$ under another Brouwerian axiom. Yet, there seems to be no positive argument for this presumption in the literature (for, monotonicity was not mentioned explicitly in the original texts and there might be other ways to restrict bar induction consistently with other Brouwerian axioms) and $\mathcal{C}$-BI for $\mathcal{C}\not\supseteq\Sigma_1^0\vee\Pi_1^0$ still survives. However we do not need to enter into such discussion, since our result will be same for $\mathcal{C}$-BI and $\mathcal{C}$-BI$_D$, and hence for any variant inbetween, including $(\mathcal{C},\mathcal{D})$-BI$_M$. Actually below we see: Π_1^0-BI is finitistically justifiable and this is optimal in the sense that Σ_1^0-BI$_D$ is not.

Lemma 2.26. $\mathbf{EL}_0^-+\mathcal{C}$-LEM$+(\exists^0\mathcal{C}\vee\forall^0\neg\mathcal{C})$-BI $\vdash\exists^0\mathcal{C}$-LEM. In particular $\mathbf{EL}_0^-+(\Sigma_1^0\vee\Pi_1^0)$-BI $\vdash$ LPO.

Proof. Let $C[x]$ be $\mathcal{C}$ and $B[u]:\equiv(|u|=1\wedge\neg C[u(0)])\vee(|u|=0\wedge(\exists xC[x]\vee\forall x\neg C[x]))$. If $\forall xB[u{*}\langle x\rangle]$ then $|u|=0\wedge\forall x\neg C[x]$ and $B[u]$. $\mathcal{C}$-LEM yields $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]$ by $C[\alpha(0)]\to B[\alpha\restriction 0]$ and $\neg C[\alpha(0)]\to B[\alpha\restriction 1]$. ☐

Lemma 2.27. (1) (i) $\mathbf{EL}_0^-+\mathcal{C}$-BI $\vdash(\mathcal{C},\mathcal{L}_{\mathrm{F}})$-BI$_M$; (ii) $\mathbf{EL}_0^-+\mathcal{C}$-BI$_D\vdash(\mathcal{C},\Delta_0^0)$-BI$_M$; (iii) $\mathbf{EL}_0^-+(\mathcal{C},\Delta_0^0)$-BI$_M\vdash\mathcal{C}$-BI$_D$. (2) $\mathbf{EL}_0^-+\mathcal{C}$-BI$_D\vdash\mathcal{C}$-Ind. (3) $\mathbf{EL}_0^-+\exists^0\mathcal{C}$-DNE$+\mathcal{C}$-DC^0 $\vdash\neg\mathcal{C}$-BI. (4) $\mathbf{EL}_0^-+(\mathcal{D},\mathsf{B}\exists^0\mathcal{C})$-BI$_M\vdash(\mathcal{D},\exists^0\mathcal{C})$-BI$_M$. (5) (i) $\mathbf{EL}_0^-+\mathcal{C}$-BI$_D\vdash\forall^0\mathcal{C}$-BI$_D$; (ii) $\mathbf{EL}_0^-+(\mathcal{C},\mathcal{D})$-BI$_M\vdash(\forall^0\mathcal{C},\mathcal{D})$-BI$_M$; and (iii) $\mathbf{EL}_0^-+\mathcal{C}$-BI $\vdash\forall^0\mathcal{C}$-BI.

Proof. (1) (i) Trivial. (ii) Easy by 2.10(d). (iii) Let $B[u]:\equiv(\exists x\le|u|)(\alpha(u\restriction x)=0)$. Then $\mathsf{Bar}[\underline{0},\{u\colon\alpha(u)=0\}]$ implies $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]$, and $B[u{*}\langle x\rangle]$ implies $B[u]\vee\alpha(u{*}\langle x\rangle)=0$ and $B[u]\vee A[u{*}\langle x\rangle]$ if $\forall u(\alpha(u)=0\to A[u])$. So $\forall x(B[u{*}\langle x\rangle]\vee A[u{*}\langle x\rangle])$ implies $B[u]\vee\forall xA[u{*}\langle x\rangle]$ and $B[u]\vee A[u]$ if $\forall xA[u{*}\langle x\rangle]\to A[u]$.

Let C be $\mathcal{C}$. (2) Assume $C[0]$ and $(\forall x<n)(C[x]\to C[x{+}1])$. Take $\alpha(u)=|u|\dot{-}n$ and $A[u]:\equiv C[n{-}|u|]$. (3) Assume (a) $\mathsf{Bar}[\underline{0},\{u\colon\neg C[u]\}]$ and (b) $\forall u(\forall x\neg C[u{*}\langle x\rangle]\to\neg C[u])$. Let $B[u,v]:\equiv C[v]\wedge u\subset v\wedge|v|=|u|{+}1$. By $\exists^0\mathcal{C}$-DNE with (b), $\forall u,v(B[u,v]\to\exists wB[v,w])$. If $C[\langle\,\rangle]$, as $\exists vB[\langle\,\rangle,v]$, $\mathcal{C}$-DC^0 yields α with $\forall nB[\alpha(n),\alpha(n{+}1)]$ and $\alpha(0)=\langle\,\rangle$ and, for $\beta(n):=(\alpha(n{+}1))(n)$, Δ_0^0-Ind shows $\alpha(n)=\beta\restriction n$ and so $\forall nC[\beta\restriction n]$ contradicting (a). (4) Let $B[u]:\equiv(\exists x,y<|u|)C[u\restriction y,x]$. Obviously $B[u]\to B[u{*}v]$. If $\mathsf{Bar}[\underline{0},\{u\colon\exists xC[u,x]\}]$ then $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]$, and $B[u]$ implies $\exists xC[u,x]$, if $\forall u,v(\exists xC[u,x]\to\exists xC[u{*}v,x])$. (5) (i) Let $[v]_0^2:=\langle(v(0))_0^2,\dots,(v(|v|{-}1))_0^2\rangle$ and $A[y,u]:\equiv C[[u]_0^2,((\langle y\rangle{*}u)(|u|))_1^2]$. If $\mathsf{Bar}[\underline{0},\{u\colon\alpha(u)=0\}]$ and $\forall u(\alpha(u)=0\to\forall zC[u,z])$ then $\mathsf{Bar}[\underline{0},\{u\colon\alpha([u]_0^2)=0\}]$ and $\forall u(\alpha([u]_0^2)=0\to A[y,u])$. If $\forall u(\forall x,zC[u{*}\langle x\rangle,z]\to\forall zC[u,z])$, then $\forall xA[y,u{*}\langle x\rangle]$, i.e., $\forall xC[[u]_0^2{*}\langle(x)_0^2\rangle,(x)_1^2]$ yields $\forall zC[[u]_0^2,z]$, and so $A[y,u]$. Thus $A[y,\langle\,\rangle]$ by $\mathcal{C}$-BI$_D$ for any y. Hence $\forall zC[\langle\,\rangle,z]$. (ii) (iii) Similar. ☐

Corollary 2.28. (1) $\mathbf{EL}_0+$MP $\vdash\Pi_1^0$-BI. (2) $\mathbf{EL}_0^-+\Sigma_n^0$-BI$_D\vdash\Pi_{n+1}^0$-Ind. (3) $\mathbf{EL}_0^-\vdash\mathcal{C}$-BI$_D\leftrightarrow(\mathcal{C},\Sigma_1^0)$-BI$_M$.

2.5.2 fan theorem

Definition 2.29 (Fan and $\mathcal{C}$-FT, $\mathcal{C}$-BFT, $\mathcal{C}$-WFT)**.** For a class $\mathcal{C}$ of formulae, define the following where $\mathsf{Fan}[\gamma] :\equiv \forall u(\gamma(u)=0 \to \exists x(\gamma(u*\langle x\rangle)=0) \wedge \exists n \forall x(\gamma(u*\langle x\rangle)=0 \to x<n))$ and $u<\beta :\equiv (\forall k<|u|)(u(k)<\beta(k))$.
($\mathcal{C}$-FT) $\mathsf{Fan}[\gamma] \wedge \mathsf{Bar}[\gamma,\{u\colon B[u]\}] \to \exists m \forall \alpha(\forall k(\gamma(\alpha\restriction k)=0) \to (\exists n<m)B[\alpha\restriction n])$;
($\mathcal{C}$-BFT) $\mathsf{Fan}[\gamma] \wedge \forall u(\gamma(u)=0 \to u<\beta) \wedge \mathsf{Bar}[\gamma,\{u\colon B[u]\}] \to \exists m \forall \alpha(\forall k(\gamma(\alpha\restriction k)=0) \to (\exists n<m)B[\alpha\restriction n])$;
($\mathcal{C}$-WFT) $(\forall \alpha<\underline{2})\exists n B[\alpha\restriction n] \to \exists m(\forall \alpha<\underline{2})(\exists n<m)B[\alpha\restriction n]$, all for B from $\mathcal{C}$.

$\mathcal{C}$-WFT consists of the instances of $\mathcal{C}$-FT with γ defined by $\gamma(u)=0$ iff $u<\underline{2}$. This is a classical contrapositive of weak König's lemma. 2.20 is enhanced as (1) in the following (cf. [37, X.4 and IV.1.4]).

Lemma 2.30. (1) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{LEM}+\Sigma^0_1\text{-}\mathsf{Bdg}+\Delta^0_0\text{-}\mathsf{AC}^{00}+\Delta^0_0\text{-}\mathsf{BFT}$ is interpreted by $\mathfrak{g}$ in $\mathbf{WKL}_0^*$.
(2) (i) $\mathbf{EL}_0^-+(\exists^0\forall^0\mathsf{B}\exists^0\mathcal{C},\mathsf{B}\exists^0\mathcal{C})\text{-}\mathsf{BI}_M \vdash \mathcal{C}\text{-}\mathsf{FT}$; and (ii) $\mathbf{EL}_0^-+(\exists^0\mathsf{B}\forall^0\mathsf{B}\exists^0\mathcal{C},\mathsf{B}\exists^0\mathcal{C})\text{-}\mathsf{BI}_M \vdash \mathcal{C}\text{-}\mathsf{WFT}$.
(3) (i) $\mathbf{EL}_0^-+\mathsf{B}\exists^0\mathcal{C}\text{-}\mathsf{FT}\vdash\exists^0\mathcal{C}\text{-}\mathsf{FT}$ and (ii) $\mathbf{EL}_0^-+\mathsf{B}\exists^0\mathcal{C}\text{-}\mathsf{BFT}\vdash\exists^0\mathcal{C}\text{-}\mathsf{BFT}$. (4) $\mathbf{EL}_0^-+\mathsf{B}\forall^0\mathcal{C}\text{-}\mathsf{WFT}\vdash\mathcal{C}\text{-}\mathsf{Bdg}$.

Proof. Take C from $\mathcal{C}$. (2) (i) Let $A[u,\gamma] :\equiv \exists n \forall v((\forall k\le|v|)(\gamma(u*(v\restriction k))=0) \wedge |v|=n \to (\exists k<|u*v|)C[(u*v)\restriction k]))$ from $\exists^0\forall^0\mathsf{B}\exists^0\mathcal{C}$. If $\mathsf{Fan}[\gamma]$ then $\forall^0\mathsf{B}\exists^0\mathcal{C}\text{-}\mathsf{Bdg}$, which is by 2.27(2), yields $\forall x A[u*\langle x\rangle] \to A[u]$. (ii) Similar.
(3) Let $B[u] :\equiv (\exists x,k<|u|)C[u\restriction k,x]$. As $\exists k,x C[\alpha\restriction k,x] \to \exists n B[\alpha\restriction n]$, if $\mathsf{Bar}[\gamma,\{u\colon \exists x C[u,x]\}]$ then $\mathsf{Bar}[\gamma,\{u\colon B[u]\}]$.
(4) Let $B[u,m] :\equiv |u|\ge m \wedge (\forall x<m)(u\restriction x=\underline{0}\restriction x \wedge u(x)>0 \to C[x,|u|-m])$. Then $(\forall x<m)\exists y C[x,y]$ implies $(\forall \alpha<\underline{2})\exists k B[\alpha\restriction k,m]$ and, by $\mathsf{B}\forall^0\mathcal{C}\text{-}\mathsf{WFT}$, $\exists n(\forall \alpha<\underline{2})(\exists k<n)B[\alpha\restriction k,m]$, i.e., $\exists n(\forall x<m)(\exists y<n)C[x,y]$. ☐

Thus, classically Σ^0_1-BFT is finitistically justifiable. This is optimal in the sense that $(\mathbf{ACA}_0)^{\mathfrak{ch}}$ classically follows from Π^0_1-WFT as shown in [7] (cf. 4.7(1)); and from Δ^0_0-FT as in [37, Theorem III.7.2] (cf. 4.8(1)). Though [37, Theorem III.7.2] relies on Σ^0_1-Ind, it does not matter as seen in the next proposition.

Proposition 2.31. $\mathbf{EL}_0^-+\Delta^0_0\text{-}\mathsf{FT}$ proves Σ^0_1-Ind.

Proof. Let C be Δ^0_0. Assume $\exists y C[0,y]$ and $(\forall x<n)(\exists y C[x,y] \to \exists y C[x{+}1,y])$. With Δ^0_0-LNP, by replacing $C[x,y]$ with $C[x,y] \wedge (\forall z<y)\neg C[x,z]$ we may assume $(C[x,y]\wedge C[x,z]) \to y=z$. Define γ and $B[x,u]$ by

$$\gamma(u)=0 \leftrightarrow (\forall k<|u|)(u(k)\neq 0 \to k\le n \wedge (\forall l\le k)(u(l)\neq 0 \wedge C[l,u(l)-1])); \qquad B[x,u] :\equiv |u|>u\restriction(x{+}1).$$

Assume $\gamma(u)=0$. If $(\forall k<|u|)(u(k)\neq 0) \wedge |u|\le n$, then $\forall x(\gamma(u*\langle x\rangle)=0 \leftrightarrow x=0 \vee x=y{+}1)$ for y with $C[|u|,y]$, yielded by $C[|u|-1,u(|u|-1)-1]$ if $|u|>0$; otherwise $\forall x(\gamma(u*\langle x\rangle)=0 \leftrightarrow x=0)$. So $\mathsf{Fan}[\gamma]$. As $\forall \alpha \exists m B[n,\alpha\restriction m]$, Δ^0_0-FT yields m with $\forall \alpha(\forall k(\gamma(\alpha\restriction k)=0) \to (\exists k<m)B[n,\alpha\restriction k])$. By Δ^0_0-Ind on $k\le n{+}1$ we prove $(\exists u<m)D[k,u]$ for $D[k,u] :\equiv |u|=k \wedge (k\neq 0 \to u(k-1)\neq 0) \wedge \gamma(u)=0$. If $D[k,v]$, the assumption yields y with $C[k,y]$; then $D[k{+}1,u]$ for $u:=v*\langle y{+}1\rangle$, and $(\exists k<m)B[n,(u*\underline{0})\restriction k]$ which implies $u\le(u*\underline{0})\restriction(n{+}1)<k<m$. ☐

2.5.3 (weak) continuity principles

Definition 2.32 ($\mathcal{C}$-WC^i, $\mathcal{C}$-WC!^{i})**.** For a class $\mathcal{C}$ of formulae and $i=0,1$, $\mathcal{C}$-WC^i is defined as follows, and $\mathcal{C}$-WC!^{i} is defined with $\exists$ replaced by $\exists!$ in the premises.
($\mathcal{C}$-WC^0) $\forall \alpha \exists x A[\alpha,x] \to \forall \alpha \exists x,m \forall \beta A[(\alpha\restriction m)*\beta,x]$ for A from $\mathcal{C}$;
($\mathcal{C}$-WC^1) $\forall \alpha \exists \gamma A[\alpha,\gamma] \to \forall \alpha \exists \gamma(A[\alpha,\gamma] \wedge \forall n \exists m \forall \beta \exists \delta A[(\alpha\restriction m)*\beta,(\gamma\restriction n)*\delta])$ for A from $\mathcal{C}$. [13]

We can see that $(\exists^1\mathcal{C})$-WC!^{0} implies $(\exists^1\mathcal{C})$-WC!^{1}, by considering $A[\alpha,x,n] :\equiv \exists\gamma(B[\alpha,\gamma]\wedge\gamma\restriction n=x)$. Thus, with 2.33(1)(iii) below, $\mathcal{L}_{\mathrm{F}}$-WC!^{1} and $\mathcal{L}_{\mathrm{F}}$-WC!^{0} are equivalent.[14] Informally this is an easy consequence of the universality (in the sense of category theory) of the product topology with which Baire space is equipped.

$\mathcal{C}$-WC^i asserts the existence of a continuous *branch cut*, not the continuity of all branch cuts. We cannot show the equivalence between $\mathcal{L}_{\mathrm{F}}$-WC^1 and $\mathcal{L}_{\mathrm{F}}$-WC^0, because of the results mentioned in f.n.14.

By 2.14, Σ^0_1-WC!^{1} is vacuous and $\mathbf{EL}_0^- \vdash \Sigma^0_1$-WC^1. Classically this is optimal by 2.33(2)(ii) below.

[13] As $\mathcal{L}_{\mathrm{F}}$-WC^1 has turned out to be refuted by Kripke's schema (KS) (see, e.g., [14, p.246]), a formalization of creative subject (CS), its status as an axiom of INT is questionable. Though Vesley [44] proposed an alternative formalization consistent with $\mathcal{L}_{\mathrm{F}}$-WC^1, it does not seem to represent any informal idea of CS but just a technical substitute for KS in a similar way as WFT is a substitute for BI. (Namely, it follows from KS and suffices for concrete uses of CS by Brouwer.) Once $\mathcal{L}_{\mathrm{F}}$-WC^1 thus becomes doubtful, we can no longer fully trust $\mathcal{L}_{\mathrm{F}}$-WC^0, because any argument for the latter, basically appealing to the meaning of $\exists$ in Intuitionism, cannot avoid the former. This is one reason why we take only $\mathcal{C}$-WC!^{i} in Figures 1 and 2 (see also f.n.11). However, for us it matters only when we discuss which axioms characterize INT (to be weakened for our purpose), and we can use models (or interpretations) satisfying $\mathcal{L}_{\mathrm{F}}$-WC^i: as declared in f.n.5 we confine our study to "objective Intuitionism".

[14] Hence the consistency of $\mathcal{L}_{\mathrm{F}}$-WC!^{1} with Kripke's schema follows from that of $\mathcal{L}_{\mathrm{F}}$-WC^0, which is known.

Lemma 2.33. (1) Over $\mathbf{EL}_0^-$, (i) Σ^0_1-WC^1; (ii) $\mathcal{C}$-WC^1 implies $\mathcal{C}$-WC^0; and (iii) $(\mathcal{C}\wedge\Pi^0_1)$-$\mathsf{WC!}^1$ implies $\mathcal{C}$-$\mathsf{WC!}^0$. (2) (i) $\mathbf{EL}_0^-$+Π^0_1-WC^0+LLPO is inconsistent; and (ii) $\mathbf{EL}_0^-$+Π^0_1-$\mathsf{WC!}^0$+LPO is inconsistent.

Proof. (1) As (ii) is easier, we prove (iii). For A from $\mathcal{C}$, let $B[\alpha,\gamma]:\equiv A[\alpha,\gamma(0)]\wedge\gamma\ominus 1=\underline{0}$. Then $\forall\alpha\exists!xA[\alpha,x]$ implies $\forall\alpha\exists!\gamma B[\alpha,\gamma]$. $(\mathcal{C}\wedge\Pi^0_1)$-$\mathsf{WC!}^1$ yields $\forall\alpha\exists\gamma,m(B[\alpha,\gamma]\wedge\forall\beta\exists\delta B[(\alpha{\restriction}m)*\beta,(\gamma{\restriction}1)*\delta])$.
(2)(i) Let $A[\alpha,i]:\equiv\exists n(\alpha{\restriction}n=\underline{0}{\restriction}n\wedge\alpha(n)>0\wedge n=2\cdot\lfloor n/2\rfloor+i)$. As $\neg(A[\alpha,0]\wedge A[\alpha,1])$, by LLPO, $\forall\alpha\exists i\neg A[\alpha,i]$. Π^0_1-WC^0 yields i and n with $\forall\beta\neg A[(\underline{0}{\restriction}n)*\beta,i]$. Thus $\neg A[(\underline{0}{\restriction}n)*\underline{1},i]\wedge\neg A[(\underline{0}{\restriction}(n+1))*\underline{1},i]$, a contradiction.
(ii) Let $A[\alpha,n]:\equiv(n=0\rightarrow\alpha=\underline{0})\wedge(n>0\rightarrow\alpha(n-1)>0\wedge\alpha{\restriction}(n-1)=\underline{0}{\restriction}(n-1))$. LPO and Δ^0_0-LNP imply $\forall\alpha\exists!nA[\alpha,n]$. Π^0_1-$\mathsf{WC!}^0$, applied to $\underline{0}$, leads a contradiction similarly. ☐

2.5.4 summary: maximal fragments in the classical setting

Proposition 2.34. (i) $\mathbf{EL}_0^-$+$\mathcal{L}_{\mathrm{F}}$-LEM+$\Sigma^0_1\{$-Bdg,-BFT,-AC^{00},-AC^{01},-WC^0,-$\mathsf{WC}^1\}$ is interpreted by $\mathfrak{g}$ in $\mathbf{WKL}_0^*$; and (ii) so is $\mathbf{EL}_0$+$\mathcal{L}_{\mathrm{F}}$-LEM+$\Pi^0_1\{$-BI,-$\mathsf{Ind}\}$+$\Sigma^0_1\{$-DC^1,-DC^0,-Ind,-BFT,-AC^{00},-AC^{01},-WC^0,-$\mathsf{WC}^1\}$ in $\mathbf{WKL}_0$.

Proof. Σ^0_1-AC^{00},-DC^0 yield Σ^0_1-AC^{01},-DC^1 by 2.14. The rest is by 2.30(1)(3), 2.16(2)(i)(4), 2.28(1), 2.33(1). ☐

These fragments are optimal (in the classical setting) in the following sense: Δ^0_0-DC^i yields Σ^0_1-Ind by 2.16(2)(i)(3)(i); Σ^0_1-$\mathsf{DC!}^1$ is vacuous by 2.14; Π^0_1-$\mathsf{AC!}^{00}$, Π^0_1-BFT and Δ^0_0-FT imply $(\mathbf{ACA}_0)^{\mathfrak{ch}}$ where all Π^0_1-$\mathsf{DC!}^1$, Π^0_1-$\mathsf{DC!}^0$ and Π^0_1-$\mathsf{AC!}^{01}$ imply Π^0_1-$\mathsf{AC!}^{00}$ by 2.16(2)(v)(vi); and Π^0_1-$\mathsf{WC!}^0$ is inconsistent by 2.33(2)(ii).

One of our main results is that for this optimality LPO suffices instead of the full classical logic or $\mathcal{L}_{\mathrm{F}}$-LEM.

2.5.5 continuous choice and remarks on choice axioms along functions

Notation 2.35. $\alpha=\beta|\gamma$ denotes a Π^0_2 formula $\forall x\exists y(\beta(\langle x\rangle*(\gamma{\restriction}y))=\alpha(x)+1\wedge(\forall z<y)(\beta(\langle x\rangle*(\gamma{\restriction}z))=0))$.

Definition 2.36 (generalized continuous choice/bounding; $\mathcal{C}$-CC^i, $\mathcal{C}$-CB^i and $\mathcal{C}$-$\mathsf{CC!}^i$)**.** For classes $\mathcal{C}$ and $\mathcal{D}$ of formulae, define the following axiom schemata where A is any from $\mathcal{C}$ and B from $\mathcal{D}$.

(($\mathcal{C},\mathcal{D}$)-GCC^0) $\forall\alpha(B[\alpha]\rightarrow\exists xA[\alpha,x])\rightarrow\exists\gamma\forall\alpha(B[\alpha]\rightarrow\exists\delta(\delta=\gamma|\alpha\wedge A[\alpha,\delta(0)]))$;
(($\mathcal{C},\mathcal{D}$)-GCB^0) $\forall\alpha(B[\alpha]\rightarrow\exists xA[\alpha,x])\rightarrow\exists\gamma\forall\alpha(B[\alpha]\rightarrow\exists\delta(\delta=\gamma|\alpha\wedge(\exists y<\delta(0))A[\alpha,y]))$;
(($\mathcal{C},\mathcal{D}$)-GCC^1) $\forall\alpha(B[\alpha]\rightarrow\exists\beta A[\alpha,\beta])\rightarrow\exists\gamma\forall\alpha(B[\alpha]\rightarrow\exists\delta(\delta=\gamma|\alpha\wedge A[\alpha,\delta]))$;
(($\mathcal{C},\mathcal{D}$)-GCB^1) $\forall\alpha(B[\alpha]\rightarrow\exists\beta A[\alpha,\beta])\rightarrow\exists\gamma\forall\alpha(B[\alpha]\rightarrow\exists\delta(\delta=\gamma|\alpha\wedge(\exists\beta<\delta)A[\alpha,\beta]))$.

($\mathcal{C},\mathcal{D}$)-$\mathsf{GCC!}^i$ is defined with $\exists$ replaced by $\exists!$ in the premise; $\mathcal{C}$-CC^i, $\mathcal{C}$-CB^i and $\mathcal{C}$-$\mathsf{CC!}^i$ are by setting $B\equiv\top$.

$\mathcal{C}$-CC^1 could be seen as the conjunction of $\mathcal{C}$-AC^{11} the axiom of function-function choice for $\mathcal{C}$ properties and $\mathcal{C}$-$\mathsf{CC!}^1$ asserting that any $\mathcal{C}$-definable functional is represented as $\alpha\mapsto\gamma|\alpha$ for some γ.

Even while $\mathcal{C}$-AC^{1i}'s are not formalizable in our $\mathcal{L}_{\mathrm{F}}$, it is plausible to think: (1) $\mathcal{C}$-$\mathsf{AC!}^{1i}$ implies $\mathcal{C}$-$\mathsf{AC!}^{0i}$; and (2) $\mathcal{C}$-AC^{1i}'s follow from $\mathcal{C}$-CC^i and $\mathcal{C}$-$\mathsf{AC!}^{1i}$'s from $\mathcal{C}$-$\mathsf{CC!}^i$ if all the classes in the axioms of the system are closed under Σ^0_1 definable total functions. For, "imaginary" choice functionals would be of the base complexity but, for (2), be coded by $\alpha|\beta$, which is Σ^0_1 definable as far as $(\alpha|\beta){\downarrow}$. As Σ^0_1-AC^{00} makes $\mathbf{EL}_0^-$ satisfy this condition by overwriting 2.10(d), we can "imaginarily" evaluate the strength of $\mathcal{C}$-AC^{1i}, by that of $\mathcal{C}$-CC^i+Σ^0_1-AC^{00} from above and $\mathcal{C}$-$\mathsf{AC!}^{0i}$ from below. We *could* thus add $\mathcal{L}_{\mathrm{F}}$-AC^{1i}'s (as we can add $\mathcal{L}_{\mathrm{F}}$-CC^i) in 1.4; Σ^0_1-AC^{1i}'s in 2.5.4 by 2.37(1); and claim that LPO+Π^0_1-$\mathsf{AC!}^{1i}$'s are non-justifiable by 4.8(iii) and 2.16(vi).

Similarly, we could consider that $\mathcal{C}$-AC^{10} (and so $\mathcal{C}$-AC^{11}) makes $\mathcal{C}$-WC^0 and $\mathcal{C}$-$\mathsf{WC!}^0$ be equivalent. From 2.33(2)(i) we *could* claim that neither Π^0_1-AC^{11} nor Π^0_1-AC^{10} can be added to the combination of Brouwerian axioms finitistically justifiable or guaranteed jointly with LLPO, while $\mathcal{L}_{\mathrm{F}}$-$\mathsf{AC!}^{1i}$ can with Σ^0_1-GDM and MP.

Lemma 2.37. (1) $\mathbf{EL}_0^-\vdash\Sigma^0_1$-$\mathsf{CC}^1$. (2) Over $\mathbf{EL}_0^-$, (i) $\mathcal{C}$-CC^1 implies $\mathcal{C}$-WC^0; (ii) $(\mathcal{C}\wedge\Pi^0_1)$-$\mathsf{CC!}^1$ implies $\mathcal{C}$-$\mathsf{WC!}^0$. (3) Over $\mathbf{EL}_0^-$+Σ^0_1-Bdg, (i) $\mathcal{C}$-CC^1 yields $\mathcal{C}$-WC^1; (ii) $\mathcal{C}$-$\mathsf{CC!}^1$ yields $\mathcal{C}$-$\mathsf{WC!}^1$. (4) $\mathbf{EL}_0^-$+$\mathcal{D}$-CB^0+$\mathcal{C}$-$\mathsf{BI}_D\vdash(\mathcal{C},\mathcal{D})$-$\mathsf{BI}_M$.

