A note on the independence of premiss rule

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Abstract

In this paper, we prove that certain theories of (many-sorted) intuitionistic predicate logic is closed under the independence of premiss rule (IPR). As corollaries, we show that **HA** and $\mathbf{HA}^\omega$ extended by some non-classical axioms and non-constructive axioms are closed under IPR.

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1 Introduction

It has been proved using various techniques that intuitionistic arithmetic **HA** is closed under the *independence of premiss rule*

$$\text{IPR: } \vdash A \to \exists x B \Rightarrow \vdash \exists x(A \to B) \qquad (x \text{ not free in } A,\ A \text{ Rasiowa-Harrop}).$$

Using Kleene or Aczel slash relation, it is closed under IPR for sentences A and $\exists x B$; see [7, 3.5.16], and using a tricky technique due to Visser [8, part 6, 3.1], it is closed under the following form of IPR

$$\vdash \neg A \to \exists x B \Rightarrow \vdash \exists x(\neg A \to B);$$

see also [7, 3.5.5]. Note that, in **HA**, $\vdash A \leftrightarrow \neg\neg A$ for each Rasiowa-Harrop formula A; see [4, 5.18]. Troelstra [6, 3.7] made use of a realizability interpretation to deal with closure under

IPR$^\omega$: $\vdash A \to \exists x^\sigma B \Rightarrow\, \vdash \exists x^\sigma(A \to B)$ (x not free in A, A Rasiowa-Harrop)

and showed that some extensions of intuitionistic arithmetic in all finite types $\mathbf{HA}^\omega$ is closed under IPR$^\omega$ restricted to negative A.

In this paper, we introduce a variant of Visser's technique which is simpler with a technique in [2, 3], and prove that certain theories of (many-sorted) intuitionistic predicate logic **IQC** is closed under IPR. We make use of *schemata* to characterize these theories; see [5] and also [7, 2.3.13]. As corollaries, we show that **HA** and $\mathbf{HA}^\omega$ extended by non-classical axioms such as choice axioms, Church's thesis, continuity axioms, the fan theorem and bar induction, and non-constructive axioms such as the comprehension axiom for negated formulae are all closed under IPR and IPR$^\omega$.

2 Closure under IPR

We use the standard language of (many-sorted) first-order predicate logic containing $\wedge, \vee, \to, \bot, \forall$ and $\exists$ as primitive logical operators. *Prime formulae* are atomic formulae or $\bot$. In the following, we use $\vdash$ for deducibility in intuitionistic logic.

Definition 1. We shall use $*$ as a symbol for the propositional symbol that acts as a *place holder*. A translation $(\cdot)^*$ is defined inductively by

1. $P^* \equiv * \to P$ for P prime;
2. $(A \circ B)^* \equiv A^* \circ B^*$ for $\circ \in \{\wedge, \vee, \to\}$;
3. $(QxA)^* \equiv QxA^*$ for $Q \in \{\forall, \exists\}$.

Lemma 2. $\bot^* \vdash A^*$.

Proof. By induction on the complexity of A. For $A \equiv B \to C$, since $\bot^* \vdash C^*$ by the induction hypothesis, we have $\bot^* \vdash B^* \to C^*$. □

Proposition 3. *If* $\Gamma \vdash A$*, then* $\Gamma^* \vdash A^*$*, where* $\Gamma^* \equiv \{B^* \mid B \in \Gamma\}$*.*

Proof. By induction on the height of deduction trees of $\Gamma \vdash A$ in the natural deduction system for intuitionistic logic; see [7, 2.1]. For the induction step where the final rule applied is the intuitionistic absurdity rule $\bot_i$, use Lemma 2. □

Definition 4. We introduce certain predicate symbols $\nu_1, \nu_2, \nu_3, \dots$ (being outside of our standard language), also called *place holders*, to deal with schemata as syntactic objects similar to formulae. *Schemata* are inductively defined by

1. a prime formula is a schema;
2. if ν is an n-ary place holder and $t_1, \dots, t_n$ are terms, then $\nu(t_1, \dots, t_n)$ is a schema;
3. if α and β are schemata, then $\alpha \circ \beta$ is a schema for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. if α is a schema, then $Qx\alpha$ is a schema for $Q \in \{\forall, \exists\}$.

Formulae are schemata without place holders.

For example, the *induction schema* is given by a schema

$$\nu(0) \wedge \forall x(\nu(x) \rightarrow \nu(Sx)) \rightarrow \forall x \nu(x),$$

where ν is a unary place holder.