Proof. (1) For A from Σ^0_1, let $\forall\alpha\exists\beta A[\alpha,\beta]$. Take D by 2.14 and γ as follows. Then $\forall\alpha\exists\beta(\beta=\gamma|\alpha\wedge A[\alpha,\beta])$.

$$\gamma(y)=\begin{cases}(v*\underline{0})(z)+1 & \text{if } y=\langle z\rangle*u \text{ and if } v:=|u| \text{ satisfies } D[u{\restriction}|v|,v]\\ 0 & \text{otherwise.}\end{cases}$$

(2) Easy. (3) Let $C[x,y]:\equiv\gamma(\langle x\rangle*(\alpha{\restriction}y))>0\wedge(\forall z<y)(\gamma(\langle x\rangle*(\alpha{\restriction}y))=0)$. If $\exists\delta(\delta=\gamma|\alpha)$, then $(\forall x<n)\exists yC[x,y]$ and Σ^0_1-Bdg yields m with $(\forall x<n)(\exists y<m)C[x,y]$. Then $\forall\beta(\beta{\restriction}m=\alpha{\restriction}m\rightarrow(\gamma|\beta){\restriction}n=(\gamma|\alpha){\restriction}n)$.
(4) Let B from $\mathcal{D}$ and assume $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]\equiv\forall\alpha\exists nB[\alpha{\restriction}n]$. $\mathcal{D}$-CB^0 yields γ with $\forall\alpha(\exists k<(\gamma|\alpha)(0))B[\alpha{\restriction}k]$. Define $\beta(u)=0\leftrightarrow(\exists k\le|u|)(\gamma(\langle 0\rangle*(u{\restriction}k))\neq 0\wedge|u|\ge\gamma(\langle 0\rangle*(u{\restriction}k))-1)$. Then $\mathsf{Bar}[\underline{0},\{u\colon\beta(u)=0\}]$. For any α, there is n with $\beta(\alpha{\restriction}n)=0$, which implies $n\ge(\gamma|\alpha)(0)$ and so $B[\alpha{\restriction}n]$, if $\forall u,v(B[u]\rightarrow B[u*v])$ holds. ☐

2.5.6 remarks on axiom schemata for decidable properties

In the context of Intuitionism, one of the most important constraints on properties is decidability: A is called *decidable* or *detachable* if $\forall x(A[x] \vee \neg A[x])$. In other words, we can decide, for any x, if $A[x]$ holds or not.

This is not syntactical and so inadequate for our way of defining axiom schemata, similarly to the non-syntactical constraints Δ^0_{n+1} in classical arithmetic. For, it might be the case that $\forall x(A[x,y] \vee \neg A[x,y])$ holds for some y but $\forall x(A[x,z] \vee \neg A[x,z])$ does not for another z. Thus the constraint is on the abstract $\{x\colon A[x,y]\}$ rather than on the formula A, as the constraint Bar (Def.2.25), where an abstract $\{\vec{x}\colon B[\vec{x},\vec{y}]\}$ is a formula $B[\vec{x},\vec{y}]$ with designated free variables $\vec{x}$. By 2.8(1), Δ^0_0 abstracts are decidable, but not vice versa.

Below are some related schemata, where D, E and U stand for 'decidable', 'existential' and 'universal' respectively. In some literature MP and LLPO refer to $\mathsf{E\text{-}DNE}$ and $\mathsf{E\text{-}GDM}$ (restricted to $z=2$) respectively.

Definition 2.38. $A[\vec{x},\vec{y}]$ is called *decidable in* $\vec{x}$ if $\mathsf{D}[\{\vec{x}\colon A[\vec{x},\vec{y}]\}] :\equiv \forall\vec{x}(A[\vec{x},\vec{y}] \vee \neg A[\vec{x},\vec{y}])$.

($\mathsf{E\text{-}DNE}$) $\mathsf{D}[\{x\colon A[x]\}] \to (\neg\neg\exists x A[x] \to \exists x A[x])$;

($\mathsf{E\text{-}GDM}$) $\mathsf{D}[\{x,y\colon A[x,y,z]\}] \to (\neg(\forall x<z)\exists y A[x,y,z] \to (\exists x<z)\forall y \neg A[x,y,z])$;

($\mathsf{EU\text{-}Ind}$) $\mathsf{D}[\{x,y,z\colon A[x,y,z]\}] \wedge \exists y\forall z A[0,y,z] \wedge (\forall x<n)(\exists y\forall z A[x,y,z] \to \exists y\forall z A[x{+}1,y,z]) \to \exists y\forall z A[n,y,z]$;

($\mathsf{U\text{-}BI}$) $\mathsf{D}[\{u,y\colon B[u,y]\}] \wedge \mathsf{Bar}[\underline{0},\{u\colon \forall y B[u,y]\}] \wedge \forall u(\forall x,y B[u*\langle x\rangle,y] \to \forall y B[u,y]) \ \to\ \forall y B[\langle\,\rangle,y]$.

In what follows, however, we will not consider these for the following reason. In the upper bound proofs, we always have full choice $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{AC}^{00}$, with which decidable properties are equivalently Δ^0_0, i.e., $\mathsf{D}(\{x\colon A[x,\vec{y}]\})$ implies $\exists\alpha\forall x(\alpha(x)=0 \leftrightarrow A[x,\vec{y}])$. For lower bounds, we can obtain all the expected results for the corresponding weaker syntactical classes (e.g., Δ^0_0 instead of D, Σ^0_1 instead of E). Thus our results for syntactic classes can automatically be enhanced for these schemata. So the schemata listed above (as well as $\mathsf{EU\text{-}DC}^0$ and $\mathsf{E\text{-}DC}^1$ defined similarly) are all finitistically justifiable jointly with $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{AC}^{0i}$ $(i=0,1)$, $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{FT}$ and $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{CC!}^1$.

3 Upper Bounds: Functional Realizability

3.1 Preliminaries for upper bound proofs

We will need two equivalences, which are among the folklore in classical second order arithmetic. We here sharpen these in the intuitionistic context (Cor.3.3 and Cor.3.9) with some related fundamental results.

3.1.1 bounded comprehension

The first equivalence to be sharpen is between induction and bounded comprehension. This was mentioned in [37, Exercise II.3.13]. For this equivalence, we need a semi-classical principle. For the equivalence in the purely intuitionistic setting, we need to replace the induction $\mathcal{C}\text{-}\mathsf{Ind}$ by the least number principle $\mathcal{C}\text{-}\mathsf{LNP}$.

Definition 3.1 ($\mathcal{C}\text{-}\mathsf{BCA}$, $\Delta^0_0(\mathcal{C})$, $\Sigma^0_1(\mathcal{C})$, $\Pi^0_1(\mathcal{C})$)**.** For a class $\mathcal{C}$ of formulae,

($\mathcal{C}\text{-}\mathsf{BCA}$) $\exists u(|u|=n \wedge (\forall k<n)(u(k)=0 \leftrightarrow A[k]))$ for any A from $\mathcal{C}$.

$\Delta^0_0(\mathcal{C})$ denotes the smallest class $\mathcal{D} \supseteq \mathcal{C}$ closed under $\wedge, \vee, \to, \mathsf{B}\exists^0, \mathsf{B}\forall^0$. $\Sigma^0_1(\mathcal{C}) \equiv \exists^0\Delta^0_0(\mathcal{C})$ and $\Pi^0_1(\mathcal{C}) \equiv \forall^0\Delta^0_0(\mathcal{C})$.

Lemma 3.2. (i) $\mathbf{EL}^-_0 + \mathsf{B}\forall^0\mathcal{C}\text{-}\mathsf{LNP}$ proves $\mathcal{C}\text{-}\mathsf{BCA}$; (ii) $\mathbf{EL}^-_0 + \mathcal{C}\text{-}\mathsf{BCA}$ proves $\mathcal{C}\text{-}\mathsf{Ind}$, $\mathcal{C}\text{-}\mathsf{LEM}$ and $\mathcal{C}\text{-}\mathsf{LNP}$; and (iii) $\mathbf{EL}^-_0 + \mathcal{C}\text{-}\mathsf{BCA}$ proves $\Delta^0_0(\mathcal{C})\text{-}\mathsf{BCA}$. Hence (iv) $\mathcal{C}\text{-}\mathsf{BCA}$ and $\mathsf{B}\forall^0\mathcal{C}\text{-}\mathsf{LNP}$ are equivalent over $\mathbf{EL}^-_0$.

Proof. Let A be from $\mathcal{C}$. (i) We may assume $|u| \le |v| \wedge (\forall k<|u|)(u(k) \le v(k)) \to u \le v$ by changing way of coding if necessary. Let $B[u] :\equiv |u|=n \wedge (\forall k<n)(u(k)=0 \to A[k])$ which is $\mathsf{B}\forall^0\mathcal{C}$ (cf. Notation 2.9(3)). $\mathsf{B}\forall^0\mathcal{C}\text{-}\mathsf{LNP}$ yields v with $B[v] \wedge (\forall u<v)\neg B[u]$. It remains to show $(\forall k<n)(A[k] \to v(k)=0)$. For $k<n$ with $A[k]$, if $v(k)\neq 0$, then u defined by $u(k)=0$ and $u(l)=v(l)$ for $l\neq k$ satisfies $u<v$ and $B[u]$, a contradiction.
(ii) $\mathcal{C}\text{-}\mathsf{BCA}$ yields u with $(\forall x\le n)(u(x)=0 \leftrightarrow A[x])$. If $A[0]$ and $(\forall x<n)(A[x] \to A[x{+}1])$, then $u(0)=0$ and $(\forall x<n)(u(x)=0 \to u(x{+}1)=0)$ which, with $\Delta^0_0\text{-}\mathsf{Ind}$, yields $u(n)=0$ and so $A[n]$. The others are similar.
(iii) We show $\exists u(|u|=n \wedge (\forall k<n)(u(x)=0 \leftrightarrow A[(x)^k_0,...,(x)^k_{k-1}]))$ by induction on A. Consider the case of $A[\vec{x}] \equiv (Qy<t[\vec{x}])B[\vec{x},y]$. The induction hypothesis yields v with $(\forall z<|v|)(v(z)=0 \leftrightarrow B[(z)^{k+1}_0,...,(z)^{k+1}_k])$ and $|v|=(n,t[(n)^k_0,...,(n)^k_{k-1}])$. Then $(\forall x<n)(\forall y<(|v|)^2_1)(v(((x)^k_0,...,(x)^k_{k-1},y))=0 \leftrightarrow B[(x)^k_0,...,(x)^k_{k-1},y])$. Take u with $(\forall x<n)(u(x)=0 \leftrightarrow (Qy<t[(x)^k_0,...,(x)^k_{k-1}])v((x)^k_0,...,(x)^k_{k-1},y)=0)$. This is what we need. $\square$

Corollary 3.3. (1) $\mathbf{EL}_0^- \vdash \Pi^0_n\textsf{-BCA} \leftrightarrow \Pi^0_n\textsf{-LNP}$. (2) $\mathbf{EL}_0^- + \Sigma^0_n\textsf{-Ind} + \Sigma^0_{n+1}\textsf{-DNE} \vdash \Sigma^0_n\textsf{-BCA} \wedge \Delta^0_0(\Sigma^0_n)\textsf{-Ind}$.
(3) $\mathbf{EL}_0^- + \Sigma^0_n\textsf{-Ind} + \Sigma^0_n\textsf{-LEM} \subseteq \mathbf{EL}_0^- + \Sigma^0_n\textsf{-BCA} \subseteq \mathbf{EL}_0^- + \Sigma^0_n\textsf{-Ind} + \Sigma^0_{n+1}\textsf{-DNE}$.

Proof. (2) We have $\mathsf{B}\forall^0\neg\Pi^0_n \subseteq \Sigma^0_n$ by 2.8(3) and 2.24(1)(i), and $\mathsf{B}\exists^0(\Pi^0_n \wedge \mathsf{B}\forall^0\neg\Pi^0_n) \subseteq \Sigma^0_{n+1}$. By 2.8(2)(i), $\mathbf{EL}_0^- + \Sigma^0_n\textsf{-Ind} + \Sigma^0_{n+1}\textsf{-DNE}$ proves $\Pi^0_n\textsf{-LNP}$ and so $\Pi^0_n\textsf{-BCA}$ which with $\Sigma^0_n\textsf{-DNE}$ implies $\Sigma^0_n\textsf{-BCA}$. ☐

The statements (2) and (3) refine the corresponding classical results: $\Sigma^0_n\textsf{-Ind}$ implies $\Delta^0_0(\Sigma^0_n)\textsf{-Ind}$ (e.g., [17, Ch.I, 2.14 Lemma]); and $\Sigma^0_n\textsf{-Ind}$ is equivalent to $\Sigma^0_n\textsf{-BCA}$. Since $\Sigma^0_n\textsf{-BCA}$ easily follows from $\mathcal{L}_{\mathrm{F}}\textsf{-Ind} + \Sigma^0_n\textsf{-LEM}$, in the usual intuitionistic context with full induction, $\Sigma^0_n\textsf{-BCA}$ is equivalent to $\Sigma^0_n\textsf{-LEM}$. In our context however we need some trick to adjust the proof above to Σ^0_n to show this (cf. [27, Lemma 37]) while we saw that it is equivalent to $\mathsf{B}\Pi^0_{n+1}\textsf{-LNP}$, to $\Delta^0_0(\Sigma^0_n)\textsf{-LNP}$ and to $\Pi^0_n\textsf{-LNP} + \Sigma^0_n\textsf{-DNE}$. As $\mathcal{L}_{\mathrm{F}}\textsf{-Ind} + \Sigma^0_n\textsf{-LEM}$ is known not to imply $\Sigma^0_{n+1}\textsf{-DNE}$ (by [1]), the second $\subseteq$ in (3) is proper. We do not know if so is the first.

Our proof refines [17, Ch.I, 2.13 Lemma] and differs from that suggested in [37]. The latter proof is based on *pigeon-hole principle* (PHP), and does not solve the question above either. Whereas we applied the least number principle to sequence u's or large numbers in the sense of 1.8, in the proof by PHP the induction is applied to k's with $k < |u|$ or small numbers.[15] Thus the difference between these two proofs could be essential in the further studies mentioned in 1.8,[16] but not so essential for the purpose of the present paper.

3.1.2 bounded König's lemma

The other equivalence is between weak König's lemma (WKL) and Π^0_1 axiom of choice (for sets). The implication from the former to the latter was in [25, Lemma 3.6], and the converse is a consequence of the equivalence known in constructive reverse mathematics (e.g., [18, Proposition 16.18 and Theorem 16.21]).

Definition 3.4 ($u<\alpha$, $\mathcal{C}$-BKL, $\mathcal{C}$-WKL)**.** Let $u < \alpha :\equiv (\forall k < |u|)(u(k) < \alpha(k))$. For a class $\mathcal{C}$ of formulae,
($\mathcal{C}$-BKL) $\forall n(\exists u < \alpha)(|u| = n \wedge (\forall k \le n)A[u{\restriction}k]) \rightarrow (\exists\gamma < \alpha)\forall n A[\gamma{\restriction}n]$ for any A from $\mathcal{C}$;
($\mathcal{C}$-WKL) $\forall n(\exists u < \underline{2})(|u| = n \wedge (\forall k \le n)A[u{\restriction}k]) \rightarrow (\exists\gamma < \underline{2})\forall n A[\gamma{\restriction}n]$ for any A from $\mathcal{C}$.

Lemma 3.5. (1) For A from $\forall^0\mathcal{C}$ there is a formula B from $\mathsf{B}\forall^0\mathcal{C}$ such that (a) $\forall n B[\beta{\restriction}n] \rightarrow \forall n A[\beta{\restriction}n]$ and that (b) $\forall n(\exists u < \alpha)(|u| = n \wedge (\forall k \le n)A[u{\restriction}k]) \rightarrow \forall n(\exists u < \alpha)(|u| = n \wedge (\forall k \le n)B[u{\restriction}k])$.
(2) Therefore, over $\mathbf{EL}_0^- + \mathsf{B}\forall^0\mathcal{C}\textsf{-BKL}$, (i) $\forall^0\mathcal{C}\textsf{-BKL}$ holds; (ii) $(\exists\beta < \alpha)\forall n A[\beta{\restriction}n]$ is $\forall^0(\mathsf{B}\exists^0\mathsf{B}\forall^0\mathcal{C})$ if A is $\forall^0\mathcal{C}$.
(3) $\mathbf{EL}_0^- + \mathcal{D}\textsf{-BKL} + \mathsf{B}\exists^0\mathcal{D}\textsf{-Ind} + \mathsf{B}\exists^0\mathsf{B}\forall^0\neg\mathcal{C}\textsf{-LEM} + \mathsf{B}\exists^0\mathcal{C}\textsf{-DNE} \vdash \mathcal{C}\textsf{-BFT}$ for $\mathcal{D} \equiv \mathsf{B}\forall^0(\mathcal{C} \rightarrow \mathsf{B}\exists^0\mathcal{C})$.

Proof. Let C be $\mathcal{C}$. (1) Say $A[u] \equiv \forall x C[u, x]$. Define $B[u] :\equiv (\forall x, k < |u|)C[u{\restriction}k, x]$. For (a), if $\forall n B[\beta{\restriction}n]$, then, for n and x, $B[\beta{\restriction}(n{+}x{+}1)]$ implies $C[\beta{\restriction}n, x]$. As $\forall u((\forall k \le |u|)A[u{\restriction}k] \rightarrow (\forall k \le |u|)B[u{\restriction}k])$, (b) holds.
(3) Let $D[v] :\equiv (\exists k \le |v|)C[v{\restriction}k]$ and $B[u] :\equiv \gamma(u) = 0 \wedge (\forall v < \beta)(\gamma(v) = 0 \wedge |v| = |u| \wedge D[u] \rightarrow D[v])$ from $\mathcal{D}$. Assume $\mathsf{Fan}[\gamma]$, $\mathsf{Bar}[\gamma, C]$ and $\forall u(\gamma(u) = 0 \rightarrow u < \beta)$. We show $\exists u(\gamma(u) = 0 \wedge |u| = n \wedge (\forall k \le n)B[u{\restriction}k])$ by $\mathsf{B}\exists^0\mathcal{D}\textsf{-Ind}$ on n. If $n = 0$ this is trivial. Assume $|v| = n \wedge (\forall k \le n)B[v{\restriction}k]$. $\mathsf{B}\exists^0\mathsf{B}\forall^0\neg\mathcal{C}\textsf{-LEM}$ gives two cases: if $\gamma(w) = 0 \wedge |w| = n{+}1 \wedge \neg D[w]$ then $(\forall k \le n{+}1)\neg D[w{\restriction}k]$ and $(\forall k \le n{+}1)B[w{\restriction}k]$; if no such w exists, as $\forall w(\gamma(w) = 0 \wedge |w| = n{+}1 \rightarrow D[w])$ by $\mathsf{B}\exists^0\mathcal{C}\textsf{-DNE}$, $\mathsf{Fan}[\gamma]$ yields x with $\gamma(v{*}\langle x\rangle) = 0 \wedge B[v{*}\langle x\rangle]$. $\mathcal{D}$-WKL yields β with $\forall k B[\beta{\restriction}k]$. $\mathsf{Bar}[\gamma, C]$ gives n with $C[\beta{\restriction}n]$. Then $\forall v(\gamma(v) = 0 \wedge |v| = n \rightarrow (\exists k \le n)C[v{\restriction}k])$ by $B[\beta{\restriction}n]$. ☐

Compare (2)(i) with 2.30(3)(ii). A similar argument was also used for 2.27(4) (and will be in 3.51(2)).

As an instance of (3) with $\mathcal{C} \equiv \Delta^0_0$, $\mathbf{EL}_0^- + \Delta^0_0\textsf{-BKL} \vdash \Delta^0_0\textsf{-BFT}$. This was shown in [19], but the essentially same proof had been given: e.g., the proof of [23, 4.7 Proposition 2) "$\rightarrow$"] with g instantiated with the particular g defined just below (++) on p.1263 is exactly the same proof, and there might be earlier proofs.

Lemma 3.6. (i) $\mathbf{EL}_0^- + \neg\mathcal{C}\textsf{-BKL} + \mathsf{B}\exists^0\mathcal{C}\textsf{-GDM} \vdash \exists^0\mathcal{C}\textsf{-GDM}$; (ii) $\mathbf{EL}_0^- + \mathsf{B}\forall^0\mathcal{C}\textsf{-BKL} + \mathsf{B}\exists^0(\mathsf{B}\forall^0\mathcal{C})\textsf{-Ind} \vdash \forall^0\mathcal{C}\textsf{-BAC}^{00}$;
(iii) $\mathbf{EL}_0^- + \mathcal{D}\textsf{-DNE} + \mathcal{D}\textsf{-Ind} + \forall^0\neg\mathcal{E}\textsf{-2AC}^{00} + \exists^0\mathcal{E}\textsf{-DM} \vdash \mathcal{C}\textsf{-WKL}$ for $\mathcal{D} \equiv \mathsf{B}\exists^0\mathsf{B}\forall^0\mathcal{C}$ and $\mathcal{E} \equiv \mathcal{D} \wedge \neg\mathcal{D}$.

Proof. Let A be $\mathcal{C}$. (i) Let $C[u] :\equiv |u| > 0 \rightarrow \neg A[u(0), |u|{-}1]$. Assume $\neg(\forall x < m)\exists y A[x, y]$. For any n, by $(\mathsf{B}\exists^0\mathcal{C})\textsf{-GDM}$, $\neg(\forall x < m)(\exists y < n)A[x, y]$ implies $(\exists x < m)(\forall y < n)\neg A[x, y]$. For such $x < m$, $\langle x\rangle{*}(\underline{0}{\restriction}n{-}1)$ witnesses $\exists u(u < \underline{m} \wedge |u| = n \wedge (\forall k \le n)C[u{\restriction}k])$. $\neg\mathcal{C}$-BKL yields $\beta < \underline{m}$ with $\forall n C[\beta{\restriction}n]$, and $\forall y\neg A[\beta(0), y]$.
(ii) Assume $\forall x(\exists y < \alpha(x))\forall z A[x, y, z]$. Let $B[u] :\equiv (\forall x, z < |u|)A[x, u(x), z]$. For n, $\mathsf{B}\exists^0(\mathsf{B}\forall^0\mathcal{C})\textsf{-Ind}$ on $k \le n$ shows $(\exists u < \alpha)(|u| = k \wedge (\forall x < k)(\forall z < n)A[x, u(x), z])$. $\mathsf{B}\forall^0\mathcal{C}$-BKL yields β with $\forall x B[\beta{\restriction}x]$. So $\forall x, z A[x, \beta(x), z]$.

[15] Actually bounded comprehension in [37] is the existence of set with the condition to which only finite segment is relevant.
[16] Also, the dissolution of the distinction between large and small numbers is essential for the proof of 2.37(1).

(iii) Assume $\forall n(\exists u < \underline{2})(|u| = n \wedge (\forall k \leq n)A[u\restriction k])$. Define a $\mathcal{D}$ formula B and an $\mathcal{E}$ formula C by

$$B[k,u] :\equiv (\exists v < \underline{2})(|v| = k \wedge (\forall l \leq |u|+k)A[(u{*}v)\restriction l]); \qquad C[n,u,x] :\equiv B[n, u{*}\langle 1-x\rangle] \wedge \neg B[n, u{*}\langle x\rangle].$$

Suppose $\exists nC[n,u,0] \wedge \exists nC[n,u,1]$, say $C[n,u,0] \wedge C[m,u,1]$. We may assume $n \geq m$. $C[n,u,0]$ implies $B[n, u{*}\langle 1\rangle]$ and so $B[m, u{*}\langle 1\rangle]$ contradicting $C[m,u,1]$. Thus $\exists^0\mathcal{E}$-DM yields $\forall n\neg C[n,u,0] \vee \forall n\neg C[n,u,1]$.

$\forall^0\neg\mathcal{E}$-$\mathsf{2AC}^{00}$ yields $\gamma < \underline{2}$ with $\forall u, n\neg C[n,u,\gamma(u)]$. We can show $(\exists v < \underline{2}\restriction n)(\forall k < n)(v(k) = \gamma(v\restriction k))$ by induction on n. Thus Δ^0_0-$\mathsf{2AC}^{00}$ yields $\beta < \underline{2}$ with $\forall k(\beta(k) = \gamma(\beta\restriction k))$ and so $\forall n, k\neg C[n, \beta\restriction k, \beta(k)]$.

We prove $B[n-k, \beta\restriction k]$ by $\mathcal{D}$-Ind on $k \leq n$. For $k = 0$, this is by assumption. For $k < n$, if $B[n-k, \beta\restriction k]$, say $|v| = n-k \wedge (\forall l \leq n)A[((\beta\restriction k){*}v)\restriction l]$ then $B[n-k-1, (\beta\restriction k){*}\langle v(0)\rangle]$. We may assume $v(0) = 1-\beta(k)$. By $\neg C[n-k-1, \beta\restriction k, \beta(k)]$ we have $\neg\neg B[n-k-1, (\beta\restriction k){*}\langle \beta(k)\rangle]$. Apply $\mathcal{D}$-DNE. Thus $B[0, \beta\restriction n]$, and $A[\beta\restriction n]$. ☐

Via $\mathfrak{g}$ and $\mathfrak{ch}$ from Subsection 2.3, Π^0_1-$\mathsf{2AC}^{01}$ corresponds to Π^0_1-AC and Π^0_1-$\mathsf{2AC}^{00}$ to Σ^0_1 separation. Hence (iii) with $\mathcal{C} \equiv \Delta^0_0$ refines the classical fact that Σ^0_1 separation implies WKL (cf. [37, Lemma IV.4.4]).

Replacing $\forall^0\neg\mathcal{E}$-$\mathsf{2AC}^{00}$ and $\exists^0\mathcal{E}$-DM by $\forall^0\neg\mathcal{E}$-BAC^{00} and $\exists^0\mathcal{E}$-GDM in (iii), we can prove $\mathcal{C}$-BKL. However, in a straightforward manner (or as in [37, Lemma IV.1.4]) we can show $\mathsf{EL}^-_0 + \mathcal{C}$-$\mathsf{WKL} \vdash \mathcal{C}$-$\mathsf{BKL}$.

Corollary 3.7. Over $\mathbf{EL}^-_0 + \Delta^0_0(\mathcal{C})\{$-$\mathsf{DNE}$,-$\mathsf{GDM}$,-$\mathsf{Ind}\}$, the following are equivalent:
(a) $\Pi^0_1(\mathcal{C})$-BKL; (b) $\Delta^0_0(\mathcal{C})$-BKL; (c) $\Sigma^0_1(\mathcal{C})$-GDM+$\Pi^0_1(\mathcal{C})$-BAC^{00}; (d) $\Sigma^0_1(\mathcal{C})$-DM+$\Pi^0_1(\mathcal{C})$-$\mathsf{2AC}^{00}$; (e) $\Delta^0_0(\mathcal{C})$-WKL.

Lemma 3.8. $\mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{BKL}$ proves Π^0_1-BAC^{01}.

Proof. Let A be Π^0_1. By 2.14, we may assume $A[x,\beta] \equiv \forall yC[x, \beta\restriction y]$ where C is Δ^0_0. Let $(u)_x(y) = u((x,y))$ for $(x,y) < |u|$ and define $B[u] :\equiv (\forall x < |u|)(\forall y < |(u)_x|)C[x, (u)_x\restriction y]$. Assume $\forall x(\exists \beta < (\gamma)_x)A[x,\beta]$.

By assumption, $(\forall x < n)(\exists v)D[x,n,v]$ where $D[x,n,v] \equiv v < (\gamma)_x \wedge |v| = n \wedge (\forall y < n)C[x, v\restriction y]$. By induction on $m \leq n$, we can show $(\exists w < \gamma\restriction(m,n))(w < \gamma \wedge (\forall x < m)D[x,n,(w)_x])$. Setting $m = n$ we have $(\exists u < \gamma)(|u| = n \wedge B[u])$. Δ^0_0-BKL yields $\beta < \gamma$ with $\forall nB[\beta\restriction n]$, and $\forall x, yC[x, (\beta)_x\restriction y]$, i.e., $\forall xA[x, (\beta)_x]$. ☐

Corollary 3.9. Π^0_1-BKL; Π^0_1-BAC^{01}+Σ^0_1-GDM; Π^0_1-$\mathsf{2AC}^{00}$+LLPO; and Δ^0_0-WKL are equivalent over $\mathbf{EL}^-_0$.

3.2 Functional realizability

3.2.1 general theory of Lifschitz's realizability

A general and abstract machinery for Lifschitz's realizability is provided by a theory **CDL** of combinators and $\in_{\mathrm{L}}$. This could be seen as a subsystem of *explicit mathematics* with classes[17] from [16]: all individuals are also classes and comprehension is much more restricted than elementary, with some modification on case distinction. Since the use of undefined terms is essential, we have to modify the first order logic as follows.

Definition 3.10 (logic of partial terms (cf. [5, VI.1]))**.** The first order logic of partial terms is formulated by the usual axioms and inference rules of the first order (intuitionistic or classical) logic, but

(i) a new unary predicate (treated as a logical symbol) $\downarrow$, called *definedness predicate*, is added;
(ii) $\forall$- and $\exists$-axioms (if formulated in Hilbert-style) are replaced by $\forall xA[x] \wedge t\downarrow \to A[t]$ and $A[t] \wedge t\downarrow \to \exists xA[x]$;
(iii) the equality axioms are formulated only with free variables and only for atomic formulae;
(iv) strictness axiom: $A[t] \to t\downarrow$ for any *atomic* formula A (which includes $t[s]\downarrow \to s\downarrow$ for any term $t[x]$).