Definition 5. Let α be a schema, and let $B_1, \dots, B_k$ be formulae. Let $\nu_1, \dots, \nu_k$ be place holders, and let $\vec{x_1}, \dots, \vec{x_k}$ be sequences of variables with lengths the arities of $\nu_1, \dots, \nu_k$, respectively. Then a schema

$$\alpha[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k]$$

is defined by

1. $P[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k] \equiv P$ for P prime;
2. $\nu(t_1, \dots, t_n)[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k] \equiv B_i[y_1/t_1, \dots, y_n/t_n]$ if $\nu \equiv \nu_i$ and $\vec{x_i} \equiv y_1, \dots, y_n$, and $\nu(t_1, \dots, t_n)$ otherwise;
3. $(\alpha \circ \beta)[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k] \equiv$
   $\alpha[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k] \circ \beta[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k]$
   for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. $(Qx\alpha)[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k] \equiv Qx(\alpha[\nu_1/\lambda\vec{x_1}.B_1, \dots, \nu_k/\lambda\vec{x_k}.B_k])$ for $Q \in \{\forall, \exists\}$.

We simply write $\alpha[\nu_1/B_1, \ldots, \nu_k/B_k]$ or even $\alpha[B_1, \ldots, B_k]$ for

$$\alpha[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k]$$

whenever possible, if it does not cause confusion. An *instance* of a schema α with place holders exactly $\vec{\nu}$ is a formula $\alpha[\vec{\nu}/\vec{B}]$ or $\alpha[\vec{B}]$.

Remark 6. In using the substitution notations $A[*/B]$ and $\alpha[\vec{\nu}/\vec{B}]$, we shall assume that B and $\vec{B}$ are free for $*$ and $\vec{\nu}$ in A and α, respectively, or we assume that a suitable renaming of bound variables is carried out.

Lemma 7. *Let α be a schema. Then* $\vdash * \to (\alpha[\vec{B}]^* \leftrightarrow \alpha[\vec{B^*}])$.

Proof. By induction on the complexity of α. ☐

Definition 8. We define the class $\mathcal{RH}$ of *Rasiowa-Harrop schemata* as follows. Let P range over prime formulae, ρ and ρ' over $\mathcal{RH}$, and α over arbitrary schemata. Then $\mathcal{RH}$ is inductively generated by the clause

$$P, \rho \wedge \rho', \forall x\rho, \alpha \to \rho \in \mathcal{RH}.$$

Proposition 9. *If $\rho \in \mathcal{RH}$, then* $\vdash (* \to \rho[\vec{B^*}]) \to \rho[\vec{B}]^*$.

Proof. By induction on $\mathcal{RH}$. For $\rho \equiv \alpha \to \rho'$, since $\vdash * \to (\alpha[\vec{B}]^* \to \alpha[\vec{B^*}])$ by Lemma 7 and $\vdash (* \to \rho'[\vec{B^*}]) \to \rho'[\vec{B}]^*$ by the induction hypothesis, we have

$$\dfrac{\dfrac{\dfrac{(* \to \rho'[\vec{B^*}]) \to \rho'[\vec{B}]^* \qquad \dfrac{\dfrac{\dfrac{[* \to (\alpha[\vec{B^*}] \to \rho'[\vec{B^*}])] \quad [*]}{\alpha[\vec{B^*}] \to \rho'[\vec{B^*}]} \qquad \dfrac{\dfrac{* \to (\alpha[\vec{B}]^* \to \alpha[\vec{B^*}]) \quad [*]}{\alpha[\vec{B}]^* \to \alpha[\vec{B^*}]} \qquad [\alpha[\vec{B}]^*]}{\alpha[\vec{B^*}]}}{\rho'[\vec{B^*}]}}{* \to \rho'[\vec{B^*}]}}{\rho'[\vec{B}]^*}}{\alpha[\vec{B}]^* \to \rho'[\vec{B}]^*}}{(* \to (\alpha[\vec{B^*}] \to \rho'[\vec{B^*}])) \to (\alpha[\vec{B}]^* \to \rho'[\vec{B}]^*)}$$

☐

Definition 10. If Γ is a set of schemata, and A is a formula, we write $\Gamma \vdash A$ if A can be derived from instances of schemata in Γ, and Γ^* is the collection of all B^* where B is an instance of a schema in Γ. A set Γ of schemata is *closed under* $(\cdot)^*$ if $\Gamma \vdash A^*$ for each instance A of a schema in Γ.

Theorem 11. *Let Γ be a set of schemata closed under $(\cdot)^*$. Then*

$$\Gamma \vdash A \rightarrow \exists x B \Rightarrow \Gamma \vdash \exists x(A \rightarrow B)$$

for each formula A in $\mathcal{RH}$ with $x \notin \mathrm{FV}(A)$.

Proof. Let $A \in \mathcal{RH}$, and suppose that $\Gamma \vdash A \rightarrow \exists x B$. Then $\Gamma^* \vdash A^* \rightarrow \exists x B^*$ by Proposition 3, and, since $\vdash B^* \rightarrow (* \rightarrow B)$ by Lemma 7, we have

$$\Gamma^* \vdash A^* \rightarrow \exists x(* \rightarrow B).$$

Since Γ is closed under $(\cdot)^*$ we have $\Gamma \vdash A^* \rightarrow \exists x(* \rightarrow B)$, so we have $\Gamma \vdash (A^*)[*/A] \rightarrow \exists x(A \rightarrow B)$. Since $\vdash (* \rightarrow A) \rightarrow A^*$ by Proposition 9, we have $\vdash (A \rightarrow A) \rightarrow (A^*)[*/A]$, and hence $\vdash (A^*)[*/A]$. Therefore $\Gamma \vdash \exists x(A \rightarrow B)$. □

Definition 12. We define simultaneously classes $\mathcal{S}$ and $\mathcal{W}$ of schemata as follows. Let ρ range over $\mathcal{RH}$, ν over expressions $\nu_k(t_1 \ldots t_n)$ (ν_k an n-ary place holder symbol), γ and γ' over $\mathcal{S}$, and δ and δ' over $\mathcal{W}$. Then $\mathcal{S}$ and $\mathcal{W}$ are inductively generated by the clauses

$$\rho, \nu, \gamma \wedge \gamma', \gamma \vee \gamma', \forall x \gamma, \exists x \gamma, \delta \rightarrow \gamma \in \mathcal{S};$$
$$\nu, \delta \wedge \delta', \delta \vee \delta', \forall x \delta, \exists x \delta, \gamma \rightarrow \delta \in \mathcal{W}.$$

Proposition 13.

1. *If $\gamma \in \mathcal{S}$, then $\vdash \gamma[\vec{B^*}] \rightarrow \gamma[\vec{B}]^*$.*
2. *If $\delta \in \mathcal{W}$, then $\vdash \delta[\vec{B}]^* \rightarrow \delta[\vec{B^*}]$.*

Proof. By simultaneous induction on $\mathcal{S}$ and $\mathcal{W}$. For $\rho \in \mathcal{RH}$, since

$$\vdash (* \rightarrow \rho[\vec{B^*}]) \rightarrow \rho[\vec{B}]^*$$

by Proposition 9, we have $\vdash \rho[\vec{B^*}] \rightarrow \rho[\vec{B}]^*$. □

Corollary 14. *If Γ is a set of schemata in $\mathcal{S}$, then Γ is closed under $(\cdot)^*$.*

3 Theories closed under IPR

Since universal closures of atomic formulae and their negations are in $\mathcal{RH}$, and the induction schema is in $\mathcal{S}$, the axioms and the axiom schemata of **HA** (see [6, 1.3.3–1.3.4] and [7, 3.3.1]) are in $\mathcal{S}$. Note that, although the replacement schema REPL in [7, 2.1.8] has to be restricted to prime formulae to be in $\mathcal{S}$, REPL can be derivable from the restricted schema. Similarly, it is straightforward to see that the axioms and the axiom schemata of $\mathbf{HA}^\omega$ (see [6, 1.6.7] and [7, 9.1.7]) are in $\mathcal{S}$. Therefore we have the following.

Theorem 15. *Let* Γ *and* Γ^ω *be sets of schemata in* $\mathcal{S}$ *(in the languages of* **HA** *and* $\mathbf{HA}^\omega$*, respectively). Then* $\mathbf{HA} + \Gamma$ *is closed under* IPR*, and* $\mathbf{HA}^\omega + \Gamma^\omega$ *is closed under* IPR^ω*.*

The additional axioms for the extensional system $\mathbf{E}\text{-}\mathbf{HA}^\omega$ and the intensional system $\mathbf{I}\text{-}\mathbf{HA}^\omega$ of $\mathbf{HA}^\omega$ are in $\mathcal{S}$. Furthermore, the following are non-classical schemata and non-constructive schemata in $\mathbf{HA}^\omega$ which are in $\mathcal{S}$; see [7, 4.2, 4.3, 4.6 and 4.7-8] for choice axioms, Church's thesis, continuity axioms, the fan theorem and bar induction, and see [7, 4.3.4 Remark and 1.3.6] and [4, 7.6] for LPO and WLPO, and $\mathrm{CA}^\omega_\neg$ (note that $\exists$-PEM and $\forall$-PEM are used in [7] instead of LPO and WLPO).

Choice axioms

$\mathrm{AC}_{\sigma,\tau}$: $\forall x^\sigma \exists y^\tau \nu(x,y) \to \exists z^{\sigma\to\tau} \forall x^\sigma \nu(x, zx)$;

DC_σ: $\forall x^\sigma \exists y^\sigma \nu(x,y) \to \forall x^\sigma \exists z^{0\to\sigma}(z0 = x \wedge \forall n^0 \nu(zn, z(Sn)))$;

RDC_σ: $\forall x^\sigma[\nu_1(x) \to \exists y^\sigma(\nu_1(y) \wedge \nu_2(x,y))]$
$\to \forall x^\sigma[\nu_1(x) \to \exists z^{0\to\sigma}(z0 = x \wedge \forall n^0(\nu_1(zn) \wedge \nu_2(zn, z(Sn))))]$.