Notice that (iii) includes $x = x$ and so (iv) yields $x\downarrow$. Thus free variables vary only over "defined" objects. This logic is called E^+-logic with equality in [40, Ch.1, 2.4], where $\downarrow$ is called the existence predicate.

Definition 3.11 ($\mathcal{L}_{\mathrm{Cb}}$, $\mathcal{L}_{\mathrm{CD}}$, $\mathcal{L}_{\mathrm{CDL}}$)**.** (1) The language $\mathcal{L}_{\mathrm{Cb}}$ has $=$ as the only predicate symbol; one binary function symbol $|$; constant symbols k, s, p, p_0 and p_1. $\mathcal{L}_{\mathrm{CD}}$ is the expansion with constants z, o and d; and a unary relation symbol Bo. $\mathcal{L}_{\mathrm{CDL}}$ expands $\mathcal{L}_{\mathrm{CD}}$ with a binary predicate symbol $\in_{\mathrm{L}}$ and constant symbols g, u, r, f and c. Variables of these languages are denoted by $\alpha, \beta, \gamma, ..., \xi, \eta, ...$ (except λ) possibly with subscripts.
(2) (i) $st :\equiv s|t$; $st_0...t_n :\equiv (...(st_0)...)t_n$; $\langle s,t\rangle := \mathsf{p}st$ and $\langle s,t,t'\rangle := \mathsf{p}s(\mathsf{p}tt')$. (ii) $s \simeq t :\equiv (s\downarrow) \vee (t\downarrow) \to s = t$.
(3) (i) For a term t and a variable ξ, another term $\lambda\xi.t$, without occurrences of ξ, is defined inductively:
(a) $\lambda\xi.\eta :\equiv \mathsf{k}\eta$ if $\xi \not\equiv \eta$; (b) $\lambda\xi.\xi :\equiv \mathsf{skk}$; (c) $\lambda\xi.c :\equiv \mathsf{k}c$ for a constant c; (d) $\lambda\xi.st :\equiv \mathsf{s}(\lambda\xi.s)(\lambda\xi.t)$;
(4) (i) $\lambda\eta_0 \cdots \eta_n.t :\equiv \lambda\eta_0.(...(\lambda\eta_n.t)...)$; (ii) $\mathsf{fix} := \lambda\zeta.(\lambda\xi\eta.\zeta(\xi\xi)\eta)(\lambda\xi\eta.\zeta(\xi\xi)\eta)$.

[17]The notion of class in explicit mathematics has been called *type* in the later references of explicit mathematics.

Definition 3.12 (**Cb**, **CD**, **CDL**)**.** The theory **Cb** of $\mathcal{L}_{\mathrm{Cb}}$ is generated over intuitionistic logic of partial terms by axioms (k), (s), (p). **CD** is **Cb**+(zo)+(d) in $\mathcal{L}_{\mathrm{CD}}$, and **CDL** is **Cb**+(g)+(u)+(r) in $\mathcal{L}_{\mathrm{CDL}}$. [18]

(k) $\mathsf{k}\alpha\beta=\alpha$; (s) $\mathsf{s}\alpha\beta\!\downarrow\wedge\ \mathsf{s}\alpha\beta\gamma\simeq\alpha\gamma(\beta\gamma)$; (p) $\mathsf{p}_0(\mathsf{p}\alpha\beta)=\alpha\wedge\mathsf{p}_1(\mathsf{p}\alpha\beta)=\beta\wedge\mathsf{p}_0\alpha\!\downarrow\wedge\ \mathsf{p}_1\alpha\!\downarrow$;
(zo) $\mathrm{Bo}[\alpha]\leftrightarrow(\alpha=\mathsf{z}\vee\alpha=\mathsf{o})$; (d) $\mathsf{d}\beta\gamma\mathsf{z}=\beta\wedge\mathsf{d}\beta\gamma\mathsf{o}=\gamma$; (g) $\mathsf{g}\alpha\!\downarrow\wedge\ (\xi\in_{\mathrm{L}}\mathsf{g}\alpha\leftrightarrow\xi=\alpha)$;
(u) $\mathsf{u}\alpha\!\downarrow\wedge\ (\xi\in_{\mathrm{L}}\mathsf{u}\alpha\leftrightarrow(\exists\beta\in_{\mathrm{L}}\alpha)(\xi\in_{\mathrm{L}}\beta))$; (r) $(\forall\eta\in_{\mathrm{L}}\alpha)(\beta\eta\!\downarrow)\to\mathsf{r}\alpha\beta\!\downarrow\wedge\ \forall\xi(\xi\in_{\mathrm{L}}\mathsf{r}\alpha\beta\leftrightarrow(\exists\eta\in_{\mathrm{L}}\alpha)(\xi=\beta\eta))$.

In **CDL** we can consider an object as a code of sets of objects with $\in_{\mathrm{L}}$, and g, u and r give the codes of singletons, unions and direct images under operations. The constants f and c are used only in the extensions.

Definition 3.13 (**CDLc**, **CDLf**)**.** (1) **CDLc** is an extension of **CDL** by $\exists!\xi(\xi\in_{\mathrm{L}}\alpha)\to(\mathsf{c}\alpha\!\downarrow\wedge\ \mathsf{c}\alpha\in_{\mathrm{L}}\alpha)$.
(2) **CDLf** is an extension of **CDL** by $(\exists\xi\in_{\mathrm{L}}\alpha)(\mathsf{p}_0\xi=\eta)\to\mathsf{f}\alpha\eta\!\downarrow\wedge\ \forall\xi(\xi\in_{\mathrm{L}}\mathsf{f}\alpha\eta\leftrightarrow\xi\in_{\mathrm{L}}\alpha\wedge\mathsf{p}_0\xi=\eta)$.

Thus c "chooses" an element if the set is a singleton and f gives the code of inverse images along projection if inhabited. While these were not needed in the definition nor in the proofs of basic properties below, they will be essential to generalize the "featured" properties of Lifschitz's realizability (c in 3.29 and f in 3.31).

Lemma 3.14. (1) For any $\mathcal{L}_{\mathrm{Cb}}$ term $t[\xi]$, $\mathbf{Cb}\vdash(\lambda\xi.t[\xi])\!\downarrow\wedge\,(\lambda\xi.t[\xi])s\simeq t[s]$. (2) $\mathbf{Cb}\vdash\mathsf{fix}\,\zeta\!\downarrow\wedge\,\mathsf{fix}\,\zeta\eta\simeq\zeta(\mathsf{fix}\,\zeta)\eta$.

$\mathbb{N}$ with Kleene bracket $nm\simeq\{n\}(m)$ is a model of **CD**. We can trivially extend it to **CDL** by interpreting $\in_{\mathrm{L}}$ as $=$ (only singletons are codable), but also by interpreting $n\in_{\mathrm{L}}m$ as $n<(m)_1^2\wedge\pi[(m)_0^2,n]$ where π is universal Π_1 (the codable are bounded Π^0_1), and we can interpret g, u and r accordingly, as well as c and f.

In $\mathbf{r}_{\mathrm{L}}$-realizability defined below, a realizer of existence statement is a (code of) inhabited sets of pairs of witnesses and realizers of the instantiated statements. Within the trivial model of **CDL**, $\mathbf{r}_{\mathrm{L}}$-realizability is the usual number-realizability; and in the other aforementioned model it is Lifschitz's (number) realizability.

Below let $\mathcal{L}$ and $\mathcal{L}'$ be first order languages sharing the set of variables, and let $\mathcal{L}'$ expand $\mathcal{L}_{\mathrm{CDL}}$.

Definition 3.15 ($\alpha\ \mathbf{r}_{\mathrm{L}}\ A$, $\mathbf{r}_{\mathrm{L}}$-realizable, b_A)**.** (1) For atomic $\mathcal{L}$ formulae A, fix $\mathcal{L}'$ formulae $\alpha\,\mathbf{r}_{\mathrm{L}}\,A$ whose free variables are α and those in A, where $\alpha\ \mathbf{r}_{\mathrm{L}}\ \bot\ :\equiv\ \bot$. Extend $\alpha\ \mathbf{r}_{\mathrm{L}}\ A$ for an arbitrary $\mathcal{L}$ formula A by

$\alpha\ \mathbf{r}_{\mathrm{L}}\ (A\wedge B)\ :\equiv\ (\mathsf{p}_0\alpha\ \mathbf{r}_{\mathrm{L}}\ A)\wedge(\mathsf{p}_1\alpha\ \mathbf{r}_{\mathrm{L}}\ B)$; $\alpha\ \mathbf{r}_{\mathrm{L}}\ (A\to B)\ :\equiv\ \forall\beta(\beta\ \mathbf{r}_{\mathrm{L}}\ A\to\alpha\beta\!\downarrow\wedge\ \alpha\beta\ \mathbf{r}_{\mathrm{L}}\ B)$;
$\alpha\ \mathbf{r}_{\mathrm{L}}\ (A\vee B)\ :\equiv\ \exists\eta(\eta\in_{\mathrm{L}}\alpha)\wedge(\forall\xi\in_{\mathrm{L}}\alpha)(\mathrm{Bo}[\mathsf{p}_0\xi]\wedge(\mathsf{p}_0\xi=\mathsf{z}\to\mathsf{p}_1\xi\ \mathbf{r}_{\mathrm{L}}\ A)\wedge(\mathsf{p}_0\xi=\mathsf{o}\to\mathsf{p}_1\xi\ \mathbf{r}_{\mathrm{L}}\ B))$;
$\alpha\ \mathbf{r}_{\mathrm{L}}\ \forall\xi A[\xi]\ :\equiv\ \forall\xi(\alpha\xi\!\downarrow\wedge\ \alpha\xi\ \mathbf{r}_{\mathrm{L}}\ A[\xi])$; $\alpha\ \mathbf{r}_{\mathrm{L}}\ \exists\xi A[\xi]\ :\equiv\ \exists\eta(\eta\in_{\mathrm{L}}\alpha)\wedge(\forall\xi\in_{\mathrm{L}}\alpha)(\mathsf{p}_1\xi\ \mathbf{r}_{\mathrm{L}}\ A[\mathsf{p}_0\xi])$.

An $\mathcal{L}$ theory T is called $\mathbf{r}_{\mathrm{L}}$-*realizable* in an $\mathcal{L}'$ theory T' if, for any A in T, $T'\vdash\exists\alpha(\alpha\,\mathbf{r}_{\mathrm{L}}\,A)$.
(2) Fix $\mathcal{L}_{\mathrm{CDL}}$ terms $\mathsf{b}_{A[\vec\eta]}$ for atomic $A[\vec\eta]$'s. Extend b_A to arbitrary A by $\mathsf{b}_{A\wedge B}:\equiv\lambda\vec\eta\alpha.\mathsf{p}(\mathsf{b}_A\vec\eta(\mathsf{r}\alpha\mathsf{p}_0))(\mathsf{b}_B\vec\eta(\mathsf{r}\alpha\mathsf{p}_1))$; $\mathsf{b}_{B\to A}:\equiv\lambda\vec\eta\alpha\beta.\mathsf{b}_A\vec\eta(\mathsf{r}\alpha(\lambda\zeta.\zeta\beta))$; $\mathsf{b}_{\forall\xi A[\vec\eta,\xi]}:\equiv\lambda\vec\eta\alpha\xi.\mathsf{b}_{A[\vec\eta,\xi]}\vec\eta(\mathsf{r}\alpha(\lambda\zeta.\zeta\xi))$; and $\mathsf{b}_{\exists\xi A[\vec\eta,\xi]},\mathsf{b}_{A\vee B}:\equiv\lambda\vec\eta\alpha.\mathsf{u}\alpha$.

Strictly, b_A is defined for abstracts A rather than formulae. We write $\mathsf{b}_{C[\vec\alpha]}$ also for $\mathsf{b}_{C[\vec\alpha]}\vec\eta$ with the free variables $\vec\eta$ implicit (i.e., other than $\vec\alpha$'s) in $C[\vec\alpha]$. We will not need the definition of b_A but the following.

Lemma 3.16. For an $\mathcal{L}'$ theory T', if $\mathbf{CDL}+T'\vdash\exists\xi(\xi\in_{\mathrm{L}}\alpha)\wedge(\forall\xi\in_{\mathrm{L}}\alpha)(\xi\,\mathbf{r}_{\mathrm{L}}A[\vec\eta])\to(\mathsf{b}_A\vec\eta\alpha)\!\downarrow\wedge\ \mathsf{b}_A\vec\eta\alpha\,\mathbf{r}_{\mathrm{L}}A[\vec\eta]$ for any atomic $\mathcal{L}$ formula A, then it holds for an arbitrary $\mathcal{L}$ formula A.

Proposition 3.17. Assume the premise of 3.16. If $A[\vec\eta]$ follows from sentences B_1, ..., B_n intuitionistically, then $\mathbf{CDL}\vdash\forall\beta_1,...,\beta_n(\beta_1\ \mathbf{r}_{\mathrm{L}}\ B_1\wedge...\wedge\beta_n\ \mathbf{r}_{\mathrm{L}}\ B_n\to t\beta_1...\beta_n\!\downarrow\wedge\ t\beta_1...\beta_n\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta A[\vec\eta])$ for a closed $\mathcal{L}_{\mathrm{CDL}}$-term t.

Proof. Consider a Hilbert-style calculus. The negative part is almost trivial as follows. $\lambda\vec\eta.\mathsf{k}$, $\lambda\vec\eta.\mathsf{s}$, $\lambda\vec\eta.\mathsf{p}_i$, $\lambda\vec\eta.\mathsf{p}$ and $\lambda\vec\eta\xi\alpha.\alpha\xi$ realize the universal closures of the axioms $\forall\vec\eta(A{\to}B{\to}A)$, $\forall\vec\eta((A{\to}(B{\to}C)){\to}(A{\to}B){\to}(A{\to}C))$, $\forall\vec\eta(A_0{\wedge}A_1{\to}A_i)$, $\forall\vec\eta(A_0{\to}A_1{\to}A_0{\wedge}A_1)$ and $\forall\vec\eta,\xi(\forall\zeta A[\zeta]\to A[\xi])$ respectively. If $s\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(C{\to}A)$ and $t\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta C$ then $\lambda\vec\eta.s\vec\eta(t\vec\eta)\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta A$, and if $t\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta,\zeta(C{\to}A[\zeta])$ then $\lambda\vec\eta\alpha\zeta.t\vec\eta\zeta\alpha\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(C\to\forall\zeta A[\zeta])$.

$\lambda\vec\eta\xi\gamma.\mathsf{g}(\langle\xi,\gamma\rangle)\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta,\xi(A[\xi]\to\exists\zeta A[\zeta])$. If $t\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta,\zeta(A[\zeta]\to C)$ then $\lambda\vec\eta\gamma.\mathsf{b}_C\vec\eta(\mathsf{r}\gamma(\lambda\xi.t\vec\eta(\mathsf{p}_0\xi)(\mathsf{p}_1\xi)))$ realizes $\forall\vec\eta(\exists\zeta A[\zeta]\to C)$: if $\gamma\ \mathbf{r}_{\mathrm{L}}\ \exists\zeta A[\zeta]$ then $(\forall\xi\in_{\mathrm{L}}\gamma)(\mathsf{p}_1\xi\ \mathbf{r}_{\mathrm{L}}\ A[\mathsf{p}_0\xi])$ and $(\forall\xi\in_{\mathrm{L}}\gamma)(t\vec\eta(\mathsf{p}_0\xi)(\mathsf{p}_1\xi)\!\downarrow\wedge\, t\vec\eta(\mathsf{p}_0\xi)(\mathsf{p}_1\xi)\ \mathbf{r}_{\mathrm{L}}\ C)$, i.e., $(\forall\xi'\in_{\mathrm{L}}\mathsf{r}\gamma(\lambda\xi.t\vec\eta(\mathsf{p}_0\xi)(\mathsf{p}_1\xi)))(\xi'\ \mathbf{r}_{\mathrm{L}}\ C)$, and similarly $\exists\xi'(\xi'\in_{\mathrm{L}}\mathsf{r}\gamma(\lambda\xi.\beta\vec\eta(\mathsf{p}_0\xi)(\mathsf{p}_1\xi)))$; apply 3.16.

$\lambda\vec\eta\gamma.\mathsf{g}(\langle\mathsf{z},\gamma\rangle)\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(A\to A\vee B)$ and $\lambda\vec\eta\gamma.\mathsf{g}(\langle\mathsf{o},\gamma\rangle)\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(B\to A\vee B)$. If $s\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(A\to C)$ and $t\ \mathbf{r}_{\mathrm{L}}\ \forall\vec\eta(B\to C)$ then, similarly we can show that $\lambda\vec\eta\alpha.\mathsf{b}_C\vec\eta(\mathsf{r}\alpha(\lambda\xi.\mathsf{d}(s\vec\eta(\mathsf{p}_1\xi))(t\vec\eta(\mathsf{p}_1\xi))(\mathsf{p}_0\xi)))$ realizes $\forall\vec\eta(A\vee B\to C)$. ☐

Therefore $A\mapsto\exists\alpha(\alpha\,\mathbf{r}_{\mathrm{L}}\,A)$ can be considered as an interpretation of intuitionistic logic (i.e., the theory axiomatized by $\emptyset$) over $\mathcal{L}$ to extensions of **CDL** in the sense of 1.2. The theme of this section is to clarify: with which axioms in $\mathcal{L}'$, which axioms in $\mathcal{L}$ can be interpreted in this way.

[18] With the totality $\forall\alpha,\beta(\alpha|\beta\!\downarrow)$, we can define p and p_i by d, z and o. However, without it we cannot obtain $\mathsf{p}_0\alpha\!\downarrow\wedge\ \mathsf{p}_1\alpha\!\downarrow$.

3.2.2 Kleene's second model $\mathfrak{k}$

We will need functional realizability and so a functional model of **CD**, called Kleene's second model. Though [40, Ch.9, 4.1] gave a construction in an abstract way, it seems easier for us to give an explicit definition, in order to check if the construction is possible in our context of weak induction.

Notation 3.18 $(u|v)$**.** $(u|v)(x)$ is $u(\langle x\rangle * (v\restriction y)) - 1$ if $y = \min\{z : u(\langle x\rangle * (v\restriction z)) > 0\}$, and is undefined if there is no such y. "$(u|v)(x)$ is defined" is Δ^0_0. If $u \subseteq u'$, $v \subseteq v'$ and $(u|v)(k)$ is defined, then $(u|v)(k) = (u'|v')(k)$.

Definition 3.19 $(A^{\mathfrak{k}})$**.** For an $\mathcal{L}_{\mathrm{Cb}}$ term t and $\mathcal{L}_{\mathrm{Cb}}$ formula A, define $\mathcal{L}_{\mathrm{F}}$ formulae $[\![t]\!](\xi)$ and $A^{\mathfrak{k}}$ by
$[\![\alpha]\!](\xi) :\equiv \xi = \alpha$; $[\![c]\!](\xi) :\equiv \xi = c^{\mathfrak{k}}$ for a constant c; $[\![st]\!](\xi) :\equiv \exists\eta,\zeta([\![s]\!](\eta) \wedge [\![t]\!](\zeta) \wedge \xi = \eta|\zeta)$; $(s\downarrow)^{\mathfrak{k}} :\equiv \exists\xi([\![s]\!](\xi))$;
$(s = t)^{\mathfrak{k}} :\equiv \exists\xi([\![s]\!](\xi) \wedge [\![t]\!](\xi))$; $\quad \perp^{\mathfrak{k}} :\equiv \perp$; $\quad (A\Box B)^{\mathfrak{k}} :\equiv A^{\mathfrak{k}} \Box B^{\mathfrak{k}}$ ($\Box \equiv \wedge, \to, \vee$); $\quad (Q\xi A)^{\mathfrak{k}} :\equiv Q\xi A^{\mathfrak{k}}$ ($Q \equiv \forall, \exists$).
where $\xi = \eta|\zeta$ is from 2.35 and where $c^{\mathfrak{k}}$'s are defined as follows by Δ^0_0 bounded search in $\mathbf{EL}^-_0$ from 2.10:

$$\mathsf{p}_i{}^{\mathfrak{k}}(x) = \begin{cases} (w(y))^2_i + 1 & \text{if } x = \langle y\rangle * w \\ & \quad\text{and } |w| = y+1; \\ 0 & \text{otherwise.} \end{cases} \qquad \mathsf{k}^{\mathfrak{k}}(x) = \begin{cases} u(y) + 2 & \text{if } x = \langle\langle y\rangle\rangle * u \text{ and } |u| = y+1, \\ 1 & \text{if } x = \langle v\rangle \text{ and } |v| \neq 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathsf{p}^{\mathfrak{k}}(x) = \begin{cases} (u(y), v(y)) + 2 & \text{if } x = \langle\langle y\rangle * v\rangle * u \text{ and } |u| = |v| = y+1; \\ 0 & \text{if } x = \langle\langle y\rangle * v\rangle * u \text{ and } |u| \neq |v| = y+1; \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathsf{s}^{\mathfrak{k}}(x) = \begin{cases} ((u|w)|(v|w))(y) + 3 & \text{if } x = \langle\langle\langle y\rangle * w\rangle * v\rangle * u \text{ and } |u| = |v| = |w| \\ & \quad\text{and } (\forall z \leq y)\text{“}(u|w)(z),\ (v|w)(z) \text{ and } ((u|w)|(v|w))(z) \text{ are defined”}; \\ 2 & \text{if } x = \langle\langle\langle y\rangle * w\rangle * v\rangle * u \text{ and } |u| = |v| = |w| \text{ but otherwise;} \\ 1 & \text{if } x = \langle\langle\langle y\rangle * w\rangle * v\rangle,\ 0 < |w| \text{ and } |v| \neq |w|; \\ 2 & \text{if } x = \langle\langle\langle y\rangle\rangle * v\rangle \text{ with } |v| > 0 \text{ or } x = \langle\langle\langle\rangle\rangle * v\rangle \text{ or } x = \langle\langle\rangle\rangle; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.20. $\mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{AC}^{00} \vdash (\mathbf{Cb})^{\mathfrak{k}} \wedge ((\mathsf{p}\alpha\beta)^{\mathfrak{k}} = (\alpha, \beta))$.

Proof. Let $\overline{\alpha}n$ denote $\alpha\restriction n$. We can easily see $(\mathsf{p}_i\alpha)^{\mathfrak{k}}(x) = (\alpha(x))^2_i$, and using the following we can show $(\mathsf{k}\alpha\beta)^{\mathfrak{k}}(x) = \alpha(x)$ and $(\mathsf{p}\alpha\beta)^{\mathfrak{k}}(x) = (\alpha(x), \beta(x))$ and the first conjunct of (s).

$$(\mathsf{k}\alpha)^{\mathfrak{k}}(x) = \begin{cases} \alpha(y) + 1 & \text{if } x = \langle y\rangle, \\ 0 & \text{otherwise} \end{cases}; \qquad \text{and} \qquad (\mathsf{p}\alpha)^{\mathfrak{k}}(x) = \begin{cases} (\alpha(y), v(y)) + 1 & \text{if } x = \langle y\rangle * v \wedge |v| = y+1; \\ 0 & \text{otherwise;} \end{cases}$$

$$(\mathsf{s}\alpha)^{\mathfrak{k}}(x) = \begin{cases} ((\overline{\alpha}k|w)|(v|w))(y) + 2 & \text{if } x = \langle\langle y\rangle * w\rangle * v \text{ and, for } k := |v| = |w|, \\ & (\forall z \leq y)\text{“}(\overline{\alpha}k|w)(z),\ (v|w)(z) \text{ and } ((\overline{\alpha}k|w)|(v|w))(z) \text{ are defined”}; \\ 1 & \text{if } x = \langle\langle y\rangle * w\rangle * v \text{ and } |v| = |w| \text{ but otherwise;} \\ 0 & \text{if } x = \langle\langle y\rangle * w\rangle * v,\ 0 < |w| \text{ and } |v| \neq |w|; \\ 1 & \text{if } x = \langle\langle y\rangle\rangle * v \text{ with } |v| > 0 \text{ or } x = \langle\langle\rangle\rangle * v \text{ or } x = \langle\,\rangle; \end{cases}$$

$$(\mathsf{s}\alpha\beta)^{\mathfrak{k}}(x) = \begin{cases} ((\overline{\alpha}k|w)|(\overline{\beta}k|w))(y) + 1 & \text{if } x = \langle y\rangle * w \text{ and, for } k := |w|, \\ & (\forall z \leq y)\text{“}(\overline{\alpha}k|w)(z),\ (\overline{\beta}k|w)(z) \text{ and } ((\overline{\alpha}k|w)|(\overline{\beta}k|w))(z) \text{ are defined”}; . \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\mathsf{s}\alpha\beta\gamma)^{\mathfrak{k}}\downarrow$. Then $(\mathsf{s}\alpha\beta\gamma)^{\mathfrak{k}}(y) = ((\overline{\alpha}k|\overline{\gamma}k)|(\overline{\beta}k|\overline{\gamma}k))(y)$, where k is a least such that $(\overline{\alpha}k|\overline{\gamma}k)(z)$, $(\overline{\beta}k|\overline{\gamma}k)(z)$ and $((\overline{\alpha}k|\overline{\gamma}k)(\overline{\beta}k|\overline{\gamma}k))(z)$ are defined for all $z \leq y$. By 3.18, $((\alpha|\gamma)|(\beta|\gamma))(y)$ is defined and is $((\overline{\alpha}k|\overline{\gamma}k)|(\overline{\beta}k|\overline{\gamma}k))(y)$. Σ^0_1-AC^{00} yields $(\alpha|\gamma)\downarrow$, $(\beta|\gamma)\downarrow$, $((\alpha|\gamma)|(\beta|\gamma))\downarrow$ and $(\mathsf{s}\alpha\beta\gamma)^{\mathfrak{k}} = ((\alpha|\gamma)|(\beta|\gamma))$. Conversely let $((\alpha|\gamma)|(\beta|\gamma))\downarrow$, which implies $(\alpha|\gamma)\downarrow$ and $(\beta|\gamma)\downarrow$. For x, by 2.16(3)(ii) Δ^0_0-AC^{00} yields k with $(\overline{\alpha}k|\overline{\gamma}k)(y)$, $(\overline{\beta}k|\overline{\gamma}k)(y)$ and $((\overline{\alpha}k|\overline{\gamma}k)|(\overline{\beta}k|\overline{\gamma}k))(y)$ are defined for all $y \leq x$. Then $(\mathsf{s}\alpha\beta\gamma)^{\mathfrak{k}}(x) = ((\overline{\alpha}k|\overline{\gamma}k)|(\overline{\beta}k|\overline{\gamma}k))(x)$. Thus $(\mathsf{s}\alpha\beta\gamma)^{\mathfrak{k}}\downarrow$. $\square$

Lemma 3.21. (1) (i) For a Σ^0_1 formula A, $\mathbf{EL}^-_0$ proves that: if $\forall x, y, z, \alpha(A[x,y,\alpha] \wedge A[x,z,\alpha] \to y = z)$ holds, then there is γ_A with (a) $\forall\alpha((\gamma_A|\alpha)\downarrow \leftrightarrow \exists\beta\forall x A[x, \beta(x), \alpha]\,)$ and (b) $\forall\alpha((\gamma_A|\alpha)\downarrow \to \forall x A[x, (\gamma_A|\alpha)(x), \alpha]\,)$; and (ii) for a Σ^0_1 formula A, $\mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{AC}^{00}$ proves that there is γ_A with (b) and $\forall\alpha((\gamma_A|\alpha)\downarrow \leftrightarrow \forall x\exists y A[x,y,\alpha])$.
(2) For a Π^0_1 formula $B[\xi,\eta,\gamma]$, $\mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{BKL}$ proves that there is χ_B such that for any α, β and ξ, $(\chi_B|\alpha|\beta)\downarrow \wedge\ \forall\xi((\exists\eta < \alpha)B[\xi,\eta,\beta] \leftrightarrow \forall n((\chi_B|\alpha|\beta)(\xi\restriction n) = 0))$.

Proof. (1)(ii) follows from (i) and Δ^0_0-LNP. (i) By 2.14, take C from Δ^0_0 with $A[x,y,\alpha]\leftrightarrow\exists kC[x,y,\alpha\restriction k]$. Let

$$\gamma_A(w)=\begin{cases} y{+}1 & \text{if } w=\langle x\rangle * v \text{ and } y<|v|\wedge(\exists k<|v|)C[x,y,v\restriction k]\\ 0 & \text{if there are no such } x,v,y\end{cases}.$$

(2) By 3.5(2)(ii) and 2.14, let $\forall\xi,\alpha,\beta((\exists\eta<\alpha)B[\xi,\eta,\beta]\leftrightarrow\neg\exists nC[\xi\restriction n,(\alpha,\beta)])$ where C is Δ^0_0. (1)(i) with 2.10(d) yields γ with $(\gamma|(\alpha,\beta))\downarrow$ and $\forall u((\gamma|(\alpha,\beta))(u)=0\leftrightarrow\neg C[u,(\alpha,\beta)])$. Set $\chi_B:=\lambda\alpha\beta.\gamma|(\mathsf{p}|\alpha|\beta)$. ☐

Here (1) formalizes the famous fact: any continuous functional can be represented by an operation in Kleene's second model (cf. [22, Subsec. 5.2]). (2) is a preliminary for van Oosten's model treated in 3.2.3.