Church's thesis

CT: $\forall a^1 \exists x^0 \forall y^0 \exists z^0 (Txyz \wedge az = Uz)$;

CT_0: $\forall x^0 \exists y^0 \nu(x,y) \to \exists k^0 \forall x^0 \exists y^0 (Tkxy \wedge \nu(x, Uy))$.

Continuity axioms

WC-N: $\forall a^1 \exists x^0 \nu(a,x) \to \forall a^1 \exists x^0 \exists y^0 \forall b^1 \nu(\overline{a}x * b, y)$;

C-N: $\forall a^1 \exists x \nu(a,x) \to \exists c^1[c \in K_0 \wedge \forall a^1 \exists m^0 \exists n^0 (c(\overline{a}m) = n+1 \wedge \nu(a,n))]$,
where $c \in K_0 \Leftrightarrow \forall a^1 \exists n^0 (c(\overline{a}n) \neq 0) \wedge \forall n^0 \forall m^0 (cn \neq 0 \to cn = c(n * m))$.

The fan theorem and bar induction

FAN: $\forall a^1 \exists x^0 \nu(\overline{(a \sqcap 1)}x) \to \exists z^0 \forall a^1 \exists y^0 (y \leq z \wedge \nu(\overline{(a \sqcap 1)}y)$,
where $(a \sqcap 1)n \equiv \min\{an, 1\}$.

$\mathrm{BI_M}$: $[\forall a^1 \exists x^0 \nu(\overline{a}x) \wedge \forall n^0 \forall m^0 (\nu(n) \to \nu(n * m)) \wedge$
$\forall n^0 (\forall y^0 \nu(n * \langle y \rangle) \to \nu(n))] \to \nu(\langle\rangle)$.

Non-constructive axioms

LPO: $\forall a^1 (\exists x^0 (ax = 0) \vee \neg \exists x^0 (ax = 0))$;

WLPO: $\forall a^1 (\forall x^0 (ax = 0) \vee \neg \forall x^0 (ax = 0))$;

$\mathrm{CA}_\neg^\sigma$: $\exists y^{\sigma \to 0} \forall x^\sigma (yx = 0 \leftrightarrow \neg \nu(x))$.

Corollary 16. *A theory extending* $\mathbf{HA}^\omega$ *with the above schemata is closed under* IPR^ω*.*

Corollary 17. $\mathbf{HA} + \mathrm{CT}_0$*,* $\mathbf{HA} + \mathrm{LPO}$ *and* $\mathbf{HA} + \mathrm{WLPO}$ *are closed under* IPR*, where* LPO *and* WLPO *are respectively the following schemata in* $\mathbf{HA}$*:*

LPO*:* $\exists x (t(x) = 0) \vee \neg \exists x (t(x) = 0)$;

WLPO*:* $\forall x (t(x) = 0) \vee \neg \forall x (t(x) = 0)$.

Remark 18. Note that $\mathbf{HA} + \mathrm{CT}_0 + \mathrm{MP_{PR}}$ is not closed under IPR, where

$\mathrm{MP_{PR}}$: $\neg\neg \exists x A(x) \to \exists x A(x)$ (A primitive recursive);

see [1, Corollary 1] and the proof of [6, 3.2.27 (i)].

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References

[1] Makoto Fujiwara, Hajime Ishihara and Takako Nemoto, *Some principles weaker than Markov's principle*, preprint, 2015.

[2] Hajime Ishihara, *A note on the Gödel-Gentzen translation*, MLQ Math. Log. Q. **46** (2000), 135–137.

[3] Hajime Ishihara, *Some conservative extension results on classical and intuitionistic sequent calculi*, In U. Berger et al. eds., Logic, Construction, Computation, Ontos Verlag, Heusenstamm, 2013, pp. 289–304.

[4] Urlich Kohlenbach, *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*, Springer, Heidelberg-Berlin, 2008.

[5] Daniel Leivant, *Syntactic translations and provably recursive functions*, J. Symbolic Logic **50** (1985), 682–688.

[6] Anne S. Troelstra, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Springer, Berlin, 1973.

[7] Anne S. Troelstra and Dirk van Dalen, *Constructivism in Mathematics*, Vol. I and II, North-Holland, Amsterdam, 1988.

[8] Albert Visser, *Aspects of diagonalization and provability*, Ph.D. thesis, Rijksuniversiteit Utrecht, 1981.

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