Definition 3.22 ($\mathfrak{k}$)**.** Expand $\mathfrak{k}$ to $\mathcal{L}_{\mathrm{CDL}}$ by $\mathrm{Bo}[\alpha]^{\mathfrak{k}}:\equiv\alpha<\underline{2}\wedge\forall x,y(\alpha(x)=\alpha(y))$ and $\xi\in^{\mathfrak{k}}_{\mathsf{L}}\alpha:\equiv\alpha=\xi$ with $\mathsf{z}^{\mathfrak{k}}:=\underline{0}$; $\mathsf{o}^{\mathfrak{k}}:=\underline{1}$; $\mathsf{d}^{\mathfrak{k}}:=\lambda\xi\eta\zeta.\gamma_A(\mathsf{p}\xi(\mathsf{p}\eta\zeta))$; $\mathsf{g}^{\mathfrak{k}},\mathsf{u}^{\mathfrak{k}},\mathsf{c}^{\mathfrak{k}}:=\lambda\xi.\xi$; $\mathsf{f}^{\mathfrak{k}}:=\mathsf{k}$; and $\mathsf{r}^{\mathfrak{k}}:=\lambda\xi\eta.\eta|\xi$, where γ_A is as in 3.21(1)(i) above applied to A from Δ^0_0 such that $A[x,y,\mathsf{p}^{\mathfrak{k}}\xi(\mathsf{p}^{\mathfrak{k}}\eta\zeta)]\leftrightarrow((\zeta(0)=0\rightarrow y=\xi(x))\wedge(\zeta(0)\neq 0\rightarrow y=\eta(x)))$.

Proposition 3.23. $\mathbf{EL}^-_0+\Delta^0_0$-$\mathsf{AC}^{00}\vdash(\mathbf{CDL}\mathsf{c})^{\mathfrak{k}}+(\mathbf{CDL}\mathsf{f})^{\mathfrak{k}}$.

3.2.3 van Oosten's model $\mathfrak{o}$

Under $\mathfrak{k}$, only singletons are codable and so $\mathbf{r}_{\mathsf{L}}$-realizability is the usual function realizability. On the other hand, under $\mathfrak{o}$ due to van Oosten [29, §5], α codes the sets of infinite paths through the "bounded" tree $\{u<(\alpha)^2_1:\forall n((\alpha)^2_0(u\restriction n)=0)\}$ so that bounded König's lemma could be $\mathbf{r}_{\mathsf{L}}$-realizable. We have to check if it works in our context of weak induction. This is not easy. Indeed Dorais [13, Remark 4.10] tried to weaken induction in van Oosten's argument but required Π^0_1-Bdg. We show that it is not needed and Δ^0_0-Ind suffices.

Definition 3.24 ($\mathfrak{o}$ and π_A)**.** (1) Let $\mathfrak{o}$ coincide with $\mathfrak{k}$ on $\mathcal{L}_{\mathrm{CD}}$, and $\xi\in^{\mathfrak{o}}_{\mathsf{L}}\alpha:\equiv\xi<(\alpha)^2_1\wedge\forall n((\alpha)^2_0(\xi\restriction n)=0)$.
(2) For any Π^0_1 formula $A[\xi,\eta,\gamma]$, define $\pi_A:\equiv\lambda\alpha\beta\gamma.\mathsf{p}|(\chi_A|\beta|\gamma)|\alpha$ where χ_A is from 3.21(2).

Then $\pi_A|\alpha|\beta|\gamma$ codes the bounded Π^0_1 set $\{\xi<\alpha:(\exists\eta<\beta)A[\xi,\eta,\gamma]\}$, as stated in the next lemma (2), whereas (1) gives us the necessary bound to make the arguments (for 3.26) work only with Δ^0_0-Ind. This will be essential to define the interpretation of r in 3.26, and, in later parts, r will gives the necessary bounds.

Lemma 3.25. (1) $\mathbf{EL}^-_0+\mathsf{MP}+\Delta^0_0$-$\mathsf{AC}^{00}+\Delta^0_0$-$\mathsf{BKL}$ proves that there is ζ such that

$$(\forall\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)((\beta|\eta)\downarrow)\rightarrow\zeta|(\alpha,\beta)\downarrow\wedge(\forall\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)((\beta|\eta)<\zeta|(\alpha,\beta)\wedge\forall k(\exists n<(\zeta|(\alpha,\beta))(k))(\beta(\langle k\rangle*(\eta\restriction n))>0)).$$

(2) For A from Π^0_1, $\mathbf{EL}^-_0+\Delta^0_0$-$\mathsf{BKL}\vdash\forall\alpha,\beta,\gamma((\pi_A|\alpha|\beta|\gamma)\downarrow\wedge\forall\xi(\xi\in^{\mathfrak{o}}_{\mathsf{L}}\pi_A|\alpha|\beta|\gamma\ \leftrightarrow\ \xi<\alpha\wedge(\exists\eta<\beta)A[\xi,\eta,\gamma]))$.

Proof. Since (2) is immediate, we prove (1). Let $C[u,k,\alpha,\beta]:\equiv(\exists x,w<|u|)\neg((\alpha)^2_0(u\restriction x)=0\wedge\beta(\langle k\rangle*(u\restriction w))=0)$ and $D[k,y,\alpha,\beta]:\equiv(\forall u<(\alpha)^2_1)(|u|=y\rightarrow(\exists l\leq|u|)C[u\restriction l,k,\alpha,\beta])$, where $u<\gamma$ is defined in 3.4. Now we have

$$\begin{aligned}
(\forall\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)((\beta|\eta)\downarrow)\ &\leftrightarrow\ \forall k(\forall\eta<(\alpha)^2_1)(\forall x((\alpha)^2_0(\eta\restriction x)=0)\rightarrow\exists w(\beta(\langle k\rangle*(\eta\restriction w))>0)) && \text{(by }\Delta^0_0\text{-}\mathsf{AC}^{00}\text{)}\\
&\leftrightarrow\ \forall k(\forall\eta<(\alpha)^2_1)\neg(\forall x((\alpha)^2_0(\eta\restriction x)=0)\wedge\neg\exists w(\beta(\langle k\rangle*(\eta\restriction w))>0)) && \text{(by }\mathsf{MP}\text{)}\\
&\leftrightarrow\ \forall k(\forall\eta<(\alpha)^2_1)\exists x,w\neg((\alpha)^2_0(\eta\restriction x)=0\wedge\beta(\langle k\rangle*(\eta\restriction w))=0) && \text{(by }\mathsf{MP}\text{)}\\
&\leftrightarrow\ \forall k(\forall\eta<(\alpha)^2_1)\exists lC[\eta\restriction l,k,\alpha,\beta]\ \leftrightarrow\ \forall k\neg(\exists\eta<(\alpha)^2_1)\forall l\neg C[\eta\restriction l,k,\alpha,\beta]\\
&\leftrightarrow\ \forall k\neg\forall y(\exists u<(\alpha)^2_1)(|u|=y\wedge(\forall l\leq y)\neg C[u\restriction l,k,\alpha,\beta])\ \leftrightarrow\ \forall k\exists yD[k,y,\alpha,\beta]. && \text{(by }\Delta^0_0\text{-}\mathsf{BKL},\mathsf{MP}\text{)}
\end{aligned}$$

3.21(1)(ii) yields γ with $\forall kD[k,(\gamma|(\alpha,\beta))(k),\alpha,\beta]$. Then $\forall k(\forall\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)(\exists n<(\gamma|(\alpha,\beta))(k))(\beta(\langle k\rangle*(\eta\restriction n))>0)$.
Thus ζ with $(\zeta|(\alpha,\beta))(k)=\max(\,(\gamma|(\alpha,\beta))(k),\ \beta\restriction(\langle k\rangle*((\alpha)^2_1\restriction(\gamma|(\alpha,\beta))(k)))\,)+1$ is what we need. ☐

Proposition 3.26. $\mathbf{EL}^-_0+\mathsf{MP}+\Delta^0_0$-$\mathsf{AC}^{00}+\Delta^0_0$-$\mathsf{BKL}\vdash(\mathbf{CDL}\mathsf{c}+\mathbf{CDL}\mathsf{f})^{\mathfrak{o}}$ with suitable $\mathsf{g}^{\mathfrak{o}}$, $\mathsf{u}^{\mathfrak{o}}$, $\mathsf{r}^{\mathfrak{o}}$, $\mathsf{c}^{\mathfrak{o}}$ and $\mathsf{f}^{\mathfrak{o}}$.

Proof. $\alpha=\xi$ is Π^0_1. $(\exists\beta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)(\xi\in^{\mathfrak{o}}_{\mathsf{L}}\beta)$, $(\exists\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)(\xi=\beta|\eta)$ and $\xi\in^{\mathfrak{o}}_{\mathsf{L}}\alpha\wedge\mathsf{p}_0\xi=\eta$ are equivalent, respectively, to $\xi<\alpha\wedge(\exists\beta<\alpha)((\beta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)\wedge(\xi\in^{\mathfrak{o}}_{\mathsf{L}}\beta))$, $\xi<\zeta|(\alpha,\beta)\wedge(\exists\eta\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)(\xi=\beta|\eta)$ and $\xi<\alpha\wedge\xi\in^{\mathfrak{o}}_{\mathsf{L}}\alpha\wedge(\xi)^2_0=\eta$, where ζ is from 3.25(1) and $\xi=\beta|\eta$ is equivalently Π^0_1 with the bound $\zeta|(\alpha,\beta)$. 3.25(2) yields $\mathsf{g}^{\mathfrak{o}}$, $\mathsf{u}^{\mathfrak{o}}$, $\mathsf{r}^{\mathfrak{o}}$ and $\mathsf{f}^{\mathfrak{o}}$.

Let $u\perp\xi:\equiv(\exists k<|u|)(u(k)\neq\xi(k))$; recall $u<\beta:\equiv(\forall k<|u|)(u(k)<\beta(k))$ and $(\beta\ominus n)(k)=\beta(k{+}n)$.

Assume $\exists!\xi(\xi\in^{\mathfrak{o}}_{\mathsf{L}}\alpha)$ and $\xi\in^{\mathfrak{o}}_{\mathsf{L}}\alpha$. Then $(\forall u<(\alpha)^2_1)(u\perp\xi\rightarrow\neg(\exists\eta<(\alpha)^2_1\ominus|u|)\forall n((\alpha)^2_0((u*\eta)\restriction n)=0)$. By Δ^0_0-BKL, $(\forall u<(\alpha)^2_1)(u\perp\xi\rightarrow\neg\forall m(\exists v<(\alpha)^2_1\ominus|u|)(|v|=m\wedge(\forall n<m{+}|u|)((\alpha)^2_0((u*v)\restriction n)=0)))$, and, by MP, $(\forall u<(\alpha)^2_1)(u\perp\xi\rightarrow B[u,\alpha])$ where $B[u,\alpha]:\equiv\exists m(\forall v<(\alpha)^2_1\ominus|u|)(|v|=m\rightarrow(\exists n<m{+}|u|)((\alpha)^2_0((u*v)\restriction n)>0))$. Thus $\xi\restriction n$ is the only w with $C[n,w,\alpha]:\equiv w<(\alpha)^2_1\wedge|w|=n\wedge(\forall u<(\alpha)^2_1)(|u|=n\wedge u\neq w\rightarrow B[u,\alpha])$ since $\forall n\neg B[\xi\restriction n,\alpha]$. C is equivalently Σ^0_1 with Σ^0_1-Bdg which is by Δ^0_0-AC^{00} with 2.16(3)(ii). Apply 3.21(1)(ii) to $D[n,y,\alpha]:\equiv\exists w(C[n{+}1,w,\alpha]\wedge w(n)=y)$; then $\forall nD[n,(\gamma_D|\alpha)(n),\alpha]$, i.e., $(\gamma_D|\alpha)(n)=\xi(n)$. Set $\mathsf{c}^{\mathfrak{o}}=\gamma_D$. ☐

3.2.4 characterizing axioms of realizability

As in 3.2.1 let $\mathcal{L}'$ expand $\mathcal{L}_{\mathrm{CDL}}$ via some interpretation, but atomic $\mathcal{L}_{\mathrm{CDL}}$ formulae may be non-atomic in $\mathcal{L}'$, as in $\mathfrak{k}$ or $\mathfrak{o}$. As Δ^0_0 is non-sense, 2.9(3) is not applicable here. General treatment here will help us in [28].

Definition 3.27 ($A^{\mathbf{r}_\mathrm{L}}$, canonicalized, $N(\mathcal{C})$, $RH(\mathcal{C})$, $\mathcal{R}$)**.** (1) To an $\mathcal{L}$ formula A, assign an $\mathcal{L}'$ formula $A^{\mathbf{r}_\mathrm{L}}$ by $A^{\mathbf{r}_\mathrm{L}} :\equiv \exists\alpha(\alpha\,\mathbf{r}_\mathrm{L}\,A)$ for atomic A; $(A\Box B)^{\mathbf{r}_\mathrm{L}} :\equiv A^{\mathbf{r}_\mathrm{L}}\Box B^{\mathbf{r}_\mathrm{L}}$ for $\Box\equiv\wedge,\rightarrow,\vee$; $(QxA)^{\mathbf{r}_\mathrm{L}} :\equiv QxA^{\mathbf{r}_\mathrm{L}}$ for $Q\equiv\forall,\exists$.
(2) $A[\vec\eta]$, without other parameters, is called $\mathbf{r}_\mathrm{L}$-*canonicalized* by c_A if $\forall\vec\eta,\alpha(\alpha\,\mathbf{r}_\mathrm{L}A[\vec\eta]\rightarrow c_A\vec\eta\!\downarrow\wedge\, c_A\vec\eta\,\mathbf{r}_\mathrm{L}A[\vec\eta])$.
(3) A formula is called (i) $N(\mathcal{C})$ or *negative in* $\mathcal{C}$ if it is built up from $\mathcal{C}$ formulae by $\wedge$, $\rightarrow$ and $\forall$; and
(ii) *Rasiowa-Harrop in* $\mathcal{C}$ or $RH(\mathcal{C})$ if it is built up from $\mathcal{C}$ by $\wedge$, $\forall$ and $A\rightarrow$- with arbitrary formulae A.
(4) $\mathcal{R}$ is the class of $\mathcal{L}'$ formulae negative in $\{\exists\xi(\xi\in_\mathrm{L}\alpha),\ \xi\in_\mathrm{L}\alpha,\ \alpha\beta\!\downarrow,\ \gamma=\alpha\beta,\ \mathrm{Bo}[\alpha]\}\cup\{\alpha\,\mathbf{r}_\mathrm{L}\,A\mid A \text{ is } \mathcal{L}\text{-atomic}\}$.
(5) For classes $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{L}'$ formulae, define the following where C and D are any from $\mathcal{C}$ and $\mathcal{D}$ respectively:
($(\mathcal{C},\mathcal{D})$-$\mathsf{GC}_\mathrm{L}$) $\forall\alpha(D[\alpha]\rightarrow\exists\beta C[\alpha,\beta])\rightarrow\exists\gamma\forall\alpha(D[\alpha]\rightarrow\gamma\alpha\!\downarrow\wedge\,\exists\xi(\xi\in_\mathrm{L}\gamma\alpha)\wedge(\forall\xi\in_\mathrm{L}\gamma\alpha)C[\alpha,\xi])$;
($(\mathcal{C},\mathcal{D})$-$\mathsf{GC!}$) $\forall\alpha(D[\alpha]\rightarrow\exists!\beta C[\alpha,\beta])\rightarrow\exists\gamma\forall\alpha(D[\alpha]\rightarrow\gamma\alpha\!\downarrow\wedge\, C[\alpha,\gamma\alpha])$.

Lemma 3.28. Assume the premise of 3.16. (1) If $\mathcal{C}$ formulae are $\mathbf{r}_\mathrm{L}$-canonicalized, then so are $RH(\mathcal{C})$ ones.
(2) For an $\mathcal{L}$ formula A, (i) $\alpha\,\mathbf{r}_\mathrm{L}\,A$ is in $\mathcal{R}$; and (ii) $\mathbf{CDL}$+$(\mathcal{R},\mathcal{R})$-$\mathsf{GC}_\mathrm{L}\vdash A^{\mathbf{r}_\mathrm{L}}\leftrightarrow\exists\alpha(\alpha\,\mathbf{r}_\mathrm{L}\,A)$.

Proof. It is easy to see (1) and (2)(i). We see (2)(ii) by induction on A. The atomic, $\wedge$, $\vee$ cases are obvious.

By induction hypothesis, $(B\rightarrow C)^{\mathbf{r}_\mathrm{L}}$ is equivalent to $\exists\beta(\beta\,\mathbf{r}_\mathrm{L}\,B)\rightarrow\exists\gamma(\gamma\,\mathbf{r}_\mathrm{L}\,C)$, i.e., $\forall\beta((\beta\,\mathbf{r}_\mathrm{L}\,B)\rightarrow\exists\gamma(\gamma\,\mathbf{r}_\mathrm{L}\,C))$. Obviously $\alpha\,\mathbf{r}_\mathrm{L}(B\rightarrow C)$ implies this. Conversely, by (2)(i), the above with $(\mathcal{R},\mathcal{R})$-$\mathsf{GC}_\mathrm{L}$ yields α such that $\forall\beta(\beta\,\mathbf{r}_\mathrm{L}\,B\rightarrow(\alpha\beta)\!\downarrow\wedge\,\exists\xi(\xi\in_\mathrm{L}\alpha\beta)\wedge(\forall\xi\in_\mathrm{L}\alpha\beta)(\xi\,\mathbf{r}_\mathrm{L}\,C))$. Thus $\lambda\beta.\mathsf{b}_C(\alpha\beta)\ \mathbf{r}_\mathrm{L}\ (B\rightarrow C)$.

If $\alpha\ \mathbf{r}_\mathrm{L}\ \forall\xi A[\xi]$ then $\forall\xi(\alpha\xi\ \mathbf{r}_\mathrm{L}\ A[\xi])$ and so $\forall\xi A[\xi]^{\mathbf{r}_\mathrm{L}}$ by induction hypothesis. If $\forall\xi A[\xi]^{\mathbf{r}_\mathrm{L}}$ then $\forall\xi\exists\gamma(\gamma\,\mathbf{r}_\mathrm{L}\,A[\xi])$ and so $(\mathcal{R},\mathcal{R})$-$\mathsf{GC}_\mathrm{L}$ yields α with $\forall\xi(\alpha\xi\!\downarrow\wedge\,\exists\eta(\eta\in_\mathrm{L}\alpha\xi)\wedge(\forall\eta\in_\mathrm{L}\alpha\xi)(\eta\ \mathbf{r}_\mathrm{L}\,A[\xi]))$. Thus $\lambda\xi.\mathsf{b}_A\xi(\alpha\xi)\,\mathbf{r}_\mathrm{L}\ \forall\xi A[\xi]$.

If $\alpha\,\mathbf{r}_\mathrm{L}\,\exists\xi A[\xi]$, then $(\exists\xi\in_\mathrm{L}\alpha)(\mathsf{p}_1\xi\,\mathbf{r}_\mathrm{L}\,A[\mathsf{p}_0\xi])$ and by induction hypothesis $A[\mathsf{p}_0\xi]^{\mathbf{r}_\mathrm{L}}$ and so $\exists\eta A[\eta]^{\mathbf{r}_\mathrm{L}}$. Conversely, if $A[\eta]^{\mathbf{r}_\mathrm{L}}$, the induction hypothesis yields α with $\alpha\ \mathbf{r}_\mathrm{L}\ A[\eta]$ and so $\mathsf{g}(\mathsf{p}\eta\alpha)\ \mathbf{r}_\mathrm{L}\ \exists\xi A[\xi]$. ☐

Lemma 3.29. In $\mathbf{CDL}\mathsf{c}$, if $(\xi=\eta)^{\mathbf{r}_\mathrm{L}}\rightarrow\xi=\eta$ and $\mathcal{D}$ formulae are canonicalized, then $(\mathcal{L},\mathcal{D})$-$\mathsf{GC!}$ is realizable.

Proof. Assume $\zeta\ \mathbf{r}_\mathrm{L}\ \forall\alpha(D[\alpha]\rightarrow\exists!\beta C[\alpha,\beta])$. For α with $\zeta'\,\mathbf{r}_\mathrm{L}\,D[\alpha]$, (a) $\zeta\alpha(c_D\alpha)\!\downarrow$, (b) $\mathsf{p}_0(\zeta\alpha(c_D\alpha))\ \mathbf{r}_\mathrm{L}\ \exists\beta C[\alpha,\beta]$ and (c) $\mathsf{p}_1(\zeta\alpha(c_D\alpha))\ \mathbf{r}_\mathrm{L}\ \forall\beta,\beta'(C[\alpha,\beta]\wedge C[\alpha,\beta']\rightarrow\beta=\beta')$. Let $\gamma:=\lambda\alpha.\mathsf{c}(\mathsf{r}(\mathsf{p}_0(\zeta\alpha(c_D\alpha)))\mathsf{p}_0)$.

If $\eta,\eta'\in_\mathrm{L}\mathsf{p}_0(\zeta\alpha(c_D\alpha))$, by (b)(c), $(\mathsf{p}_0\eta=\mathsf{p}_0\eta')^{\mathbf{r}_\mathrm{L}}$. By the assumption, $\exists!\eta(\eta\in_\mathrm{L}\mathsf{r}(\mathsf{p}_0(\zeta\alpha(c_D\alpha)))\mathsf{p}_0)$ and $\gamma\alpha\!\downarrow$. For $\xi\in_\mathrm{L}\mathsf{p}_0(\zeta\alpha(c_D\alpha))$, by $\mathsf{p}_0\xi=\gamma\alpha$ and (b), $\mathsf{p}_1\xi\ \mathbf{r}_\mathrm{L}\,C[\alpha,\gamma\alpha]$. So $\mathsf{b}_C\alpha(\gamma\alpha)(\mathsf{r}(\mathsf{p}_0(\zeta\alpha(c_D\alpha)))\mathsf{p}_1)\ \mathbf{r}_\mathrm{L}\,C[\alpha,\gamma\alpha]$.

Thus $\zeta\,\mathbf{r}_\mathrm{L}\,\forall\alpha(D[\alpha]\rightarrow\exists!\beta C[\alpha,\beta])$ implies $\lambda\alpha\zeta'.\mathsf{b}_C\alpha(\gamma\alpha)(\mathsf{r}(\mathsf{p}_0(\zeta\alpha(c_D\alpha)))\mathsf{p}_1)\,\mathbf{r}_\mathrm{L}\,\forall\alpha(D[\alpha]\rightarrow C[\alpha,\gamma\alpha])$. ☐

Below we additionally assume $\mathcal{L}\equiv\mathcal{L}'$. The notions of canonicalizedness, actualizedness and completedness defined below are, although *not* implying "being realized", called "having a canonical realizer" in the literature, where the three notions do not seem to be distinguished clearly. The last two make sense only when the formula belongs to both the realized and realizing languages (i.e., $A\in\mathcal{L}\cap\mathcal{L}'$), while the first is free from such an assumption. By definition, $A^{\mathbf{r}_\mathrm{L}}\leftrightarrow A$ if all the atomic are $\mathbf{r}_\mathrm{L}$-completed.

Definition 3.30 (actualized, completed)**.** $A[\vec\eta]$ is (i) $\mathbf{r}_\mathrm{L}$-*actualized* by d_A if $\forall\vec\eta(A[\vec\eta]\leftrightarrow(d_A\vec\eta\!\downarrow\wedge\, d_A\vec\eta\ \mathbf{r}_\mathrm{L}\ A[\vec\eta]))$;
(ii) $\mathbf{r}_\mathrm{L}$-*completed* by c_A if it is $\mathbf{r}_\mathrm{L}$-canonicalized and $\mathbf{r}_\mathrm{L}$-actualized by the same c_A.

Lemma 3.31. (1) If $\in_\mathrm{L}$ is completed, so is $\exists\xi(\xi\in_\mathrm{L}$-$)$. (2) If $\mathcal{C}$ formulae are completed, so are $N(\mathcal{C})$ ones.
(3) $(\mathcal{L},\mathcal{D})$-$\mathsf{GC}_\mathrm{L}$ is $\mathbf{r}_\mathrm{L}$-realizable in $\mathbf{CDL}\mathsf{f}$ if $\in_\mathrm{L}$ is completed, $\downarrow$ actualized, and $\mathcal{D}$ formulae canonicalized.

Proof. (1) For $\xi\in_\mathrm{L}\alpha$, we have $c_{\in_\mathrm{L}}\xi\alpha\!\downarrow$ and $\langle\xi,c_{\in_\mathrm{L}}\xi\alpha\rangle\!\downarrow$. Thus $\mathsf{r}\alpha(\lambda\xi.\langle\xi,c_{\in_\mathrm{L}}\xi\alpha\rangle)\ \mathbf{r}_\mathrm{L}\ \exists\xi(\xi\in_\mathrm{L}\alpha)$ iff $\exists\xi(\xi\in_\mathrm{L}\alpha)$.
(2) By induction on $N(\mathcal{C})$ formulae. Consider $\rightarrow$ only. If $\alpha\,\mathbf{r}_\mathrm{L}\,(A\rightarrow B)$, A implies $\alpha c_A\,\mathbf{r}_\mathrm{L}\,B$, $c_B\,\mathbf{r}_\mathrm{L}\,B$ and B. If $A\rightarrow B$ then $\zeta\,\mathbf{r}_\mathrm{L}\,A$ implies $c_A\,\mathbf{r}_\mathrm{L}\,A$ and A whence B, which means $\lambda\xi.c_B\,\mathbf{r}_\mathrm{L}\,(A\rightarrow B)$.
(3) Assume $\zeta\ \mathbf{r}_\mathrm{L}\ \forall\alpha(D[\alpha]\rightarrow\exists\beta C[\alpha,\beta])$. For α with $\zeta'\,\mathbf{r}_\mathrm{L}\,D[\alpha]$, we have $\zeta\alpha(c_D\alpha)\!\downarrow\wedge(\zeta\alpha(c_D\alpha)\ \mathbf{r}_\mathrm{L}\ \exists\beta C[\alpha,\beta])$.

Let $\delta:=\lambda\zeta\alpha.\mathsf{r}(\zeta\alpha(c_D\alpha))\mathsf{p}_0$. For $\xi\in_\mathrm{L}\delta\zeta\alpha$, as $(\forall\eta\in_\mathrm{L}\mathsf{f}(\zeta\alpha(c_D\alpha))\xi)(\mathsf{p}_1\eta\ \mathbf{r}_\mathrm{L}C[\alpha,\xi])$, $\mathsf{b}_C\alpha\xi(\mathsf{r}(\mathsf{f}(\zeta\alpha(c_D\alpha))\xi)\mathsf{p}_1)$ realizes $C[\alpha,\xi]$. Since $\in_\mathrm{L}$ is completed, $(*)$ $\lambda\xi\xi'.\mathsf{b}_C\alpha\xi(\mathsf{r}(\mathsf{f}(\zeta\alpha(c_D\alpha))\xi)\mathsf{p}_1)$ realizes $(\forall\xi\in_\mathrm{L}\delta\zeta\alpha)C[\alpha,\xi]$. As $\exists\xi(\xi\in_\mathrm{L}\delta\zeta\alpha)$, (1) yields $d_{\exists\xi(\xi\in_\mathrm{L}\text{-})}(\delta\zeta\alpha)\,\mathbf{r}_\mathrm{L}\,\exists\xi(\xi\in_\mathrm{L}\delta\zeta\alpha)$. The triple of $d_\downarrow(\delta\zeta)\alpha$, $d_{\exists\xi(\xi\in_\mathrm{L}\text{-})}(\delta\zeta\alpha)$ and $(*)$ realizes $\delta\zeta\alpha\!\downarrow\wedge\exists\xi(\xi\in_\mathrm{L}\delta\zeta\alpha)\wedge(\forall\xi\in_\mathrm{L}\delta\zeta\alpha)C[\alpha,\xi]$. Thus $\exists\gamma\forall\alpha(D[\alpha]\rightarrow\gamma\alpha\!\downarrow\wedge\exists\xi(\xi\in_\mathrm{L}\gamma\alpha)\wedge(\forall\xi\in_\mathrm{L}\gamma\alpha)C[\alpha,\xi])$ is realized by $\mathsf{g}(\langle\delta\zeta,\lambda\alpha\zeta'.\langle d_\downarrow(\delta\zeta)\alpha,d_{\exists\xi(\xi\in_\mathrm{L}\text{-})}(\delta\zeta)\alpha,\lambda\xi\xi'.\mathsf{b}_C\alpha\xi(\mathsf{r}(\mathsf{f}(\zeta\alpha(c_D\alpha))\xi)\mathsf{p}_1)\rangle\rangle)$. Take $\lambda\zeta.$ of this term. ☐

Corollary 3.32. If $\xi\in_\mathrm{L}\alpha$, $\alpha\beta\!\downarrow$, $\gamma=\alpha\beta$, $\mathrm{Bo}[\alpha]$ and atomic formulae are $\mathbf{r}_\mathrm{L}$-completed in $\mathbf{CDL}\mathsf{f}$+T; T is $\mathbf{r}_\mathrm{L}$-realizable in $\mathbf{CDL}\mathsf{f}$+T; $RH(\mathcal{R})\supseteq\mathcal{D}\supseteq\mathcal{R}$; and $\mathcal{C}\supseteq\mathcal{R}$, then $\mathbf{CDL}\mathsf{f}$+$T\vdash\exists\alpha(\alpha\,\mathbf{r}_\mathrm{L}\,A)$ iff $\mathbf{CDL}\mathsf{f}$+T+$(\mathcal{C},\mathcal{D})$-$\mathsf{GC}_\mathrm{L}\vdash A$.

This generalizes the characterizations of Kleene's number realizability (by ECT); Lifschitz's (number) realizability; Kleene's functional realizability (by $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GCC}^1$) and van Oosten's functional realizability.

Moreover, this shows that $(\mathcal{L}, RH(\mathcal{R}))$-$\mathsf{GC}_{\mathrm{L}}$ follows from $(\mathcal{R}, \mathcal{R})$-$\mathsf{GC}_{\mathrm{L}}$ over $\mathbf{CDL}\mathsf{f}$.

We used f only in the proof of the last lemma and we do not know if it is definable from other constants.

3.3 Realizability of intuitionistic systems

We apply the results from the last subsection to our situation: $\mathcal{L} \equiv \mathcal{L}' \equiv \mathcal{L}_{\mathrm{F}}$ where $\mathcal{L}_{\mathrm{F}}$ is considered to include $\mathcal{L}_{\mathrm{CDL}}$ via either $\mathfrak{k}$ or $\mathfrak{o}$. Setting $\alpha\, \mathbf{r}_{\mathrm{L}}\, A :\equiv A$ and $\mathsf{b}_A := \lambda\vec{\eta}\alpha.\underline{0}$ for atomic $A[\vec{\eta}]$, we have 3.17.

Definition 3.33 ($\mathbf{r}_{\mathsf{f}}$, $\mathbf{r}'_{\mathsf{f}}$)**.** $\alpha\, \mathbf{r}_{\mathsf{f}}\, A :\equiv (\alpha\, \mathbf{r}_{\mathrm{L}}\, A)^{\mathfrak{k}}$ and $\alpha\, \mathbf{r}'_{\mathsf{f}}\, A :\equiv (\alpha\, \mathbf{r}_{\mathrm{L}}\, A)^{\mathfrak{o}}$, where $QxA[x]$ is treated as $Q\xi A[\xi(0)]$.

3.3.1 realizability of base theories

Recall $\mathbf{EL}^*_0 = \mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{AC}^{00}$. As seen in 3.2.3, for $\mathbf{r}'_{\mathsf{f}}$-realizability, it is convenient to define the following.

Definition 3.34 ($\mathbf{EL}'^*_0$, $\mathbf{EL}'_0$)**.** $\mathbf{EL}'^*_0 :\equiv \mathbf{EL}^-_0 + \Delta^0_0$-$\mathsf{AC}^{00}$+$\mathsf{MP}$+$\Delta^0_0$-$\mathsf{BKL}$ and $\mathbf{EL}'_0 :\equiv \mathbf{EL}'^*_0 + \Sigma^0_1$-$\mathsf{Ind}$.

Lemma 3.35. $N(\Sigma^0_1)$ formulae are $\mathbf{r}_{\mathsf{f}}$-completed in $\mathbf{EL}^-_0 + \Sigma^0_1$-$\mathsf{Bdg}$. $N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1)$ are $\mathbf{r}'_{\mathsf{f}}$-completed in $\mathbf{EL}'^*_0$.

Proof. The atomic are trivially completed. Let B from Δ^0_0 be completed by c_B. $\exists z B[\vec{\eta}, z]$, i.e., $\exists\beta\forall x B[\vec{\eta}, \beta(0)]$ implies $(\gamma_B|\vec{\eta})\!\downarrow \wedge B[\vec{\eta}, (\gamma_B|\vec{\eta})(0)]$ by 3.21(1)(i), i.e., $\mathsf{g}|\langle \gamma_B|\vec{\eta}, c_B|\vec{\eta}|(\gamma_B|\vec{\eta})\rangle\, \mathbf{r}_{\mathrm{L}}\, \exists z B[\vec{\eta}, z]$. By the hypothesis on c_B, $\exists\alpha(\alpha\, \mathbf{r}_{\mathrm{L}} \exists z B[\vec{\eta}, z]) \to \exists z B[\vec{\eta}, z]$. This suffices for (1) by 3.31(2) with 2.11. For (2), for A from Π^0_1, 3.25(2) yields $\forall\xi(\xi \in^{\mathfrak{o}}_{\mathrm{L}} \pi_A|\alpha|\underline{1}|\gamma \leftrightarrow \xi < \alpha \wedge A[\xi, \gamma])$. So $(\exists\xi < \alpha)A[\xi, \gamma]$ iff $\mathsf{r}|(\pi_A|\alpha|\underline{1}|\gamma)|(\lambda\xi.\langle \xi, \underline{0}, c_A|\xi|\gamma\rangle)\, \mathbf{r}'_{\mathsf{f}}\, (\exists\xi < \alpha)A[\xi, \gamma]$. ☐

Theorem 3.36. (1) $\mathbf{EL}^*_0$+$(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GCC}^1$ is $\mathbf{r}_{\mathsf{f}}$-realizable in $\mathbf{EL}^*_0$.
(2) $\mathbf{EL}'^*_0$+$(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1))\{$-$\mathsf{GCC!}^1$, -$\mathsf{GC}^{\mathfrak{o}}_{\mathrm{L}}\}$ is $\mathbf{r}'_{\mathsf{f}}$-realizable in $\mathbf{EL}'^*_0$, and so is $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1))$-$\mathsf{GCB}^1$.

Proof. Since $\in^{\mathfrak{k}}_{\mathrm{L}}$ and $\in^{\mathfrak{o}}_{\mathrm{L}}$ are $N(\Sigma^0_1)$, they are completed. Also $\alpha\beta\!\downarrow$ is completed by $\mathsf{g}|\langle \alpha|\beta, c_{\delta = \alpha|\beta}|(\alpha|\beta)|\alpha|\beta\rangle$. Thus, by 3.29 and 3.31(3), $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GCC}^1$ in (1) and $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1))\{$-$\mathsf{GCC!}^1$, -$\mathsf{GC}^{\mathfrak{o}}_{\mathrm{L}}\}$ in (2) are realizable, and so are $N(\Sigma^0_1)$ axioms of $\mathbf{EL}^-_0$. Moreover MP and Δ^0_0-BKL are $\mathbf{r}'_{\mathsf{f}}$-realizable by 3.35 as they are $N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1)$. Obviously $(\mathcal{C}, \mathcal{D})$-$\mathsf{GC}^{\mathfrak{o}}_{\mathrm{L}}$ implies $(\mathcal{C}, \mathcal{D})$-$\mathsf{GCB}^1$.

It remains to realize (d) (in 2.10) of $\mathbf{EL}^-_0$ and Δ^0_0-AC^{00}. As (d) is of the form $\exists\delta\forall x A[x, \delta(x), \alpha]$ with A from Δ^0_0, 3.21(1)(i) yields γ_A with $\mathsf{g}|\langle \gamma_A|\alpha, c_{\forall x A[x, \xi(x), \eta]}|(\gamma_A|\alpha)|\alpha\rangle\, \mathbf{r}_{\mathrm{L}}\, \exists\delta\forall x A[x, \delta(x), \alpha]$. Δ^0_0-AC^{00} is realized similarly by 3.21(1)(ii) (or see more general 3.39(ii) below). ☐

Corollary 3.37. (1) $\mathbf{EL}^*_0$+$\mathsf{S} \vdash \exists\alpha(\alpha\, \mathbf{r}_{\mathsf{f}} A)$ iff $\mathbf{EL}^*_0$+S+$(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GCC}^1 \vdash A$ for $\mathsf{S} \subseteq N(\Sigma^0_1)$.
(2) $\mathbf{EL}'^*_0$+$\mathsf{S} \vdash \exists\alpha(\alpha\, \mathbf{r}'_{\mathsf{f}}\, A)$ iff $\mathbf{EL}'^*_0$+S+$(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1))$-$\mathsf{GC}^{\mathfrak{o}}_{\mathrm{L}} \vdash A$ for $\mathsf{S} \subseteq N(\Sigma^0_1 \cup \mathsf{B}\exists^1\Pi^0_1)$.

These characterizations follow from 3.32. Among $N(\Sigma^0_1)$ schemata are MP, Σ^0_1-Ind and Π^0_2-Ind.

3.3.2 realizability of the axioms of Intuitionism with the weakest induction

While 3.37 reduces realizability to the derivability from $(\mathcal{L}_{\mathrm{F}}, \mathcal{R})$-$\mathsf{GC}_{\mathrm{L}}$, showing the latter is often as demanding as showing the former directly, as below. The folklore result 3.8 will be essential in the proof of 3.39(ii).

Proposition 3.38. $\mathcal{L}_{\mathrm{F}}$-BFT is (i) $\mathbf{r}_{\mathsf{f}}$-realizable in $\mathbf{EL}^*_0$+Δ^0_0-BFT; (ii) $\mathbf{r}'_{\mathsf{f}}$-realizable in $\mathbf{EL}'^*_0$.

Proof. As Δ^0_0-BFT is equivalently $N(\Sigma^0_1)$, it suffices to derive $\mathcal{L}_{\mathrm{F}}$-BFT from $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GC}_{\mathrm{L}}$ in the systems.

Assume $\mathsf{Fan}[\gamma]$, $\forall u(\gamma(u) = 0 \to u < \beta)$ and $(\forall\alpha < \beta)(\forall k(\gamma(\alpha{\upharpoonright}k) = 0) \to \exists k B[\alpha{\upharpoonright}k])$. Then $(\mathcal{L}_{\mathrm{F}}, N(\Sigma^0_1))$-$\mathsf{GC}_{\mathrm{L}}$ yields ζ with $(\forall\alpha < \beta)(\forall k(\gamma(\alpha{\upharpoonright}k) = 0) \to (\zeta|\alpha)\!\downarrow \wedge\, \exists\eta(\eta \in_{\mathrm{L}} \zeta|\alpha) \wedge (\forall\eta \in_{\mathrm{L}} \zeta|\alpha)B[\alpha{\upharpoonright}\eta(0)])$.

Particularly, for $\alpha < \beta$, if $\forall k(\gamma(\alpha{\upharpoonright}k) = 0)$ then both (a) $\exists m C[\alpha{\upharpoonright}m]$ and (b) $\forall m(C[\alpha{\upharpoonright}m] \to (\exists k < m)B[\alpha{\upharpoonright}k])$ hold, where $C[u] :\equiv |u| > (\zeta|u)(0)$ which is Σ^0_1, and where $\zeta|u$ is defined analogously to 3.18.

As (a) means $\mathsf{Bar}[\gamma, C]$, Σ^0_1-BFT with 2.30(3)(ii) yields n with $(\forall\alpha < \beta)(\forall k(\gamma(\alpha{\upharpoonright}k) = 0) \to (\exists m < n)C[\alpha{\upharpoonright}m])$, which, with (b), implies $(\forall\alpha < \beta)(\forall k(\gamma(\alpha{\upharpoonright}k) = 0) \to (\exists k < n)B[\alpha{\upharpoonright}k])$. Note $\mathbf{EL}'^*_0 \vdash \Delta^0_0$-$\mathsf{BFT}$ by 3.5(3). ☐

Proposition 3.39. $\mathcal{L}_{\mathrm{F}}$-AC^{00} and $\mathcal{L}_{\mathrm{F}}$-AC^{01} are (i) $\mathbf{r}_{\mathsf{f}}$-realizable in $\mathbf{EL}^*_0$; (ii) $\mathbf{r}'_{\mathsf{f}}$-realizable in $\mathbf{EL}'^*_0$.

Proof. As $\mathcal{C}$-AC^{01} yields $\mathcal{C}$-AC^{00}, it suffices to derive $\mathcal{L}_{\mathrm{F}}$-AC^{01} from $(\mathcal{L}_{\mathrm{F}}, \{\top\})$-$\mathsf{GC}_{\mathrm{L}}$ (uniformly for (i) and (ii)).

Assume $\forall x\exists\beta A[x, \beta]$, i.e., $\forall\xi\exists\beta A[\xi(0), \beta]$. We have ζ with $\forall x((\zeta|\underline{x})\!\downarrow \wedge\, \exists\eta(\eta \in_{\mathrm{L}} \zeta|\underline{x}) \wedge (\forall\eta \in_{\mathrm{L}} \zeta|\underline{x})A[x, \eta])$ by $(\mathcal{L}_{\mathrm{F}}, \{\top\})$-$\mathsf{GC}_{\mathrm{L}}$. 3.21(1)(ii) applied to $y = (\zeta|\underline{(x)^2_0})((x)^2_1)$ yields γ with $(\gamma|\zeta)\!\downarrow$ and $\forall x((\gamma|\zeta)_x = (\zeta|\underline{x}))$.

We treat (i) and (ii) separately. (i) For $\in_{\mathrm{L}} \equiv \in_{\mathrm{L}}^{\mathfrak{k}}$, obviously $\forall x A[x,(\gamma|\zeta)_x]$. (ii) For $\in_{\mathrm{L}} \equiv \in_{\mathrm{L}}^{\mathfrak{o}}$, Π^0_1-BAC^{01}, with 3.8, applied to $\forall x\exists\eta(\eta \in_{\mathrm{L}}^{\mathfrak{o}} (\gamma|\zeta)_x)$ yields α with $\forall x((\alpha)_x \in_{\mathrm{L}}^{\mathfrak{o}} (\gamma|\zeta)_x)$, which implies $\forall x A[x,(\alpha)_x]$. ☐

Theorem 3.40. (1) $\mathbf{EL}_0^-$+MP+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{CC}^1$,-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}$,-$\mathsf{BFT}\}$ is $\mathbf{r_f}$-realizable in $\mathbf{EL}_0^-$+MP+Δ^0_0-AC^{00}+Δ^0_0-BFT.
(2) $\mathbf{EL}_0'^-$+MP+Σ^0_1-GDM+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{CB}^1$,-$\mathsf{CC!}^1$,-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}$,-$\mathsf{BFT}\}$ is $\mathbf{r}'_{\mathbf{f}}$-realizable in $\mathbf{EL}_0'^*$.

As a byproduct, we have the following upper bound result for the semi-Russian axiom NCT (cf. f.n.7).

Definition 3.41 (Church's thesis CT and negative Church's thesis NCT)**.** Let $\{e\}(k)=n$ abbreviate the Σ^0_1 formula asserting that the value of the recursive function with index e at k is n (Kleene bracket).
(CT) $\forall\alpha\exists e\forall k(\alpha(k)=\{e\}(k))$; ($\mathsf{NCT}$) $\forall\alpha\neg\forall e\neg\forall k(\alpha(k)=\{e\}(k))$.

Corollary 3.42. $\mathbf{EL}_0^-$+MP+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{CC}^1$,-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}\}$+$\mathsf{NCT}$ is $\mathbf{r_f}$-realizable in $\mathbf{EL}_0^-$+MP+Δ^0_0-AC^{00}+CT.

3.3.3 realizability with Σ^0_1 induction

One may wonder if $\mathcal{C}$-FT follows from $\mathcal{C}$-BFT with $\mathcal{L}_{\mathrm{F}}$-AC^{00}, as we can take a function bounding the number n of branching in $\mathsf{Fan}[\gamma]$. This is not the case when $\mathcal{C}\equiv\Delta^0_0$ by 3.40(1) and 2.31. Here we have to distinguish two ways of bounding: depending on nodes $\forall u(\gamma(u)=0\to u\ll\delta)$ as defined below, and only on heights $\forall u(\gamma(u)=0\to u<\delta)$. Now $\mathcal{L}_{\mathrm{F}}$-AC^{00} yields the former, and we need Σ^0_1-Ind or primitive recursion to enhance it to the latter. This seems analogous to the classical fact mentioned before 2.31 that KL (König's lemma) or $\mathbf{EL}_0^-$+$\mathcal{L}_{\mathrm{F}}$-LEM+Δ^0_0-FT is consistency-wise stronger than $\mathbf{WKL}_0$ (but also than $\mathbf{EL}_0$+$\mathcal{L}_{\mathrm{F}}$-FT or $\mathbf{I\Sigma}_1$).

Definition 3.43 ($\mathcal{C}$-LBFT)**.** Let $u\ll\delta :\equiv (\forall k<|u|)(u(k)<\delta(u{\restriction}k))$ and $\mathsf{LBFan}[\gamma]:\equiv\mathsf{Fan}[\gamma]\wedge\forall u(\gamma(u)=0\to u\ll\delta)$.
($\mathcal{C}$-LBFT) $\mathsf{LBFan}[\gamma]\wedge\mathsf{Bar}[\gamma,\{u\colon B[u]\}]\to\exists m\forall\alpha(\forall k(\gamma(\alpha{\restriction}k)=0)\to(\exists n<m)B[\alpha{\restriction}n])$ for any B from $\mathcal{C}$.

Lemma 3.44. (i) $\mathbf{EL}_0$+$\mathcal{C}$-$\mathsf{BFT}\vdash\mathcal{C}$-$\mathsf{LBFT}$; and (ii) $\mathbf{EL}_0^-$+Π^0_1-AC^{00}+$\mathcal{C}$-$\mathsf{LBFT}\vdash\mathcal{C}$-$\mathsf{FT}$.

Proof. (i) Let $C[d,e,\delta]:\equiv\forall u(|u|=|d|\wedge(\forall k<|d|)(u(k)<d(k))\to\delta(u)<e)$, which is equivalently Δ^0_0. Since $\forall d\exists v(|v|=|d|+1\wedge d\subset v\wedge C[d,v(|d|),\delta])$, Δ^0_0-DC^0 yields β with $\forall nC[\beta{\restriction}n,\beta(n),\delta]$. Then $\forall u(\gamma(u)=0\to u\ll\delta)$ implies $\forall u(\gamma(u)=0\to u<\beta)$. (ii) Π^0_1-AC^{00}, applied to $\mathsf{Fan}[\gamma]$, yields δ with $\forall u(\gamma(u)=0\to u\ll\delta)$. ☐

Next let us realize Σ^0_2-DC^0, which implies Σ^0_1-DC^1 by 2.14 and Σ^0_2-Ind by 2.16(3)(i). This might be the most non-trivial part of the present paper. The trick is the use of semi-classical principle. For, the realizing theory does not need to be intuitionistic since $\mathbf{i\Sigma}_1$ and $\mathbf{I\Sigma}_1$ are known to be mutually interpretable. We do not know if Σ^0_2-Ind (or Σ^0_2-DC^0, Σ^0_1-DC^1) can be realizable directly in $\mathbf{i\Sigma}_1$. Let us start with $\mathbf{r_f}$-realizability.

Definition 3.45 (closure under $\mathcal{C}$ functions)**.** A class $\mathcal{S}$ is called *closed under $\mathcal{C}$ functions* iff (i) $\mathcal{S}\wedge\mathcal{C}\wedge\neg\mathcal{C}\subseteq\mathcal{S}$ and (ii) for C from $\mathcal{C}$ and D from $\mathcal{S}$, there is D_C from $\mathcal{S}$ with $\mathbf{EL}_0^-\vdash\exists!yC[x,y]\to(D_C[x]\leftrightarrow\exists y(C[x,y]\wedge D[x,y]))$.

Proposition 3.46. (1) $\Delta^0_0(\Sigma^0_1)\cap N(\Sigma^0_1)$ is closed under Σ^0_1 functions.
(2) If $\mathcal{S}\subseteq N(\Sigma^0_1)$ and is closed under Σ^0_1 functions, both $\exists^0\mathcal{S}$-DC^0 and $\exists^0\mathcal{S}$-Ind are $\mathbf{r_f}$-realizable in $\mathbf{EL}_0$+$\mathcal{S}$-Ind.

Proof. (1) is by induction on D: $\exists!yC[y]$ yields $\exists y(C[y]\wedge(D_1[y]\to D_2[y]))\leftrightarrow(\exists y(C[y]\wedge D_1[y])\to\exists y(C[y]\wedge D_2[y]))$.
(2) As $\mathcal{S}$-Ind is $N(\Sigma^0_1)$, it suffices derive $\mathcal{S}$-DC^0 in $\mathbf{EL}_0$+$(\mathcal{L}_{\mathrm{F}},N(\Sigma^0_1))$-$\mathsf{GCC}^1$+$\mathcal{S}$-$\mathsf{Ind}$ by 2.16(2)(i)(3)(i), 3.37(1) and 3.39(i). Let $\forall x,y(A[x,y]\to\exists zA[y,z])$ with A from $\mathcal{S}$, say $\forall x,y(A[x,y]\to(\gamma|\underline{x}|\underline{y}){\downarrow}\wedge A[\underline{y},(\gamma|\underline{x}|\underline{y})(0)])$ by $(\mathcal{L}_{\mathrm{F}},N(\Sigma^0_1))$-$\mathsf{GCC}^1$. Fix x,y with $A[x,y]$. We prove $\exists!uC[n,u]\wedge D_C[n]$ by $\mathcal{S}$-Ind on n, where
$C[n,u]:\equiv|u|=n+2\wedge u{\restriction}2=\langle x\rangle*\langle y\rangle\wedge(\forall k<n)(u(k+2)=(\gamma|\underline{u(k)}|\underline{u(k+1)})(0))$; $D[k,u]:\equiv A[u(k),u(k+1)]$,

and D_C is by the closure under Σ^0_1 functions. If it is done, Σ^0_1-AC^{00} yields β with $\forall n\exists u(C[n,u]\wedge u(n)=\beta(n))$.
As $C[0,\langle x\rangle*\langle y\rangle]$, $D_C[0]$ is by $A[x,y]$. If $\exists!uC[n,u]\wedge D_C[n]$, say $C[n,v]$, then $D_C[n]$ means $A[v(n),v(n+1)]$ and so $(\gamma|\underline{v(n)}|\underline{v(n+1)}){\downarrow}\wedge A[v(n+1),z]$ for $z=(\gamma|\underline{v(n)}|\underline{v(n+1)})(0)$. Thus $C[n+1,v*\langle z\rangle]$ and so $D_C[n+1]$. ☐

As $\Pi^0_1\subseteq\Delta^0_0(\Sigma^0_1)\cap N(\Sigma^0_1)$, $\Sigma^0_2\{$-DC^0,-$\mathsf{Ind}\}$ is $\mathbf{r_f}$-realizable in $\mathbf{EL}_0$+Σ^0_2-DNE, by 3.3(2), the other folklore.

For this argument functionality is not essential: ECT in Kleene's number realizability can substitute GCC, and so $\mathbf{i\Sigma}_2$ is realizable in $\mathbf{I\Sigma}_1$. Wehmeier [45] identified the strengths of $\mathbf{i\Sigma}_1$, $\mathbf{i\Pi}_{n+2}$ and $\mathbf{i\Sigma}_{n+3}$ by this realizability, but left $\mathbf{i\Sigma}_2$. Burr [10] identified it by another method. Our argument shows that Wehmeier's method could deal with $\mathbf{i\Sigma}_2$. If we extend this number realizability to $\mathcal{L}_{\mathrm{F}}$ in an obvious manner, Σ^0_2-DC^0 and CT are also realizable. By allowing Σ_n oracle, we can interpret $\mathbf{i\Sigma}_{n+2}$+Σ_{n+2}-DNE in $\mathbf{I\Sigma}_{n+1}$.

For $\mathbf{r}'_{\mathbf{f}}$-realizability, this does not seem to work well. We employ a more elaborated way, which works also in the first order setting, i.e., Lifschitz's number realizability, with recursive indices substituting functions. However, in this case we do not know if we can enhance Σ^0_2 to $\exists^0(\Delta^0_0(\Sigma^0_1)\cap N(\Sigma^0_1))$ as in the previous case.

Definition 3.47 (($\mathcal{C},\mathcal{D}$)-EUB)**.** The schema of *extended uniform bounding* $(\mathcal{C},\mathcal{D})$-EUB is defined as follows.
(($\mathcal{C},\mathcal{D}$)-EUB) $\forall x(D[x]\to\exists yC[x,y])\to\exists\alpha\forall n(\forall x<n)(D[x]\to(\exists y<\alpha(n))C[x,y])$ for C from $\mathcal{C}$ and D from $\mathcal{D}$.

Lemma 3.48. (Π^0_1,Π^0_1)-EUB is $\mathbf{r}'_{\mathbf{f}}$-realizable in $\mathbf{EL}'_0$+LPO.

Proof. Let C and D be Π^0_1. Let D be $\mathbf{r}'_{\mathbf{f}}$-completed by c_D, by 3.35. Define A and B as follows:

$$A[n,m,\zeta]:\equiv(\forall x<n)(D[x]\to(\zeta|\underline{x}|(c_D|\underline{x}))(0)\le m);\qquad B[\alpha]:\equiv\forall n(\forall x<n)(D[x]\to(\exists y<\alpha(n))C[x,y]).$$

As A is equivalently Σ^0_1 by LPO, 3.21(1)(ii) yields γ_A with $\forall n\exists mA[n,m,\zeta]\to(\gamma_A|\zeta)\!\downarrow\wedge\forall nA[n,(\gamma_A|\zeta)(n),\zeta]$.

Let $\zeta\,\mathbf{r}'_{\mathbf{f}}\,\forall x(D[x]\to\exists yC[x,y])$. We prove $\exists mA[n,m,\zeta]$ by induction on n. Obviously $A[0,0,\zeta]$. If $A[n,m,\zeta]$ and $D[n]$, then $c_D|\underline{n}\,\mathbf{r}'_{\mathbf{f}}\,D[n]$ and so $(\zeta|\underline{n}|(c_D|\underline{n}))\!\downarrow$ which implies $A[n{+}1,m',\zeta]$ for $m':=m+(\zeta|\underline{n}|(c_D|\underline{n}))(0)$. If $A[n,m,\zeta]$ and $\neg D[n]$, then $A[n{+}1,m,\zeta]$. By Π^0_1-LEM, we have $\exists mA[n,m,\zeta]\to\exists mA[n{+}1,m,\zeta]$.

Thus $\forall nA[n,(\gamma_A|\zeta)(n),\zeta]$. Then $(\forall x<n)(D[x]\to(\zeta|\underline{x}|(c_D|\underline{x}))(0)\le(\gamma_A|\zeta)(n)\wedge\zeta|\underline{x}|(c_D|\underline{x})\ \mathbf{r}'_{\mathbf{f}}\ \exists yC[x,y])$ and so $B[\gamma_A|\zeta]$. As B is $N(\mathsf{B}\exists^1\Pi^0_1)$, by 3.35, let B be $\mathbf{r}'_{\mathbf{f}}$-completed by c_B. Then $\mathsf{g}|\langle\gamma_A|\zeta,c_B|(\gamma_A|\zeta)\rangle\,\mathbf{r}'_{\mathbf{f}}\,\exists\alpha B[\alpha]$.

Therefore $\lambda\zeta.\mathsf{g}|\langle\gamma_A|\zeta,c_B|(\gamma_A|\zeta)\rangle$ $\mathbf{r}'_{\mathbf{f}}$-realizes the instance of (Π^0_1,Π^0_1)-EUB. ☐

Proposition 3.49. Π^0_1-DC^0 is $\mathbf{r}'_{\mathbf{f}}$-realizable in $\mathbf{EL}'_0$+LPO. Hence so are Σ^0_2-DC^0, Σ^0_1-DC^1 and Σ^0_2-Ind.

Proof. By 3.36(2) and 3.48, it suffices to derive Π^0_1-DC^0 in $\mathbf{EL}'_0+(\Pi^0_1,\Pi^0_1)$-EUB. Let A be Π^0_1, and assume $\forall x,y(A[x,y]\to\exists zA[y,z])$. Then (Π^0_1,Π^0_1)-EUB yields α with $(\forall v<n)(A[(v)^2_0,(v)^2_1]\to(\exists z<\alpha(n))A[(v)^2_1,z])$.

Fix x,y with $A[x,y]$. Δ^0_0-DC^0 yields β with $\beta\lceil 2:=\langle x{+}1\rangle*\langle y{+}1\rangle$ and $\beta(k{+}2):=\alpha((\beta(k),\beta(k{+}1)))$. Let

$$B[n,\beta]:\equiv(\exists u<\beta)C[u,n];\qquad C[u,n]:\equiv|u|=n{+}2\wedge u(0)=x\wedge(\forall k\le n)A[u(k),u(k{+}1)]$$

where $u<\beta:\equiv(\forall k<|u|)(u(k)<\beta(k))$. B is Π^0_1 by 2.23(2)(i) and 3.7. Σ^0_1-Ind and MP yield Π^0_1-Ind. It remains to see $\forall nB[n,\beta]$ by Π^0_1-Ind, as it implies $\exists\gamma(\gamma(0)=x\wedge\forall kA[\gamma(k),\gamma(k{+}1)])$ by Π^0_1-BKL with 3.5(2)(i).

Obviously $\langle x\rangle*\langle y\rangle$ witnesses $B[0,\beta]$. Let $B[n,\beta]$, say $u<\beta\wedge C[u,n]$. Since $(u(n),u(n{+}1))<(\beta(n),\beta(n{+}1))$, $A[u(n),u(n{+}1)]$ yields $z<\alpha((\beta(n),\beta(n{+}1)))=\beta(n{+}2)$ with $A[u(n{+}1),z]$. So $C[u*\langle z\rangle,n{+}1]\wedge u*\langle z\rangle<\beta$. ☐

Theorem 3.50. (1) $\mathbf{EL}_0+\Sigma^0_1$-DC^1+$\Sigma^0_2\{$-DC^0,-Ind$\}$+$\mathcal{L}_{\mathrm{F}}$-FT is $\mathbf{r}_{\mathbf{f}}$-realizable in $\mathbf{EL}_0+\Delta^0_0$-BFT+Σ^0_2-DNE.
(2) $\mathbf{EL}'_0+\Sigma^0_1\{$-GDM,-DC$^1\}$+$\Sigma^0_2\{$-DC^0,-Ind$\}$+$\mathcal{L}_{\mathrm{F}}$-FT is $\mathbf{r}'_{\mathbf{f}}$-realizable in $\mathbf{EL}'_0$+LPO.

3.3.4 realizability with Π^0_2 induction

It is natural to ask how to realize Π^0_{n+2}-Ind and Σ^0_{n+3}-Ind. As Wehmeier [45] mentioned, they are all realizable in $\mathbf{I\Sigma}_2$ by Kleene's number realizability. This remains to hold for our two kinds of functional realizability.

Lemma 3.51. (1)(i) $\Pi^0_n{\to}\Pi^0_{n+1}\subseteq\forall^0\neg\Pi^0_n$ over $\mathbf{EL}^-_0+\Sigma^0_n$-DNE; (ii) $(\forall\xi\in^{\circ}_{\mathrm{L}}\alpha)A[\xi,\alpha]$ is Π^0_2 over $\mathbf{EL}'^*_0$ if A is Π^0_2.
(2) For B from Π^0_1, $\mathbf{EL}^-_0+\Delta^0_0$-BKL $\vdash\forall n(\exists\eta<\alpha)(\forall k<n)B[k,(\eta)_k,(\eta)_{k+1},(\eta)_{k+2}]\to(\exists\eta<\alpha)\forall kB[k,(\eta)_k,(\eta)_{k+1},(\eta)_{k+2}]$.
(3) (i) $\Pi^0_\infty\subseteq\exists^1\Pi^0_1$ over $\mathbf{EL}^-_0+\Pi^0_1$-AC^{01}. Hence (ii) $\mathbf{EL}^-_0+\Pi^0_1$-AC$^{01}\vdash\Pi^0_\infty$-AC^{01} and $\mathbf{EL}^-_0+\Pi^0_1$-DC$^1\vdash\Pi^0_\infty$-DC^1.

Proof. (1)(i) Σ^0_n-DNE yields $\Pi^0_n{\to}\forall^0\Sigma^0_n=\forall^0(\Pi^0_n{\to}\Sigma^0_n)=\forall^0(\Pi^0_n{\to}\neg\neg\Sigma^0_n)=\forall^0\neg(\Pi^0_n{\wedge}\neg\Sigma^0_n)\subseteq\forall^0\neg\Pi^0_n$ by 2.24(1)(ii).
(ii) Take B from Π^0_1 with $(\xi\in^{\circ}_{\mathrm{L}}\alpha\to A[\xi,\alpha])\leftrightarrow\forall x\neg B[x,\xi,\alpha]$ by (i). Then $(\forall\xi\in^{\circ}_{\mathrm{L}}\alpha)A[\xi,\alpha]$ is equivalent to $(\forall\xi<(\alpha)^2_0)\forall x\neg B[x,\xi,\alpha]$, i.e., $\forall x\neg(\exists\xi<(\alpha)^2_0)B[x,\xi,\alpha]$ which is equivalently Π^0_2 by 3.5(2)(ii) and MP.
(2) Let C be Δ^0_0 such that $\forall\eta,k(B[k,\eta,\eta',\eta'']\leftrightarrow\forall\ell C[k,\eta\lceil\ell,\eta'\lceil\ell,\eta''\lceil\ell])$ by 2.14. Now the premise implies $\forall z(\exists u<\alpha)(|u|=z\wedge(\forall k,\ell<z)C[k,(u)_k\lceil\ell,(u)_{k+1}\lceil\ell,(u)_{k+2}\lceil\ell])$, where $(u)_k$ is as in 3.8. Apply Δ^0_0-BKL.
(3) Π^0_1-AC^{01} yields the Skolem functions for any Π^0_∞ formula under the necessary existence assumption. More precisely, we can show, by meta-induction on $k\le n$ with Π^0_1-AC^{01}, that $\forall x_k\exists y_k...\forall x_0\exists y_0C[x_n,...,x_0,y_n,...,y_0]$ is equivalent to $\exists\alpha\forall x_k,...,x_0C[x_n,...,x_0,y_n,...,y_{k+1},(\alpha)_k(x_k),...,(\alpha)_0(x_k,...x_0)]$, for any Δ^0_0-formula C. ☐

Definition 3.52 (rec)**.** Let rec be such that $\mathbf{EL}^-_0\vdash(\mathsf{rec}|\xi|\eta|\underline{0}\simeq\xi)^{\mathfrak{k}}\wedge(\mathsf{rec}|\xi|\eta|\underline{z{+}1}\simeq\eta|(\mathsf{rec}|\xi|\eta|\underline{z})|\underline{z})^{\mathfrak{k}}$.

The existence of rec is directly by 3.21(1)(ii), but it can also be constructed by fix and d as in the usual theories of operations and numbers (cf. [5, VI.2.8] and [40, Ch.9, 3.8]). However, we need Π^0_2-Ind as well as Δ^0_0-AC^{00} to imply $\forall z((\mathsf{rec}|\xi|\eta|\underline{z})\!\downarrow)$ from $\forall z((\mathsf{rec}|\xi|\eta|\underline{z})\!\downarrow\to(\mathsf{rec}|\xi|\eta|\underline{z{+}1})\!\downarrow)$. This is why we need Π^0_2-Ind.

In the following, (i) is just by constructing the realizer in this way, while (ii) requires further tricks.

Theorem 3.53. $\exists^1\Pi^0_\infty\{\text{-}\mathsf{DC}^1,\text{-}\mathsf{DC}^0,\text{-}\mathsf{Ind}\}$ are (i) $\mathbf{r_f}$-realizable in $\mathbf{EL}_0^*{+}\Pi^0_2\text{-}\mathsf{Ind}$; (ii) $\mathbf{r}'_\mathbf{f}$-realizable in $\mathbf{EL}'_0{+}\Pi^0_2\text{-}\mathsf{Ind}$.
Proof. By 2.16(2)(i)(3)(i) with 3.51(3)(ii), it suffices to realize $\Pi^0_1\text{-}\mathsf{DC}^1$. By 3.21(1)(ii) and 3.25(2), take μ so that $\eta \in_{\mathrm{L}} \mu|\xi$ iff $\mathsf{p}_0|\eta = \mathsf{p}_0|\xi \wedge \mathsf{p}_1|\eta \in_{\mathrm{L}} \mathsf{p}_1|\xi$ for any η. Then by 3.21(1)(ii) construct ϵ so that, for any $\zeta,\zeta',\beta,\gamma$,

$$\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{0} \simeq \mathsf{g}|\langle\langle\beta,\underline{0}\rangle,\langle\gamma,\zeta'\rangle\rangle; \qquad \epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z+1} \simeq \mathsf{u}|(\mathsf{r}|(\mathsf{r}|(\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z})|(\theta'|\zeta))|\mu)$$

where $\theta := \lambda\zeta\xi\xi'.\zeta|(\xi)^2_0|(\xi')^2_0|(\xi')^2_1$ and $\theta' := \lambda\zeta\xi.\langle\mathsf{p}_1|\xi,\theta|\zeta|(\mathsf{p}_0|\xi)|(\mathsf{p}_1|\xi)\rangle$. The last $\simeq$ means that, for any η,

$$\eta \in_{\mathrm{L}} \epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z+1} \text{ iff } (\exists\eta' \in \epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z})\left((\eta)^2_0 = (\eta')^2_1 \wedge (\eta)^2_1 \in_{\mathrm{L}} \zeta|((\eta')^2_0)^2_0|((\eta')^2_1)^2_0|((\eta')^2_1)^2_1\right).$$

Note that $(\alpha|...|\beta)\downarrow$ is Π^0_2 by $\Delta^0_0\text{-}\mathsf{AC}^{00}$. By 3.21(1)(ii), we can take ϵ'' such that, for any $\zeta,\zeta',\beta,\gamma$,

$$\forall z(\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z}\downarrow) \to \epsilon''|\zeta|\zeta'|\beta|\gamma\downarrow \wedge \forall z((\epsilon''|\zeta|\zeta'|\beta|\gamma)_z \simeq \mathsf{r}|(\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z})|\mathsf{p}_0).$$

For (i), set $\epsilon' = \epsilon''$ and, for (ii), by 3.25(2) take also ϵ' so that, for any $\zeta,\zeta',\beta,\gamma$,

$$\forall z(\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z}\downarrow) \to \epsilon'|\zeta|\zeta'|\beta|\gamma\downarrow \wedge \forall\eta\,(\eta \in_{\mathrm{L}} \epsilon'|\zeta|\zeta'|\beta|\gamma \leftrightarrow \eta \le \epsilon''|\zeta|\zeta'|\beta|\gamma \wedge \forall z\,((\eta)_{z+2} \in_{\mathrm{L}} \theta|\zeta|(\eta)_z|(\eta)_{z+1})).$$

Let A be Π^0_1. Assume $\zeta\,\mathbf{r}_{\mathrm{L}}\,\forall\beta,\gamma(A[\beta,\gamma] \to \exists\delta A[\gamma,\delta])$ and $\zeta'\,\mathbf{r}_{\mathrm{L}}\,A[\beta,\gamma]$. By $\Pi^0_2\text{-}\mathsf{Ind}$ on z we can show:
(a) $\epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z}\downarrow \wedge (\forall\eta \in_{\mathrm{L}} \epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z})(((\eta)^2_1)^2_1\,\mathbf{r}_{\mathrm{L}}\,A[((\eta)^2_0)^2_0,((\eta)^2_1)^2_0])$ which is equivalently Π^0_2 by 3.51(1)(ii),
(b) $(\exists\eta \le \epsilon''|\zeta|\zeta'|\beta|\gamma)(\langle(\eta)_z,(\eta)_{z+1}\rangle \in_{\mathrm{L}} \epsilon|\zeta|\zeta'|\beta|\gamma|\underline{z} \wedge (\forall k < z)((\eta)_{k+2} \in_{\mathrm{L}} \theta|\zeta|(\eta)_k|(\eta)_{k+1}))$ and
(c) $(\forall\eta \in_{\mathrm{L}} \epsilon'|\zeta|\zeta'|\beta|\gamma)(\mathsf{p}_1|(\eta)_{z+1}\,\mathbf{r}_{\mathrm{L}}\,A[\mathsf{p}_0|(\eta)_z,\mathsf{p}_0|(\eta)_{z+1}])$ which is equivalently Π^0_2 again by 3.51(1)(ii).
By 3.21(1)(ii) take ν with $(\mathsf{p}_0|(\nu|\eta))_z = \mathsf{p}_0|(\eta)_z$ and $(\mathsf{p}_1|(\nu|\eta))|\underline{z} = \mathsf{p}_1|(\eta)_{z+1}$ for any η, z. Now (c) yields $(\forall\eta \in_{\mathrm{L}} \epsilon'|\zeta|\zeta'|\beta|\gamma)(\mathsf{p}_1|(\nu|\eta)\;\mathbf{r}_{\mathrm{L}}\,\forall z A[(\mathsf{p}_0|(\nu|\eta))_z,(\mathsf{p}_0|(\nu|\eta))_{z+1}])$. To show $\mathsf{r}|(\epsilon'|\zeta|\zeta'|\beta|\gamma)|\nu\,\mathbf{r}_{\mathrm{L}}\,\exists\eta\forall z A[(\eta)_z,(\eta)_{z+1}]$, it remains to see $\exists\eta(\eta \in_{\mathrm{L}} \epsilon'|\zeta|\zeta'|\beta|\gamma)$. $\Sigma^0_1\text{-}\mathsf{AC}^{00}$ yields $\alpha = \epsilon''|\zeta|\zeta'|\beta|\gamma$. (i) is done. For (ii) apply 3.51(2) to (b). $\square$

3.3.5 realizability with full induction and full bar induction

For the sake of completeness, let us realize even stronger induction schemata, beyond $\Pi^0_\infty\text{-}\mathsf{Ind} = \Sigma^0_\infty\text{-}\mathsf{Ind}$. The self-realizability of full induction $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{Ind}$ was known (e.g., from [29]). Here we recall and hierarchize it.

Definition 3.54. (1) $\Lambda^i_{n,0} :\equiv \forall^i\Sigma^0_n$; $\Lambda^i_{n,m+1} :\equiv \forall^i(\Lambda^i_{n,m}{\to}\Sigma^0_n)$ for $i<2$. (2) $\Xi_{n,0} :\equiv \Pi^0_{n+1}$; $\Xi_{n,m+1} :\equiv \forall^1(\Xi_{n,m}{\to}\Sigma^0_n)$. (3) Θ^1_0 is the closure of Δ^0_0 under $\wedge,\vee,\forall^0,\exists^0$ and $\exists^1$; Θ^1_{m+1} is that of Θ^1_m under $\wedge,\vee,\forall^0,\exists^0,\forall^1,\exists^1$ and $\Theta^1_m{\to}(\text{-})$.

Θ^1_m is the second order analogue of Burr's Θ_m from [10]. Note that Θ^1_m's exhaust $\mathcal{L}_{\mathrm{F}}$ and $\Xi_{n,m} \subseteq \Theta^1_m$. Moreover, $\Xi_{n,m+1}$ is equivalent to $\Lambda^1_{n+1,m}$ over $\mathbf{EL}_0^-{+}\Sigma^0_{n+1}\text{-}\mathsf{DNE}$. The next is enough to generalize 3.53.

Lemma 3.55. If A is Θ^1_m then both $\alpha\,\mathbf{r_f}\,A$ and $\alpha\,\mathbf{r}'_\mathbf{f}\,A$ are equivalently $\Xi_{1,m}$ over $\mathbf{EL}_0^*$ and $\mathbf{EL}'^*_0$ respectively.
Proof. $(\alpha|...|\beta)\downarrow$ is Π^0_2 by $\Delta^0_0\text{-}\mathsf{AC}^{00}$. For $\exists,\vee$ in the case of $\mathbf{r}'_\mathbf{f}$ with $m=0$, use 3.51(1)(ii). $\Xi_{n,m}$ is closed under $\wedge$, as $(A\to B)\wedge(C\to D)$ is equivalent to $\forall n((n=0\to A){\wedge}(n>0\to C)\to(n=0\to B){\wedge}(n>0\to D))$. $\square$

Theorem 3.56. $\Theta^1_m\{\text{-}\mathsf{DC}^1,\text{-}\mathsf{DC}^0,\text{-}\mathsf{Ind}\}$ is $\mathbf{r_f}$-realizable in $\mathbf{EL}_0^*{+}\Xi_{1,m}\text{-}\mathsf{Ind}$, and $\mathbf{r}'_\mathbf{f}$-realizable in $\mathbf{EL}'_0{+}\Xi_{1,m}\text{-}\mathsf{Ind}$.
Proof. The proof is the same as 3.53, but now (a) and (c) are $\Xi_{1,m}$ if A is Θ^1_m, where $\Pi^0_2 \equiv \Xi_{1,0} \subseteq \Xi_{1,m}$. $\square$

Note $\exists^1\Pi^0_\infty = \Theta^1_0$ over $\mathbf{EL}_0^*{+}\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{AC}^{01}$ by 2.11. Thus 3.53 can be seen as the instance of 3.56 for $m=0$.

A similar strategy by 3.55 applies to bar induction. This is the last Brouwerian axiom that we realize.

Theorem 3.57. $(\Theta^1_m,\mathcal{L}_{\mathrm{F}})\text{-}\mathsf{BI}_M$ is $\mathbf{r_f}$-realizable in $\mathbf{EL}_0^*{+}\Xi_{1,m}\text{-}\mathsf{BI}_D$, and $\mathbf{r}'_\mathbf{f}$-realizable in $\mathbf{EL}'_0{+}\Xi_{1,m}\text{-}\mathsf{BI}_D$.
Proof. As $(\mathcal{L}_{\mathrm{F}},\Delta^0_0)\text{-}\mathsf{GCB}^1$ implies $\mathcal{L}_{\mathrm{F}}\text{-}\mathsf{CB}^0$, it suffices to realize $\Theta^1_m\text{-}\mathsf{BI}_D$, by 2.37(4) and 3.36. Assume that ζ $\mathbf{r}_{\mathrm{L}}$-realizes $\mathsf{Bar}[\underline{0},\{u\colon\alpha(u)=0\}] \wedge \forall u(\alpha(u)=0 \to A[u]) \wedge \forall u(\forall x A[u{*}\langle x\rangle] \to A[u])$. 3.21(1)(ii) yields γ,δ,ϵ with

$$\gamma|\xi|\underline{u}|\eta \simeq \xi|\underline{u{*}\langle\eta(0)\rangle}, \qquad \delta|\zeta|\alpha|\xi|\underline{u} \simeq \begin{cases} (\zeta)^3_1|\underline{u}|(c_{\alpha(u)=0}|\alpha|\underline{u}) & \text{if } \alpha(u)=0; \\ (\zeta)^3_2|\underline{u}|(\lambda\eta.\gamma|\xi|\underline{u}|\eta) & \text{otherwise} \end{cases}, \quad \text{and} \quad \epsilon := \lambda\zeta.\mathsf{fix}|(\delta|\zeta|\alpha).$$

Let $B[u] :\equiv (\epsilon|\zeta|\underline{u})\downarrow \wedge (\epsilon|\zeta|\underline{u}\;\mathbf{r}_{\mathrm{L}}\,A[u])$. If $\alpha(u)=0$ then $\epsilon|\zeta|\underline{u}\;\mathbf{r}_{\mathrm{L}}\,A[u] \equiv B[u]$. As $\mathsf{Bar}[\underline{0},\{u\colon\alpha(u)=0\}]$ by 3.35, it remains to show $\forall x B[u{*}\langle x\rangle] \to B[u]$: we may assume $\alpha(u)\neq 0$, and $\epsilon|\zeta|\underline{u} \simeq (\zeta)^3_2|\underline{u}|(\lambda\eta.\epsilon|\zeta|\underline{u{*}\langle\eta(0)\rangle})$. Thus $\forall x B[u{*}\langle x\rangle]$, i.e., $(\lambda\eta.\epsilon|\zeta|\underline{u{*}\langle\eta(0)\rangle})\,\mathbf{r}_{\mathrm{L}}\,\forall x A[u{*}\langle x\rangle]$ yields $\epsilon|\zeta|\underline{u}\;\mathbf{r}_{\mathrm{L}}\,A[u]$. Hence $\lambda\zeta.(\epsilon|\zeta|\langle\rangle)$ realizes $\underline{\{A\}}\text{-}\mathsf{BI}_D$. $\square$

One may wonder if this can be extended to the "bar version" of dependent choice, defined as follows:

$((\mathcal{C},\mathcal{D})\text{-}\mathsf{BDC}_M)$ $\mathsf{Bar}[\underline{0},\{u\colon B[u]\}] \wedge \forall u,v(B[u]\to B[u{*}v]) \wedge \forall u,\beta\exists\gamma A[u,\beta,\gamma]$

$$\to \forall\alpha(\forall u(B[u]\to A[u,(\alpha)_{\prec u},(\alpha)_u]) \to \exists\delta\forall u(A[u,(\delta)_{\prec u},(\delta)_u] \wedge (B[u]\to(\delta)_u = (\alpha)_u))$$

where $(\gamma)_{\prec u}$ is such that $((\gamma)_{\prec u})_x = (\gamma)_{u*\langle x\rangle}$ and A is from $\mathcal{C}$ and B from $\mathcal{D}$.

Among similar axioms are *transfinite dependent choice* [31, 32] and *bar recursion* [4, §6.4]. In our context, this extension is not proper, as $\mathbf{EL}_0^-{+}\forall^0\mathcal{C}\text{-}\mathsf{AC}^{01}{+}\mathcal{D}\text{-}\mathsf{CB}^0{+}\exists^1\forall^0\mathcal{C}\text{-}\mathsf{BI}_D \vdash (\mathcal{C},\mathcal{D})\text{-}\mathsf{BDC}_M$: apply $\exists^1\forall^0\mathcal{C}\text{-}\mathsf{BI}_D$ to $A'[u] :\equiv \exists\delta\forall v(A[u{*}v,(\delta)_{\prec u*v},(\delta)_{u*v}] \wedge (B[u{*}v]\to(\delta)_{u*v} = (\alpha)_{u*v}))$ where B can be Δ^0_0 similarly to 2.37(4).

4 Lower Bounds: Forcing and Negative Interpretations

4.1 Gödel-Gentzen negative interpretation

Gödel-Gentzen negative interpretation N, sometimes called double negation translation, is the standard way of interpreting logical symbols of classical logic intuitionistically. In arithmetic, since $\neg\neg A$ is equivalent to A for atomic A, if we consider the classical $\vee$ and $\exists$ as abbreviations defined from $\wedge, \to, \perp, \forall$, we may identify A^N with A, and intuitionistic theories are extensions of classical ones with new logical symbols $\vee$ and $\exists$ in the same sense as modal logics are extensions with $\Box$ and $\Diamond$. Here, however, we consider $\vee$ and $\exists$ are primitive symbols even in the classical theories, which extend intuitionistic ones by $\mathcal{L}_{\mathrm{F}}$-LEM.

Definition 4.1 (N)**.** For a formula A, $A^N :\equiv \neg\neg A$ for atomic A; $(A \Box B)^N :\equiv A^N \Box B^N$ for $\Box \equiv \wedge, \to$; $(A \vee B)^N :\equiv \neg(\neg A^N \wedge \neg B^N)$; $(\forall \xi A)^N :\equiv \forall \xi A^N$; and $(\exists \xi A)^N :\equiv \neg\forall\xi\neg A^N$, where $QxA[x]$ is considered as $Q\xi A[\xi(0)]$.

Lemma 4.2. (1) A^N intuitionistically follows from ${B_1}^N$, ..., ${B_n}^N$, if A classically follows from B_1, ..., B_n.
(2) $((\exists x < y)A)^N$ and $((\forall x < y)A)^N$ are equivalent to $\neg(\forall x < y)\neg A^N$ and $(\forall x < y)A^N$ respectively.
(3) $\mathbf{EL}_0^- + \Sigma_n^0$-$\mathsf{DNE} \vdash A^N \leftrightarrow A$ if A is negative in Π_{n+1}^0, i.e., built up by $\wedge$, $\to$ and $\forall$ from Π_{n+1}^0 formulae.
(4) $\mathbf{EL}_0^- + \Sigma_n^0$-$\mathsf{DNE} \vdash A \to A^N$ if A is built up by $\wedge$, $\forall$ and $\exists$ from those formulae negative in Π_{n+1}^0.

Corollary 4.3. (1) $\mathbf{EL}_0^- \vdash (\mathbf{EL}_0^-)^N$; and Π_{n+1}^0-preservingly N interprets $\mathbf{EL}_0^- + \mathcal{L}_{\mathrm{F}}$-$\mathsf{LEM}$ in $\mathbf{EL}_0^- + \Sigma_n^0$-$\mathsf{DNE}$.
(2) Over $\mathbf{EL}_0^- + \Sigma_n^0$-$\mathsf{DNE}$, (i) Π_{n+1}^0-Ind, (ii) Σ_n^0-Bdg and (iii) Σ_n^0-Ind are equivalent to their N-interpretations; if $n \geq 1$, so are (iv) $\mathcal{C}$-BI_D and (v) $(\mathcal{C}, \mathcal{D})$-$\mathsf{BI}_M$ for $\mathcal{C} \in \{\Sigma_k, \Lambda_{k,m}^i, \Xi_{k,m} \mid k \leq n\}$, $\mathcal{D} \in \{\Pi_\ell^0, \Sigma_{\ell+1}^0 \mid \ell < n\}$.

Recall 3.54, the definitions of $\Lambda_{n,m}^i, \Xi_{n,m}$. $\Lambda_{1,m}^1$ is the N-interpretation of Π_{m+1}^1 normal form, over MP.

While N will be one of our main tools for lower bound proof, it yields some result for a semi-Russian axiom KA, introduced by Veldman [42]. This asserts the existence of counterexample of Δ_0^0-WFT.

Proposition 4.4 (KA)**.** Let $\mathrm{KA}[\gamma] :\equiv (\forall \alpha < \underline{2})\exists n(\gamma(\alpha{\restriction}n) = 0) \wedge \forall m(\exists u < \underline{2})(|u| = m \wedge (\forall k < m)(\gamma(u{\restriction}k) > 0))$. Then $\mathbf{EL}_0^- + \mathsf{MP} + \Delta_0^0$-$\mathsf{AC}^{00} + \mathsf{NCT} \vdash \mathsf{KA}$, where $\mathsf{KA} :\equiv \exists\gamma\mathrm{KA}[\gamma]$.

Proof. Let $\{c\}$ be the computable counterexample, i.e., $\mathbf{EL}_0^- + \Sigma_1^0$-$\mathsf{Bdg} \vdash \mathsf{CT} \to \mathrm{KA}[\{c\}]$. Applying N to this, with 4.2(3) with $n = 1$, we have $\mathbf{EL}_0^- + \mathsf{MP} + \Sigma_1^0$-$\mathsf{Bdg} \vdash \mathsf{NCT} \to \mathrm{KA}[\{c\}]$. Δ_0^0-AC^{00} yields γ with $\{c\} = \gamma$. $\Box$

Classically $\mathcal{L}_{\mathrm{F}}$-AC^{00} implies $(\mathcal{L}_{\mathrm{F}}$-$\mathsf{CA})^{\mathfrak{cb}}$. As a refinement, it is known that, even intuitionistically, Π_1^0-AC^{00} implies $(\Sigma_1^0$-$\mathsf{CA})^{\mathfrak{cb}}$ and hence it is of the strength of $\mathbf{ACA}_0$. Here Π_1^0-AC^{00} can be weakened to Π_1^0-$\mathsf{AC!}^{00}$, and even to $\mathsf{SBAC!}$ defined below, which restricts the Π_1^0 formulae to be of a special form. With SBAC, we can refine the classical implication from KL (König's lemma) to $\mathbf{ACA}_0$ (cf. [37, Theorem III.7.2]) as follows.

Definition 4.5 (semi-bounded axiom of choice SBAC and $\mathsf{SBAC!}$)**.** SBAC is defined as follows and $\mathsf{SBAC!}$ is defined with $\exists$ replaced by $\exists!$ in the premise, where $\mathrm{SB}_{C,D,t}[x, y] :\equiv C[x, y] \vee (y < t[x] \wedge \forall z D[x, y, z])$.
(SBAC) $\forall x \exists y \, \mathrm{SB}_{C,D,t}[x, y] \to \exists\alpha\forall x \, \mathrm{SB}_{C,D,t}[x, \alpha(x)]$, for C and D both from Δ_0^0.

Lemma 4.6. (1) $\mathbf{EL}_0^- + \mathsf{LPO} + \Delta_0^0$-$\mathsf{FT} \vdash (\mathsf{SBAC})^N$. (2) $\mathbf{EL}_0^- + \mathsf{LPO} + \mathsf{SBAC!} \vdash (\Sigma_1^0$-$\mathsf{CA})^{\mathfrak{cb}}$.

Proof. (1) As in the proof of 2.31, we may assume $C[x, y] \wedge C[x, z] \to y = z$. Let $A :\equiv \mathrm{SB}_{C,D,t}$. Define γ by

$\gamma(u) = 0 \leftrightarrow (\forall n < |u|)(u(n) \neq 0 \to (\forall x \leq n)(u(x) \neq 0 \wedge (C[x, u(x){-}1] \vee (u(x) \leq t[x] \wedge (\forall z < |u|)D[x, u(x){-}1, z])))).$

We prove $\mathsf{Fan}[\gamma]$. By LPO, there are two cases: if $\neg\exists y C[|u|, y]$ then $\forall z(\gamma(u{*}\langle z\rangle) = 0 \to z \leq t[x])$; if $C[|u|, y]$ for some y then $\forall z(\gamma(u{*}\langle z\rangle) = 0 \to z \leq \max(y{+}1, t[x]))$. Obviously $\gamma(u) = 0 \to \gamma(u{*}\langle 0\rangle) = 0$.

If $\forall k(\gamma(\beta{\restriction}k) = 0)$ and $\forall x(\beta(x) \neq 0)$, then $\forall k(\forall x \leq k)(C[x, \alpha(x)] \vee (\alpha(x) < t[x] \wedge (\forall z < k)D[x, \alpha(x), z]))$ for $\alpha(x) := \beta(x){-}1$, and, as "$\forall k(\forall x \leq k)$" is same as "$\forall x(\forall k \geq x)$", $\forall x(C[x, \alpha(x)] \vee (\alpha(x) < t[x] \wedge \forall z D[x, \alpha(x), z])$ and so $\forall x A^N[x, \alpha(x)]$. Thus $\forall\alpha\neg\forall x A^N[x, \alpha(x)] \to \forall\beta(\forall k(\gamma(\beta{\restriction}k) = 0) \to (\forall x(\beta(x) \neq 0) \to \perp))$ and, by MP,

$$\forall\alpha\neg\forall x A^N[x, \alpha(x)] \to \mathsf{Bar}[\gamma, \{u \colon (\exists x < |u|)(u(x) = 0)\}]. \qquad (*)$$

Assume $(\forall x \exists y A[x, y])^N$. For any n, $\forall x \neg\forall y \neg(C[x, y] \vee (y < t[x] \wedge (\forall z < n)D[x, y, z]))^N$ and, by MP, $(\forall x < n)\exists y(C[x, y] \vee (y < t[x] \wedge (\forall z < n)D[x, y, z]))$. Σ_1^0-Ind, with 2.31, shows $\exists u(|u| = k{+}1 \wedge \gamma(u) = 0 \wedge u(k) \neq 0)$ for $k < n$. Particularly, $\exists v(|v| = n \wedge v(n) \neq 0 \wedge \gamma(v) = 0)$. By Δ_0^0-FT, $\neg\mathsf{Bar}[\gamma, \{u \colon (\exists k < |u|)(u(k) = 0)\}]$, and hence, by $(*)$, $\neg\forall\alpha\neg\forall x A^N[x, \alpha(x)]$, i.e., $(\exists\alpha\forall x A[x, \alpha(x)])^N$.

(2) Let B be Σ^0_1 of $\mathcal{L}_{\mathrm{S}}$, say $B^{\mathfrak{ch}}[x] \equiv \exists y C[x,y]$. We may assume $C[x,y] \wedge C[x,z] \to y=z$. LPO yields $\forall x \exists! y (C[x,y] \vee (y=0 \wedge \forall z \neg C[x,z]))$. $\mathsf{SBAC!}$ yields α with $\forall x (C[x,\alpha(x)] \vee (\alpha(x)=0 \wedge \forall z \neg C[x,z]))$. Because $\forall x (\exists i<2)(i=0 \leftrightarrow C[x,\alpha(x)])$, there is β with $\forall x(\beta(x)=0 \leftrightarrow C[x,\alpha(x)])$. Then $\forall x(\beta(x)=0 \leftrightarrow B^{\mathfrak{ch}}[x])$. $\square$

Thus in the presence of LPO, we cannot strengthen Δ^0_0-WFT to Δ^0_0-FT unless going beyond Finitism.

How about $\mathcal{C}$-WFT, $(\mathcal{C},\mathcal{D})$-$\mathsf{BI}_M$ or $\mathcal{C}$-BI_D? By 2.30(3)(ii) and 2.28(1), the first to ask are Π^0_1-WFT and Σ^0_1-BI_D. The below answers this with help of Σ^0_2-DNE or MP. (1) refines Berger's [7], where he relies on classical logic but with a slightly weaker variant of WFT. We weaken Σ^0_2-DNE and MP in the next subsections.

Lemma 4.7. (1) $\mathbf{EL}^-_0 + \Sigma^0_2$-$\mathsf{DNE} + \Pi^0_1$-$\mathsf{WFT} \vdash ((\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}})^N$. (2) $\mathbf{EL}^-_0 + \mathsf{MP} + \Sigma^0_1$-$\mathsf{BI}_D \vdash ((\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}})^N$.

Proof. (1) Let A be Σ^0_1, say $A[x]^{\mathfrak{ch}} \equiv \exists y C[x,y]$ with C being Δ^0_0. Recall $v<\underline{2} :\equiv (\forall k<|v|)(v(k)<2)$. Define

$$D[u] :\equiv (\forall x<|u|)(u(x)=0 \leftrightarrow (\exists y<|u|)C[x,y]); \qquad B[u] :\equiv (\forall v<\underline{2})\neg D[u*v].$$

We show $(\forall \alpha<\underline{2})(\forall k \neg B[\alpha{\restriction}k] \to \forall x(\alpha(x)=0 \leftrightarrow \exists y C[x,y]))$. Let $\forall k \neg B[\alpha{\restriction}k]$, i.e., $\forall k \neg(\forall v<\underline{2})\neg D[(\alpha{\restriction}k)*v]$. By MP, $\forall k(\exists v<\underline{2})D[(\alpha{\restriction}k)*v]$. If $\alpha(x)=0$, taking $v<\underline{2}$ with $D[(\alpha{\restriction}(x+1))*v]$, as $((\alpha{\restriction}(x+1))*v)(x)=\alpha(x)=0$, we have $(\exists y<|(\alpha{\restriction}(x+1))*v|)C[x,y]$, and $\exists y C[x,y]$. Conversely if $C[x,y]$, taking $v<\underline{2}$ with $D[(\alpha{\restriction}(x+y+1))*v]$, since $(\exists z<|(\alpha{\restriction}(x+y+1))*v|)C[x,z]$ we have $\alpha(x)=((\alpha{\restriction}(x+y+1))*v)(x)=0$.

We show $((\exists \alpha<\underline{2})\forall x(\alpha(x)=0 \leftrightarrow \exists y C[x,y]))^N$, i.e., $\neg(\forall \alpha<\underline{2})\neg\forall x(\alpha(x)=0 \leftrightarrow \exists y C[x,y])$ by MP. Suppose for contradiction $(\forall \alpha<\underline{2})\neg\forall x(\alpha(x)=0 \leftrightarrow \exists y C[x,y])$. Then, by the above, $(\forall \alpha<\underline{2})\neg\forall k\neg B[\alpha{\restriction}k]$ and, by Σ^0_2-DNE, $(\forall \alpha<\underline{2})\exists k B[\alpha{\restriction}k]$. Π^0_1-WFT yields n with $(\forall \alpha<\underline{2})(\exists k<n)B[\alpha{\restriction}k]$ and so $(\forall \alpha<\underline{2})\neg D[\alpha{\restriction}n]$. However we can construct $u<\underline{2}$ with $|u|=n$ and $(\forall k<n)(u(k)=0 \leftrightarrow (\exists y<n)C[x,y])$, a contradiction.
(2) By 4.3(1)(2)(iv), it suffices to show $(\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}}$ in $\mathbf{EL}^-_0 + \mathcal{L}_{\mathrm{F}}$-$\mathsf{LEM} + \Sigma^0_1$-$\mathsf{BI}_D$. We prove Π^0_1-AC^{00} classically by 4.6(2). Let A be Π^0_1 and $B[u] :\equiv (\exists k<|u|)\neg A[k,u(k)]$. Then $\neg B[\langle\rangle]$ and $B[u] \to B[u*v]$. Now $\forall k \exists x A[k,x]$ yields $\forall x B[u*\langle x\rangle] \to B[u]$, and, by (Σ^0_1,Σ^0_1)-BI_M with 2.28(3), also $\neg\mathsf{Bar}[\underline{0},\{u\colon B[u]\}]$, i.e., $\exists \alpha \forall n \neg B[\alpha{\restriction}n]$. $\square$

Thus, only with this famous negative interpretation N, we have the following lower bound results. For the lower bounds of Σ^0_1-Ind, Σ^0_1-BI_D and Π^0_1-WFT with LPO, more works are required as in the next subsections.

Corollary 4.8. $\mathbf{ACA}_0$ is interpretable (i) Π^0_2-preservingly in $\mathbf{EL}^-_0 + \mathsf{LPO} + \Delta^0_0$-$\mathsf{FT}$ and in $\mathbf{EL}^-_0 + \mathsf{MP} + \Sigma^0_1$-$\mathsf{BI}_D$; (ii) Π^0_3-preservingly in $\mathbf{EL}^-_0 + \Sigma^0_2$-$\mathsf{DNE} + \Pi^0_1$-$\mathsf{WFT}$; and (iii) Δ^1_0-preservingly in $\mathbf{EL}^-_0 + \mathsf{LPO} + \Pi^0_1$-$\mathsf{AC!}^{00}$.

Proof. (i) By 4.6, 4.7(2) and 4.3(1) with $n=1$. (ii) Similar. (iii) $\mathbf{EL}^-_0 + \mathsf{LPO} + \Pi^0_1$-$\mathsf{AC!}^{00}$ trivially includes $\mathbf{EL}^-_0 + \mathsf{LPO} + \mathsf{SBAC!}$ and, by 4.6(2), also $\mathbf{EL}^-_0 + (\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}}$. As $(\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}}$ implies Π^0_∞-LEM and so Δ^1_0-LEM, $\mathbf{EL}^-_0 + (\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}}$ proves $((\Sigma^0_1\text{-}\mathsf{CA})^{\mathfrak{ch}})^N \wedge \Delta^1_0$-$\mathsf{LEM}$, and so interprets Δ^1_0-preservingly $(\mathbf{ACA}_0)^{\mathfrak{ch}}$ by N. $\square$

4.2 Coquand-Hofmann forcing interpretation

Gödel-Gentzen negative interpretation N yields the Π_1 conservation of $\mathbf{PA}$ over $\mathbf{HA}$. Friedman-Dragalin translation (also known as Friedman's A-translation) was introduced to enhance it to Π_2 conservation, or equivalently to show the admissibility of MP-rule. We start by recalling this well-known technique:

$$C^A :\equiv C \vee A \text{ if } C \text{ is atomic}; \quad (C \Box D)^A :\equiv C^A \Box D^A \text{ for } \Box \equiv \wedge, \to, \vee; \quad (QxC)^A :\equiv Qx(C^A) \text{ for } Q \equiv \forall, \exists.$$

For a Σ_1 formula $A[x]$, since $\mathbf{HA} \vdash A[x]^N \leftrightarrow \neg\neg A[x]$, if $\mathbf{PA} \vdash \forall x A(x)$ then $\mathbf{HA} \vdash \neg\neg A[x]$, to which by applying $A[x]$-translation, we have $\mathbf{HA} \vdash (\neg\neg A[x])^{A[x]}$ i.e., $\mathbf{HA} \vdash (A[x] \vee A[x] \to A[x]) \to A[x]$ and hence $\mathbf{HA} \vdash \forall x A[x]$. However, this combination of the negative interpretation N and $A[x]$-translation does not necessarily preserve another Π_2 sentence $\forall x B[x]$. Thus, it does not uniformly preserve Π_2 sentences. Moreover, A-translation is not $\{\perp\}$-preserving, unless A is equivalent to $\perp$, and so does not yield the consistency-wise implication. Coquand-Hofmann forcing overcomes this disadvantage, by replacing single A with a finite set of such A's. We further generalize this technique to general $\exists^0\mathcal{C}$ but assuming $\mathcal{C}$-LEM.

Below we consider any α to code a finite set of (x,ξ)'s: e.g., $(x,\xi) \in \alpha :\equiv (\exists k<\alpha(0))((\alpha \ominus 1)_k = \langle x\rangle * \xi)$, and also (x,ξ) to code $\exists u P[x,u,\xi]$. (Thus $\exists u \mathrm{Tr}_P[u,\alpha]$ means the disjunction of all formulae "belonging to" α.) As an example, we can take P from Π^0_n so that $\exists u P[x,u,\xi]$ is a universal Σ^0_{n+1} formula.

Definition 4.9 (Tr_P, $\Vdash_P$)**.** $\mathrm{Tr}_P[u,\alpha] :\equiv (\exists (x,\xi) \in \alpha)P[x,u,\xi]$; and $\alpha \Vdash_P A :\equiv A \vee \exists u \mathrm{Tr}_P[u,\alpha]$.

Since $\mathrm{Tr}_P[u,\alpha]$ is $(\exists k<\alpha(0))P[(\alpha\ominus 1)_k(0), u, (\alpha\ominus 1)_k \ominus 1]$, we see that $\exists u \mathrm{Tr}_P[u,\alpha]$ is Σ^0_{n+1} if P is Π^0_n.

Definition 4.10 ($\Vdash_P$)**.** To an $\mathcal{L}_{\mathrm{F}}$ formula B, assign $\alpha \Vdash_P B$ as follows, where $QxB[x]$ is treated as $Q\xi B[\xi(0)]$:

$$\alpha \Vdash_P B \;:\equiv\; \alpha \Vvdash_P B \text{ for atomic ;} \qquad \alpha \Vdash_P B \to C \;:\equiv\; (\forall \beta \supseteq \alpha)((\beta \Vdash_P B) \to (\beta \Vdash_P C));$$

$$\alpha \Vdash_P B \Box C \;:\equiv\; (\alpha \Vdash_P B) \Box (\alpha \Vdash_P C) \text{ for } \Box \equiv \wedge, \vee; \qquad \alpha \Vdash_P Q\xi B \;:\equiv\; Q\xi(\alpha \Vdash_P B) \text{ for } Q \equiv \forall, \exists.$$

The connection to Friedman's A-translation is clear in the atomic case. The extension to compound formulae is by Kripke semantics, where the monotonicity is (1)(ii) of the next lemma. (2) in the lemma, asserting the $\Vdash_P$ respects intuitionistic reasonings, easily follows, and (3) corresponds to the assertion that $B^A \leftrightarrow B \vee A$ if C is Σ_1, which allowed us to show $A[x]^{A[x]} \leftrightarrow A[x] \vee A[x]$, the key fact to show MP-rule.

Lemma 4.11. (1) $\mathbf{EL}_0^-$ proves (i) $B \leftrightarrow (\emptyset \Vvdash_P B)$ and (ii) $\alpha \subseteq \beta \to (\alpha \Vvdash_P B \to \beta \Vvdash_P B) \wedge (\alpha \Vdash_P B \to \beta \Vdash_P B)$.
(2) $\mathbf{EL}_0^- \vdash (\alpha \Vdash_P B_1) \wedge ... \wedge (\alpha \Vdash_P B_n) \to (\alpha \Vdash_P C)$ if C intuitionistically follows from B_1, ..., B_n.
(3) If $C, D, \exists x \neg E, E \in \mathcal{C}$ for all subformulae $C \to D$ and $\forall x E$ of B, $\mathbf{EL}_0^- + \mathcal{C}\text{-}\mathsf{LEM} \vdash (\alpha \Vdash_P B) \leftrightarrow (\alpha \Vvdash_P B)$.
(4) If F is built up by $\wedge, \vee, \forall, \exists$ from those B's which satisfy the condition of (3), $\mathbf{EL}_0^- + \mathcal{C}\text{-}\mathsf{LEM} \vdash F \leftrightarrow (\emptyset \Vdash_P F)$.
(5) If B is as in (3), $\mathbf{EL}_0^- + \mathcal{C}\text{-}\mathsf{LEM} \vdash \alpha \Vdash_P (B \to G) \leftrightarrow (B \to \alpha \Vdash_P G)$. So $\alpha \Vdash_P ((\forall x < y) G) \leftrightarrow (\forall x < y)(\alpha \Vdash_P G)$.

Proof. (3) By induction on B. The atomic case is trivial. The case of $\wedge$ is by $(C \vee F) \wedge (D \vee F) \leftrightarrow (C \wedge D) \vee F$.

$\alpha \Vdash_P C \to D$ is, by induction hypothesis, equivalent to $(\forall \beta \supseteq \alpha)((C \vee \exists u \mathrm{Tr}_P[u, \beta]) \to (D \vee \exists u \mathrm{Tr}_P[u, \beta]))$, to $C \to (\forall \beta \supseteq \alpha)(D \vee \exists u \mathrm{Tr}_P[u, \beta])$, by $\mathcal{C}$-LEM to $\neg C \vee D \vee (\forall \beta \supseteq \alpha) \exists u \mathrm{Tr}_P[u, \beta]$ and to $(C \to D) \vee \exists u \mathrm{Tr}_P[u, \alpha]$.

By induction hypothesis, $\alpha \Vdash_P \exists x E$ is equivalent to $\exists x(E \vee \exists u \mathrm{Tr}_P[u, \alpha])$ and to $(\exists x E) \vee \exists u \mathrm{Tr}_P[u, \alpha]$. Similarly $\alpha \Vdash_P \forall x E$ is to $\forall x(E \vee \exists u \mathrm{Tr}_P[u, \alpha])$ and to $(\forall x E) \vee \exists u \mathrm{Tr}_P[u, \alpha, u]$, but by $\exists x \neg E \vee \forall x E$.
(5) If $\alpha \Vdash_P (B \to G)$ and B then, by (4) and (1)(ii), $\alpha \Vdash_P B$ and $\alpha \Vdash_P G$. If $B \to (\alpha \Vdash_P G)$ then, for $\beta \supseteq \alpha$, $\beta \Vdash_P B$ implies $B \vee \exists u \mathrm{Tr}[u, \beta]$ by (3), $(\alpha \Vdash_P G) \vee (\beta \Vdash_P \bot)$ and so, by (1)(ii), $\beta \Vdash_P G$, i.e., $\alpha \Vdash_P (B \to G)$. ☐

Corollary 4.12. (i) If B is Π^0_∞, $\mathbf{EL}_0^- \vdash B \leftrightarrow (\emptyset \Vdash_P B)$; (ii) if B is Σ^0_{n+1}, $\mathbf{EL}_0^- + \Sigma^0_n\text{-}\mathsf{LEM} \vdash (\alpha \Vdash_P B) \leftrightarrow (\alpha \Vvdash_P B)$.

Definition 4.13 (self-forcible)**.** A schema S is called *self-forcible for* $\mathcal{C}$ if, for any $P \in \mathcal{C}$, S implies $\emptyset \Vdash_P \mathsf{S}$.

Corollary 4.14. (i) $\mathbf{EL}_0^- \vdash (\emptyset \Vdash_P \mathbf{EL}_0^-)$; (ii) in $\mathbf{EL}_0^- + \Sigma^0_{k+1}\text{-}\mathsf{LEM}$, Σ^0_k-Ind and Π^0_k-Ind are self-forcible for $\mathcal{L}_{\mathrm{F}}$.

Lemma 4.15. Over $\mathbf{EL}_0^- + \Sigma^0_n\text{-}\mathsf{LEM}$, (i) Π^0_n-WFT if $n > 0$; (ii) for $\mathcal{C}, \mathcal{D} \in \{\Sigma^0_{n+2k+1}, \Lambda^1_{n+2k+1,m}, \Xi_{n+2k+1,m}, \Theta^1_m\}$ (a) $\mathcal{C}$-Ind, (b) $\mathcal{C}$-Bdg, (c) $\mathcal{C}$-AC^{0i}, (d) $\mathcal{C}$-DC^i, (e) $(\mathcal{C}, \mathcal{D})$-$\mathsf{BI}_M$ and (f) $\mathcal{C}$-BI_D (if $n = 0$) are self-forcible for Π^0_n.

Proof. We may assume $\exists u \mathrm{Tr}_P[u, \alpha] \equiv \exists l C[l, \alpha]$ with C being Π^0_n. (i) If $\alpha \Vdash_P (\forall \xi < \underline{2}) \exists k B[\xi \restriction k]$ where B is Π^0_n, then, by 4.12(ii), $(\forall \xi < \underline{2}) \exists k (\alpha \Vvdash_P B[\xi \restriction k])$, i.e., $(\forall \xi < \underline{2}) \exists k (B[\xi \restriction k] \vee \exists l C[l, \alpha])$. Thus $(\forall \xi < \underline{2}) \exists k D[\xi \restriction k]$ where $D[u] :\equiv B[u] \vee C[|u|, \alpha]$ is $\Pi^0_n \vee \Pi^0_n \subseteq \Pi^0_n$ by Σ^0_n-LEM. Then Π^0_n-WFT yields m with $(\forall \xi < \underline{2})(\exists k < m) D[\xi \restriction k]$ and so $(\forall \xi < \underline{2})(\exists k < m)(\alpha \Vvdash_P B[\xi \restriction k])$. As $\xi < \underline{2}$ is Π^0_1, by 4.11(3)(5) with $n > 0$, $\alpha \Vdash_P (\forall \xi < \underline{2})(\exists k < m) B[\xi \restriction k]$.
(ii) If B is Π^0_n, then $\beta \Vdash_P Q y_{n+2k(+1 \text{ or } 2)} ... \exists y_{n+1} B[x, y_{n+2}, ...]$ is equivalent to $Q ... \exists y_{n+1}(\beta \Vdash_P B[x, y_{n+1}, ...])$ and, by 4.12(ii), $Q ... \exists y_{n+1} \exists \ell (B[x, y_{n+1}, ...] \vee C[\ell, \beta])$. By Σ^0_n-LEM, if A is equivalently $\mathcal{C}$, so is $\beta \Vdash_P A[x]$.
(a) $\alpha \Vdash_P A[0] \wedge (\forall x < n)(A[x] \to A[x{+}1])$ implies $(\forall x < n)(\alpha \Vdash_P A[x] \to \alpha \Vdash_P A[x{+}1])$ and $\alpha \Vdash_P A[n]$ by $\mathcal{C}$-Ind.
(b) If $\alpha \Vdash_P (\forall x < m) \exists y A[x, y]$ then $(\forall x < m) \exists y (\alpha \Vdash_P A[x, y])$ and $\exists u (\forall x < m)(\exists y < u)(\alpha \Vdash_P A[x, y])$ by $\mathcal{C}$-Bdg. Thus by 4.11(5), $\exists u (\alpha \Vdash_P (\forall x < m)(\exists y < u) A[x, y])$. (c) (d) (e) Similar. (f) Use (e) and 2.28(3). ☐

In the lemma, (ii)(f) seems to require $n = 0$: a bar $\{v : \beta(v) = 0\}$ is interpreted as $\{v : \alpha \Vdash_P \beta(v) = 0\}$, i.e., $\{v : \beta(v) = 0 \vee \exists u \mathrm{Tr}[u, \alpha]\}$, to which we cannot apply $\mathcal{L}_{\mathrm{F}}$-BI_D even if $\mathsf{Bar}[\underline{0}, \{v : \beta(v) = 0 \vee \exists u \mathrm{Tr}[u, \alpha]\}]$.

The following is the central trick corresponding to that of A-translation: $(\neg\neg A[x])^{A[x]} \leftrightarrow A[x]$.

Proposition 4.16. For P from Π^0_n, $\mathbf{EL}_0^- + \Sigma^0_n\text{-}\mathsf{LEM} \vdash (\emptyset \Vdash_P (\neg \forall v \neg P[x, v, \xi] \to \exists v P[x, v, \xi]))$.

Proof. Π^0_n-LEM yields $\forall v \exists u (\neg P[x, v, \xi] \vee P[x, u, \xi])$, which is equivalent to $\forall v(\neg P[x, v, \xi] \vee \exists u P[x, u, \xi])$, i.e., $\forall v(\{(x, \xi)\} \Vvdash_P \neg P[x, v, \xi])$. By 4.12(ii) with 2.24(1)(i), we have $\{(x, \xi)\} \Vdash_P \forall v \neg P[x, v, \xi]$.

Thus, if $\alpha \Vdash_P \neg \forall v \neg P[v, x, \xi]$ then $\alpha \cup \{(x, \xi)\} \Vdash_P \bot$, i.e., $\exists u \mathrm{Tr}_P[\alpha \cup \{(x, \xi)\}, u]$ which is equivalent to $\exists u (\mathrm{Tr}_P[\alpha, u] \vee P[x, u, \xi])$, to $\exists u (\alpha \Vvdash_P P[x, u, \xi])$, and, again by 4.12(ii), to $\alpha \Vdash_P \exists u P[x, u, \xi]$. ☐

Theorem 4.17. There is a Π^0_n formula P such that $\mathbf{EL}_0^- + \Sigma^0_n\text{-}\mathsf{LEM} \vdash (\emptyset \Vdash_P \mathbf{EL}_0^- + \Sigma^0_{n+1}\text{-}\mathsf{DNE})$.

Proof. Let $P[x, u, \xi] :\equiv \forall y_n \exists y_{n-1} ... Q y_1 (\xi(x, u, y_n, y_{n-1}, ..., y_1) = 0)$. Fix A from Σ^0_{n+1}. Take C from Δ^0_0 with $A[x, \alpha] \equiv \exists u \forall y_n \exists y_{n-1} ... Q y_1 C[x, u, y_n, y_{n-1}, ..., y_1, \alpha]$. Take ξ with $(\forall x, u, \vec{y})(\xi(x, u, \vec{y}) = 0 \leftrightarrow C[x, u, \vec{y}, \alpha])$ by 2.10(d). Then $\forall x (A[x, \alpha] \leftrightarrow \exists u P[x, u, \xi])$. As this argument is in $\mathbf{EL}_0^-$, $\emptyset \Vdash_P \exists \xi \forall x (A[x, \alpha] \leftrightarrow \exists u P[x, u, \xi])$ by 4.11(2) and 4.14(i). By 4.16 with 4.11(2), we finally conclude $\emptyset \Vdash_P \neg\neg A[x, \alpha] \to A[x, \alpha]$. ☐

4.3 Combining negative and forcing interpretations

Coquand-Hofmann [11] and Avigad [3] combined the interpretation $A \mapsto \emptyset \Vdash_P A$ with the negative interpretation N. We follow this way, with the following enhancement. While they considered only the first order case where P in $\Vdash_P$ is Δ^0_0, we have considered second order cases with P being Π^0_n but assuming Σ^0_n-LEM.

Theorem 4.18. (1) (a) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM and so $\mathbf{I\Delta_0ex}$ (and $\mathbf{EFA}$) are Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-$ and (b) so are $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_1$-Bdg and $\mathbf{B\Delta_0ex}$ in $\mathbf{EL}_0^-+\Sigma^0_1$-Bdg and hence in $\mathbf{EL}_0^-+\Delta^0_0$-AC^{00}.
(2) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_1$-Ind and hence $\mathbf{I\Sigma}_1=\mathbf{I\Pi}_1$ (as well as $\mathbf{PRA}$) are interpretable (a) Π^0_1-preservingly in $\mathbf{EL}_0^-+\Pi^0_1$-Ind and hence in $\mathbf{EL}_0^-+\Delta^0_0$-BI$_D$; (b) Π^0_2-preservingly in $\mathbf{EL}_0^-+\Sigma^0_1$-Ind and hence in $\mathbf{EL}_0^-+\Delta^0_0$-FT.
(3) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_2$-Ind and so $\mathbf{I\Sigma}_2=\mathbf{I\Pi}_2$ are interpretable (a) Π^0_2-preservingly in $\mathbf{EL}_0^-+\Pi^0_2$-Ind and hence in $\mathbf{EL}_0^-+\Pi^0_2$-DC!^{0} and in $\mathbf{EL}_0^-+\Pi^0_1$-DC!^{1} and (b) Π^0_3-preservingly in $\mathbf{EL}_0^-+$LPO$+\Sigma^0_2$-Ind.
(4) $\mathbf{ACA}_0$ is interpretable (a) Π^0_2-preservingly in $\mathbf{EL}_0^-+\Sigma^0_1$-BI$_D$; and in $\mathbf{EL}_0^-+$LPO$+\Delta^0_0$-FT; (b) Π^0_3-preservingly in $\mathbf{EL}_0^-+$LPO$+\Pi^0_1$-WFT; and (c) Δ^1_0-preservingly in $\mathbf{EL}_0^-+$LPO$+\Pi^0_1$-AC!^{00}.

Proof. (1) By 4.3(1) with $n=1$, $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM is Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-+$MP. The latter is Π^0_∞-preservingly interpretable in $\mathbf{EL}_0^-$ by 4.12(i) and 4.17 with $n=0$. For (b) use additionally 4.3(2)(ii) with $n=1$ and 4.15(ii)(b) with $n=k=0$, where we can easily see $\mathbf{EL}_0^-+\Delta^0_0$-AC$^{00}\vdash\Sigma^0_1$-Bdg.
(2) (a) By 4.3(1)(2)(i) with $n=0$, $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_1$-Ind$=\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Pi^0_1$-Ind is Π^0_1-preservingly interpretable in $\mathbf{EL}_0^-+\Pi^0_1$-Ind and by 2.28(2) in $\mathbf{EL}_0^-+\Delta^0_0$-BI$_D$. (b) By 4.3(1)(2)(iii) with $n=1$, $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+$ Σ^0_1-Ind is Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-+$MP$+\Sigma^0_1$-Ind. The latter is Π^0_∞-preservingly interpretable in $\mathbf{EL}_0^-+\Sigma^0_1$-Ind by 4.12(i), 4.15(ii)(a) with $n=k=0$ and 4.17 with $n=0$, and hence in $\mathbf{EL}_0^-+\Delta^0_0$-FT by 2.31.
(3) (a) By 4.3(1)(2)(i) with $n=1$, $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Pi^0_2$-Ind is Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-+$MP$+\Pi^0_2$-Ind, and, by 4.12(i), 4.15(ii)(a) with $n=k=0$ where $\Lambda^0_{1,0}=\Pi^0_2$ and 4.17, further in $\mathbf{EL}_0^-+\Pi^0_2$-Ind. The latter is included in $\mathbf{EL}_0^-+\Pi^0_2$-DC!^{0} by 2.16(3)(i), and in $\mathbf{EL}_0^-+\Pi^0_1$-DC!^{1} by 2.16(5) with $\mathcal{C}\equiv\Delta^0_0$ and 2.16(2)(v).
(b) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_2$-Ind is Π^0_3-preservingly interpretable in $\mathbf{EL}_0^-+\Sigma^0_2$-DNE$+\Sigma^0_2$-Ind by 4.3(1)(2)(iii) with $n=2$, and further in $\mathbf{EL}_0^-+$LPO$+\Sigma^0_2$-Ind by 4.12(i), 4.15(ii)(a) with $(n,k)=(1,0)$ and 4.17 with $n=1$.
(4) (a)(b)(c) follow from 4.8(i)(ii)(iii) respectively, since $\mathbf{EL}_0^-+$MP$+\Sigma^0_1$-BI$_D$ is interpretable Π^0_∞-preservingly in $\mathbf{EL}_0^-+\Sigma^0_1$-BI$_D$ by 4.12(i), 4.15(ii)(f) with $n=k=0$ and 4.17 with $n=0$; and so is $\mathbf{EL}_0^-+\Sigma^0_2$-DNE$+\Pi^0_1$-WFT in $\mathbf{EL}_0^-+$LPO$+\Pi^0_1$-WFT by 4.12(i), 4.15(i) with $n=1$ and 4.17 with $n=1$. ☐

With the hierarchy of $\Lambda^i_{n,m}$'s from 3.54, we can hierarchize the interpretability as in 4.19 below. For (d), $(\Pi^0_{n+2+m})^N\subseteq\Lambda^1_{n+1,m}$ under Σ^0_{n+1}-DNE and by recursive indices we can interpret $\Lambda^1_{n+1,m}$ in $\Lambda^0_{n+1,m}$.

Corollary 4.19. Let $k<n$ or $k=n+1$. We can interpret Π^0_{n+2}-preservingly (a) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM in $\mathbf{EL}_0^-+\Sigma^0_n$-LEM; (b) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_k$-Bdg in $\mathbf{EL}_0^-+\Sigma^0_n$-LEM$+\Sigma^0_k$-Bdg; (c) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Sigma^0_k$-Ind in $\mathbf{EL}_0^-+\Sigma^0_n$-LEM$+\Sigma^0_k$-Ind; (d) $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-LEM$+\Pi^0_{n+m+2}$-Ind in $\mathbf{EL}_0^-+\Sigma^0_n$-LEM$+\Lambda^1_{n+1,m}$-Ind and hence in $\mathbf{EL}_0^-+\Sigma^0_n$-LEM$+\Lambda^0_{n+1,m}$-Ind.

In the first order setting, by letting $\exists u P[x,u]$ be universal Σ_{n+1}, we obtain the analogous Π_{n+2}-preserving interpretability results: (a) $\mathbf{I\Delta_0ex}+\Sigma_{n+1}$-Bdg in $\mathbf{iQ}+\Sigma_{n+1}$-Bdg$+\Sigma_n$-LEM; (b) $\mathbf{I\Sigma}_{n+1}$ in $\mathbf{i\Sigma}_{n+1}+\Sigma_n$-LEM; (c) $\mathbf{I\Pi}_{n+m+2}$ in $\mathbf{iQ}+(\Lambda^0_{n+1,m}\cap\mathcal{L}_1)$-Ind$+\Sigma_n$-LEM; and (d) $\mathbf{PA}$ in $\mathbf{HA}+\Sigma_n$-LEM. However, this does not work for $\mathbf{I\Delta_0ex}$ in $\mathbf{iQ}+\Sigma_n$-LEM, since Σ_1-Bdg seems necessary for universal formula.

We can go further to stronger theories, where Π^1_m-$\mathbf{TI}_0:\equiv\mathbf{ACA}_0+\Pi^1_m$-TI and Π^1_∞-$\mathbf{TI}_0:\equiv\bigcup_m\Pi^1_m$-$\mathbf{TI}_0$.

Theorem 4.20. Π^1_{m+1}-$\mathbf{TI}_0$ is Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-+\Lambda^1_{1,m}$-BI$_D$. So is Π^1_∞-$\mathbf{TI}_0$ in $\mathbf{EL}_0^-+\mathcal{L}_{\mathrm{F}}$-BI$_D$.

Proof. By Π^1_1 normal form, we may consider $(\Pi^1_{m+1})^N\subseteq\Lambda^1_{1,m}$ over $\mathbf{EL}_0^-+$MP. Thus, by 4.3(1)(2)(iv) with $n=1$ and 4.7(2), Π^1_{m+1}-$\mathbf{TI}_0$ is Π^0_2-preservingly interpretable in $\mathbf{EL}_0^-+$MP$+\Lambda^1_{1,m}$-BI$_D$. The latter is interpretable Π^0_∞-preservingly in $\mathbf{EL}_0^-+\Lambda^1_{1,m}$-BI$_D$ by 4.12(i), 4.15(ii)(f) with $\mathcal{C}\equiv\Lambda^1_{1,m}$ and 4.17 with $n=0$. ☐

Actually Coquand and Hofmann [11] mentioned the combination of their interpretation of $\mathbf{I\Sigma}_1$ into $\mathbf{i\Sigma}_1$ further with the modified realizability of $\mathbf{i\Sigma}_1$ in $\mathbf{PRA}^\omega$, the higher order version of primitive recursive arithmetic, as an alternative proof of Parson's Theorem: the Π^0_2 conservation of $\mathbf{I\Sigma}_1$ over $\mathbf{PRA}$. However we need cut elimination to reduce $\mathbf{PRA}^\omega$ to $\mathbf{PRA}$.[19] This kind of longer combination (of negative, forcing and realizability interpretations in this order) is called *making-a-detour method* in Subsection 5.4.

[19] Generally, there is no interpretation in the sense of f.n.2 of a finitely axiomatizable T_1, like $\mathbf{I\Sigma}_1$, in reflexive T_2 (namely T_2 proves the consistency of any finite fragment of T_2) of the same consistency strength, since otherwise Con(T_1) follows from the consistency of a finite fragment of T_2, which T_2 proves. $\mathbf{PRA}$ is reflexive by $\mathbf{PRA}\equiv_{\Pi^0_2}\mathbf{I\Sigma}_1\vdash$ Con$(\mathbf{B\Sigma}_1(\mathcal{E}^n))$; see Subsec.5.2.

5 Final Remarks

5.1 Summary of results

Cor.5.2 below is by 3.40 and 3.50, with 2.28(1). [2] gave a Π^1_1-preserving interpretation of $\mathbf{WKL}_0$ in $\mathbf{RCA}_0$, which also Π^1_1-preservingly interprets $\mathbf{WKL}^*_0$ in $\mathbf{RCA}^*_0$ (where we need to show that Σ^0_1-Bdg is $\frac{1}{2}$-forced by formalizing the argument of [38, 4.5 Lemma]). By recursive indices we can Δ^1_0-preservingly interpret $\mathbf{RCA}_0$ in $\mathbf{I\Sigma}_1$ and $\mathbf{RCA}^*_0$ in $\mathbf{B\Sigma_1 ex}$. $\mathbf{PRA} \vdash \mathrm{Con}(\mathbf{B\Sigma_1 ex})$ and $\mathbf{I\Sigma}_1$ is Π_2 reducible to $\mathbf{PRA}$ (see Subsection 5.2). Hence the combinations in 5.2(i) are finitistically guaranteed and those in 5.2(ii) are finitistically justifiable.

Definition 5.1 (functionally realizable analysis $\mathbf{FR}^*_0, \mathbf{FR}_0, \mathbf{FR}^+_m, \mathbf{FR}^{++}_m$). $\mathbf{FR}^-_0 :\equiv \mathbf{EL}^-_0$+$\mathsf{MP}$+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{CB}^1$,-$\mathsf{CC!}^1\}$; $\mathbf{FR}^*_0 :\equiv \mathbf{FR}^-_0$+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}$,-$\mathsf{WFT}\}$; $\mathbf{FR}_0 :\equiv \mathbf{FR}^*_0$+$\Sigma^0_1$-$\mathsf{DC}^1$+$\Sigma^0_2\{$-$\mathsf{Ind}$,-$\mathsf{DC}^0\}$+$\Pi^0_1$-$\mathsf{BI}$+$\mathcal{L}_{\mathrm{F}}$-$\mathsf{FT}$;
$\mathbf{FR}^+_m :\equiv \mathbf{FR}_0$+$\Theta^1_m\{$-$\mathsf{Ind}$,-$\mathsf{DC}^0$,-$\mathsf{DC}^1\}$; and $\mathbf{FR}^{++}_m :\equiv \mathbf{FR}^+_m$+$(\Theta^1_m, \mathcal{L}_{\mathrm{F}})$-$\mathsf{BI}_M$ (cf. 3.54 for the definition of Θ^1_m).

Corollary 5.2. (i) Both $\mathbf{FR}^*_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1 and $\mathbf{FR}^*_0$+Σ^0_1-GDM are Π^0_∞-preservingly interpretable in $\mathbf{WKL}^*_0$; and (ii) both $\mathbf{FR}_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1 and $\mathbf{FR}_0$+Σ^0_1-GDM are Π^0_∞-preservingly interpretable in $\mathbf{WKL}_0$.

Moreover these combinations are optimal in the sense of the hierarchies of Brouwerian axioms and of semi-classical principles: by 4.18(2) with 2.16(3)(i), $\mathbf{EL}^-_0$ together with any of Π^0_1-Ind, Δ^0_0-BI_D, Σ^0_1-Ind, Δ^0_0-$\mathsf{DC!}^0$ and Δ^0_0-FT interprets $\mathbf{I\Sigma}_1$ and hence is not provably consistent in $\mathbf{PRA}$; by 4.18(3)(4)(a), $\mathbf{EL}^-_0$ with any of Π^0_2-Ind, Π^0_2-$\mathsf{DC!}^0$, Π^0_1-$\mathsf{DC!}^1$, Σ^0_1-BI_D and LPO+Σ^0_2-Ind interprets $\mathbf{I\Sigma}_2$ and hence is not reducible to $\mathbf{PRA}$; by 4.18(4) with 2.16(2)(iv), $\mathbf{EL}^-_0$+LPO with any of Π^0_1-$\mathsf{AC!}^{00}$, Π^0_1-$\mathsf{DC!}^0$, Δ^0_0-FT and Π^0_1-WFT interprets $\mathbf{ACA}_0$; as shown in 2.33(2), $\mathbf{EL}^-_0$+LLPO+Π^0_1-WC^0 and $\mathbf{EL}^-_0$+LPO+Π^0_1-$\mathsf{WC!}^0$ are both inconsistent. (See also 2.5.5.)

Classically, $\mathbf{CFG} :\equiv \mathbf{EL}^-_0$+$\mathcal{L}_{\mathrm{F}}$-$\mathsf{LEM}$+$\Sigma^0_1\{$-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}$,-$\mathsf{WFT}$,-$\mathsf{WC}^0$,-$\mathsf{WC}^1\}$ is finitistically guaranteed, and $\mathbf{CFG}$+$\Pi^0_1\{$-BI,-$\mathsf{Ind}\}$+$\Sigma^0_1\{$-Ind,-DC^0,-$\mathsf{DC}^1\}$ is finitistically justifiable; and these are optimal, as seen in 2.5.4.

Thus we have completed Figures 1 and 2. Moreover 5.2, 3.56, 3.57, 4.18 and 4.20 with the uses of $\mathfrak{g}$, yield the below (some pairs in (d) have stronger preserving as 4.18(4)) as Avigad's [2] method preserves Π^0_2-Ind.

Corollary 5.3. (a) $\mathbf{B\Sigma_1 ex}$, $\mathbf{FR}^*_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^*_0$+Σ^0_1-GDM, $\mathbf{EL}^*_0 \equiv \mathbf{EL}^-_0$+$\Delta^0_0$-$\mathsf{AC}^{00}$ and $\mathbf{EL}^-_0$+Σ^0_1-Bdg;
(b) $\mathbf{I\Sigma}_1$, $\mathbf{FR}_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}_0$+Σ^0_1-GDM, $\mathbf{EL}^-_0$+Σ^0_1-Ind, $\mathbf{EL}^-_0$+Δ^0_0-FT, $\mathbf{EL}^-_0$+Δ^0_0-$\mathsf{DC!}^0$ and $\mathbf{EL}^-_0$+Δ^0_0-DC^1;
(c) $\mathbf{I\Sigma}_2$, $\mathbf{FR}^+_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^+_0$+Σ^0_1-GDM, $\mathbf{EL}^-_0$+Π^0_2-Ind, $\mathbf{EL}^-_0$+Π^0_2-$\mathsf{DC!}^0$, $\mathbf{EL}^-_0$+Π^0_1-$\mathsf{DC!}^1$ and $\mathbf{EL}^-_0$+LPO+Σ^0_2-Ind;
(d) $\mathbf{ACA}_0$, $\mathbf{FR}^{++}_0$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^{++}_0$+Σ^0_1-GDM, $\mathbf{EL}^-_0$+Σ^0_1-BI_D, $\mathbf{EL}^-_0$+LPO+Δ^0_0-FT, $\mathbf{EL}^-_0$+LPO+Π^0_1-WFT and $\mathbf{EL}^-_0$+LPO+Π^0_1-$\mathsf{AC!}^{00}$; are, in each case, mutually interpretable Π^0_2-preservingly.

Moreover, so are theories in (b) with $\mathbf{EL}^-_0$+Π^0_1-Ind and $\mathbf{EL}^-_0$+Δ^0_0-BI_D but only Π^0_1-preservingly.

Thus we determined the "interpretability strengths" of fragments of Brouwerian axioms for all Σ^0_n and Π^0_n with semi-classical principles below Σ^0_1-GDM. For classes beyond Π^0_∞, we have the following hierarchized interpretability, since Avigad's [2] preserves also $\Xi_{1,m}$-Ind, which is interpreted in $\mathbf{I\Sigma}_{m+2}$ by recursive indices.

Corollary 5.4. (a) $\mathbf{I\Sigma}_{m+2}$, $\mathbf{FR}^+_m$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^+_m$+Σ^0_1-GDM, $\mathbf{EL}^-_0$+$\Lambda^0_{1,m}$-Ind and $\mathbf{EL}^-_0$+$\Lambda^0_{1,m}$-$\mathsf{DC!}^i$;
(b) Π^1_{m+1}-$\mathbf{TI}_0 :\equiv \mathbf{ACA}_0$+$\Pi^1_{m+1}$-$\mathsf{TI}$, $\mathbf{FR}^{++}_{m+1}$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^{++}_{m+1}$+Σ^0_1-GDM and $\mathbf{EL}^-_0$+$\Lambda^1_{1,m}$-BI_D;
(c) Π^1_{m+1}-$\mathbf{TI}_0$+Π^1_{n+1}-Ind, $\mathbf{FR}^{++}_{m+1}$+$\mathbf{FR}^+_{n+1}$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^{++}_{m+1}$+$\mathbf{FR}^+_{n+1}$+Σ^0_1-GDM and $\mathbf{EL}^-_0$+$\Lambda^1_{1,m}$-BI_D+$\Lambda^1_{1,n}$-Ind;
(d) Π^1_∞-$\mathbf{TI}_0$, $\mathbf{FR}^{++}_\infty$+$\mathcal{L}_{\mathrm{F}}$-CC^1, $\mathbf{FR}^{++}_\infty$+Σ^0_1-GDM and $\mathbf{EL}^-_0$+$\mathcal{L}_{\mathrm{F}}$-BI_D; are mutually interpretable Π^0_2-preservingly.

Note that $\mathbf{ACA}_0$ is not interpretable in $\mathbf{PA} \equiv \mathbf{I\Sigma}_\infty$ by f.n.19. Π^1_∞-$\mathbf{TI}_0$ is known to be proof theoretically equivalent to $\mathbf{ID}_1$, $\mathbf{KP}$ and $\mathbf{CZF}$, theories of *generalized predicativity*. As $\mathbf{FR}^{++}_\infty$ contains all the Brouwerian axioms formulated in $\mathcal{L}_{\mathrm{F}}$ except $\mathcal{L}_{\mathrm{F}}$-CC^i (see f.n.11), this could be "a marriage of Intuitionism and generalized predicativity". However, these are beyond *predicativity* in Feferman's [15] sense, as Π^1_2-$\mathbf{TI}_0 \vdash \mathrm{Con}(\mathbf{ATR}_0)$ (cf. [37, Exercise VII.2.32]). With bar induction restricted to Θ^1_1, (c) with $(m,n) = (0,\infty)$ is in the predicative bound, or "a marriage of Intuitionism and predicativism", as Π^1_1-$\mathbf{TI}_0 = \Sigma^1_1$-$\mathbf{DC}_0$ by [37, Theorem VIII.5.12].

For the semi-Russian axioms, 3.42 yields the first interpretability below, where by coding functions as recursive indices we interpret $\mathbf{EL}^-_0$+$\mathcal{L}_{\mathrm{F}}$-LEM+Δ^0_0-AC^{00}+CT in $\mathbf{B\Sigma_1 ex}$. By additionally 3.46, 3.56 and 2.28(1), we have the other two. The conserves are proved in Cor.5.3. NCT is consistent with $\mathcal{L}_{\mathrm{F}}$-CC^0 which contradicts CT (see f.n.7). Thus, CT is strictly stronger than NCT and than Veldman's KA by 4.4.

Definition 5.5 (semi-Russian analysis $\mathbf{SR}^-_0, \mathbf{SR}^*_0, \mathbf{SR}_0, \mathbf{SR}^+_m$). $\mathbf{SR}^-_0 :\equiv \mathbf{EL}^-_0$+$\mathsf{NCT}$+$\mathsf{MP}$+$\mathcal{L}_{\mathrm{F}}$-$\mathsf{CC}^1$;
$\mathbf{SR}^*_0 :\equiv \mathbf{SR}^-_0$+$\mathcal{L}_{\mathrm{F}}\{$-$\mathsf{AC}^{00}$,-$\mathsf{AC}^{01}\}$; $\mathbf{SR}_0 :\equiv \mathbf{SR}^*_0$+$\Sigma^0_1$-$\mathsf{DC}^1$+$\Sigma^0_2\{$-$\mathsf{Ind}$,-$\mathsf{DC}^0\}$+$\Pi^0_1$-$\mathsf{BI}$; $\mathbf{SR}^+_m :\equiv \mathbf{SR}_0$+$\Theta^1_m\{$-$\mathsf{Ind}$,-$\mathsf{DC}^0$,-$\mathsf{DC}^1\}$.

Corollary 5.6. $\mathbf{SR}^*_0$, $\mathbf{SR}_0$ and $\mathbf{SR}^+_m$ are interpretable in $\mathbf{B\Sigma_1 ex}$, $\mathbf{I\Sigma}_1$ and $\mathbf{I\Sigma}_{m+2}$, resp., Π^0_∞-preservingly.

5.2 Supplement: $\mathbf{I\Sigma}_1 \vdash \mathrm{Con}(\mathbf{B\Sigma_1ex})$ as well as $\mathbf{I\Sigma}_1 \equiv_{\Pi^0_2} \mathbf{PRA}$ and $\mathbf{I\Sigma}_2 \vdash \mathrm{Con}(\mathbf{I\Sigma}_1)$

To conclude that theories interpretable in $\mathbf{WKL}^*_0$ are finitistically guaranteed, we used a folklore result $\mathbf{I\Sigma}_1 \vdash \mathrm{Con}(\mathbf{B\Sigma_1ex})$ (and so $\mathbf{PRA} \vdash \mathrm{Con}(\mathbf{B\Sigma_1ex})$ by Π_2-reducibility). $\mathbf{I\Sigma}_1 \vdash \mathrm{Con}(\mathbf{EFA})$ and the Π^0_2 conservation of $\mathbf{B\Sigma_1ex}$ over $\mathbf{EFA}$ are stated in [37, II.8.11, X.4.2], and the version without exp is proved in [17, Ch.IV, §4(b)]. As we cannot find a reference for the folklore, we briefly sketch a proof with some byproducts.

We formalize $\mathbf{B\Sigma_1ex}$ by the following rules on the base of one-sided sequent calculus (in which $\neg$ is a syntactical operation) for classical logic, where C is Δ^0_0 and where z is an eigenvariable in (ind).

$$\frac{(C \text{ is an axiom of } \mathbf{iQex})}{\Gamma, C}\ (\text{axiom}) \qquad \frac{\Gamma, \neg C[z], C[z{+}1]}{\Gamma, \neg C[0], C[t]}\ (\text{ind}) \qquad \frac{\Gamma, (\forall x < t)\exists y C[x,y]}{\Gamma, \exists u (\forall x < t)(\exists y < u) C[x,y]}\ (\text{bdg})$$

By the standard partial cut elimination, we may assume that all cut formulae are Σ^0_1, Π^0_1 or Δ^0_0. For a (one-sided) sequent Γ, we write $\Gamma^{(n,m)}$ for the result of replacing all the unbounded quantifiers $\forall x$ and $\exists y$ by $(\forall x < n)$ and $(\exists y < m)$ respectively in Γ. By induction on derivation with free variables at most $\vec{x}$, we can show that there is an elementary function f with $\forall n (\forall \vec{x} < n)(\bigvee \Gamma^{(n,f(n))})$. Thus, if $\mathbf{B\Sigma_1ex} \vdash \perp$ then $\perp$.

While cut elimination increases the size of proofs by superexponential, it can be executed in $\mathbf{I\Sigma}_1$. Indices of elementary function can also be dealt with in $\mathbf{I\Sigma}_1$, and the required f is constructed elementarily in the sense of indices from derivation. As $\forall n (\forall \vec{x} < n)(\bigvee \Gamma^{(n,f(n))})$ is Π^0_1, we can formalize this argument in $\mathbf{I\Sigma}_1$.

Since $\mathcal{E}^n$ indices can also be dealt with, $\mathbf{I\Sigma}_1$ proves the consistency of $\mathbf{B\Sigma}_1(\mathcal{E}^n)$, defined similarly with function symbols for $\mathcal{E}^n$ (cf. f.n.12). If we allow C to be Σ^0_1 in (ind), such f's can be primitive recursive, whose indices can be used in $\mathbf{I\Sigma}_2$. Thus $\mathbf{I\Sigma}_1$ is reducible to $\mathbf{PRA}$ over Π_2, and consistent provably in $\mathbf{I\Sigma}_2$.

It is worth mentioning that, by cut elimination, we can easily show the equivalence between first-order formulation of $\mathbf{PRA}$ and quantifier-free formulation of $\mathbf{PRA}$: proving exactly same quantifier-free formulae with free variables. Tait's [39] identification of Hilbert's Finitism is with the latter, rather than the former.

Notice that this subsection is the only part in which we use cut elimination method, and that the results do not survive for ultrafinitism mentioned in 1.8 (but survive for those accepting $\mathcal{E}^4$ from f.n.12). Actually, it is known [17, Ch.V, 5.29 Corollary] that $\mathbf{B\Sigma_1ex}$ cannot prove even the consistency of Robinson Arithmetic $\mathbf{Q}$, and hence nor of the intuitionistic variant. Thus *ultrafinitistically guaranteed* parts must be even weaker.

It is interesting that forcing and realizability, which are sometimes seen as model construction methods, require only weaker meta-theories than cut elimination, the central technique in proof theory. For, it has been considered that proof theoretic arguments require weaker meta-theories than model theoretic ones.

5.3 Further problems

Strength of c-WFT. 4.7(1) actually shows that c-WFT, a restriction of WFT to c-bars ($B[u]$'s of the form $\forall v(\beta(u{*}v) = 0)$), with $\mathbf{EL}^-_0 + \Sigma^0_2$-DNE, interprets $\mathbf{ACA}_0$. Can LPO replace Σ^0_2-DNE? c-WFT has a particular significance [6, 8], and is known to be strictly between Δ^0_0-WFT and Π^0_1-WFT (where the border lies; Fig.2).

Hierarchy of WWFT and LPO. In the constructive context, weak weak König's lemma investigated in, e.g., the first author [26], should be called *weak weak fan theorem* $\mathcal{C}$-WWFT, since it is a weakened version of $\mathcal{C}$-WFT rather than of $\mathcal{C}$-WKL. What is the strength of $\mathcal{C}$-WWFT+LPO, especially for $\mathcal{C} \equiv \Pi^0_1$?

Π^0_3 conservation of Δ^0_0-FT. While 4.18(4)(b) asserts the Π^0_3 conservation of $\mathbf{EL}^-_0 + \mathcal{L}_{\mathrm{F}}$-LEM$+\Pi^0_1$-WFT over $\mathbf{EL}^-_0$+LPO$+\Pi^0_1$-WFT, (a) asserts similar but only Π^0_2 one for Δ^0_0-FT. Can it be enhanced to Π^0_3?

Effect of WLPO. We classified the axioms of Intuitionistic Mathematics into the three categories, finitistically non-justifiable, justifiable and guaranteed ones, in the presence of any semi-classical principle beyond LPO or below Σ^0_1-GDM. Among those in the gap is WLPO $\equiv \Pi^0_1$-LEM. How is the classification in the presence of it? LPO seems essential in the lower bound proofs (i.e., 4.6(2), 4.11(3) and 2.33(2)(ii)).

Effect of Baire's category theorem. It is mentioned in 1.5 that the effect of the semi-classical principle LLPO is of our special interest because of its similar status as WKL, which plays a central role in Simpson's "partial realizations of Hilbert's Program". Simpson [36] also mentioned the role of *Baire's category theorem* (BCT).[20] What is to BCT that LLPO is to WKL? And how is the effect of it in the sense of last paragraph?

[20]A finitistic consistency proof of BCT had not, however, been given until it was given by Avigad [2] almost a decade later.

5.4 Related works

Similar investigations in set theory. While we considered the axioms in the language $\mathcal{L}_{\mathrm{F}}$, the authors are preparing a paper [28] on the same questions in the language of set theory. The abstract treatment in Subsection 3.2 will be helpful. The axiom of choice along functions can now be formulated without twist, and it is natural to consider also some set theoretic principles, e.g., replacement, collection, subset collection, extensionality and regularity or foundation. Whereas the first two correspond to unique and non-unique axioms of choice respectively, the others seem specific to set theory. As we want to have ω and to stay within the strength of **PRA**, we shall consider "weak weak" set theory in the sense of the second author [33].

Independence of negated premise. Our use of realizability allowed us to add Markov's principle MP to the upper bound results, for the realizing system **CDL** was untyped. With typed systems we can add *independence of negated premise* (($\mathcal{C}$-INP): $(\neg A \to \exists x B[x]) \to \exists x(\neg A \to B[x])$ for A from $\mathcal{C}$) instead, from which follows Vesley's [44] alternative formalization of creative subject mentioned in f.n.13. In this way, we could have a marriage of "subjective Intuitionism" and Hilbert's Finitism. Ishihara and the first author [21] developed a translation $*$ for INP-rule in the same sense as Friedman's A-translation is for MP-rule. Following the way from A-translation to Coquand-Hofmann forcing (cf. Subsection 4.2), we can define, from $*$, a forcing interpreting $\mathcal{C}$-INP for reasonable $\mathcal{C}$. This might be able to solve some of the problems in 5.3.

Complexity of Kleene's second model. In the context of **EL**, Kleene's second model $\mathfrak{k}$ can be seen as a definable extension, as the systems are not sensitive to the complexity below arithmetic Δ^1_0. However, if the system is sensitive (like those we considered), it cannot be seen so, since the atomic formulae $(\alpha|\beta)\downarrow$ and $\alpha = \beta|\gamma$ are not in the base complexity. Recently Jäger, Rosebrock and the second author [22] makes use of this unusual complexity, to separate: enumerable by operation; being the domain of an operation; and being the image of an operation. They are equivalent if we interpret 'operation' as 'partial recursive function'.

Making-a-detour method. We used realizability interpretations (as upper bound proofs) to embed intuitionistic systems into classical $\mathbf{WKL}^*_0$ and $\mathbf{WKL}_0$ and a combination of negative and forcing interpretations (as lower bound proofs) to embed classical ones into intuitionistic ones. The composition of both the directions results in an interpretation of classical ones in classical ones, of the same kind that the second author [35] (with Zumbrunnen) and [34] introduced under the name of "making a detour via intuitionistic systems". This is the third kind of such model construction methods *for classical theories* that logical connectives are interpreted non-trivially (see 1.2), after Cohen's classical forcing and Krivine's classical realizability. We would like to stress that interpretations between intuitionistic ones could help studies of classical theories.

Relation to Veldman's work. While we discussed the strength of fan theorem analogously to that of König's lemma in the classical setting at the beginning of 3.3.3, the former is not as strong as the latter. The branching $\{x : \gamma(u * \langle x\rangle) = 0\}$ of the fan γ in 4.6(1) has at most $t[|u|]+2$ elements, and hence is *almost-finite* and *bounded-in-number* (both from [42, Subsec.10.2]). With these notions Veldman looks for an axiom which is intuitionistically to (weak) fan theorem as König's lemma is classically to weak König's lemma.

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