Finite sets and infinite sets in weak intuitionistic arithmetic

Takako Nemoto

July 6, 2017

Abstract

In this paper, we consider, for a set $\mathcal{A}$ of natural numbers, the following notions of finiteness

FIN1: There are k and $m_0,,,,m_{k-1}$ such that $\mathcal{A} = \{m_0, ..., m_{k-1}\}$;

FIN2: There is an upper bound for $\mathcal{A}$;

FIN3: There is m such that $\forall \mathcal{B} \subseteq \mathcal{A}(|\mathcal{B}| < m)$;

FIN4: It is not the case that $\forall y \exists x > y(y \in \mathcal{A})$;

FIN5: It is not the case that $\forall m \exists \mathcal{B} \subseteq \mathcal{A}(|\mathcal{B}| = m)$,

and infiniteness

INF1: There are no k and $m_0, ..., m_{k-1}$ such that $\mathcal{A} = \{m_0, ..., m_{k-1}\}$;

INF2: There is no upper bound for $\mathcal{A}$;

INF3: There is no m such that $\forall \mathcal{B} \subseteq \mathcal{A}(|\mathcal{B}| < m)$;

INF4: $\forall y \exists x > y(x \in \mathcal{A})$;

INF5: $\forall m \exists \mathcal{B} \subseteq \mathcal{A}(|\mathcal{B}| = m)$.

In this paper, we systematically compare them in the method of constructive reverse mathematics. We show that the equivalence among them can be characterized by various combinations of induction axioms and non-constructive principles, including the axiom called bounded comprehension.

1 Introduction

When can we call a set "finite" and "infinite"? We consider this problem for sets of natural numbers in this paper.

In constructive mathematics (cf. [9]), several notions of finiteness for sets have been defined. In most of literature, a set $\mathcal{X}$ is said to be *finite* if there is a bijection between $\{0, ..., n-1\}$ for some n (cf. [3, Ch.1.2], [2, Ch.2.1] and [9, Ch.1.3.6]). A set $\mathcal{X}$ such that there is a surjection from $\{x : x < n\}$ to $\mathcal{X}$ is called *finitely enumerable* in [3, Ch.1.2] and *finitely indexed* in [9, Ch.1.3]. A set which is a subset of a finite set is called *subfinite* in [9, Ch.1, Exercises]. A finite set is finitely enumerable and subfinite by definition. As stated in [3, Ch1.2], a finitely enumerable set $\mathcal{X}$ is finite if and only if $\mathcal{X}$ satisfies $\forall xy \in \mathcal{X}(\neg x = y \vee \neg\neg x = y)$. However, a finitely enumerable set is neither finite nor subfinite in general. Consider a set $\{0, x\}$, where x is a real number. We can easily construct a surjective $f : \{0, 1\} \to \{0, x\}$, namely, $f(0) = 0$ and $f(1) = x$, and so it is finitely enumerable. However, if $\{0, x\}$ were finite for any real number x, we would have WLPO $\neg\exists x(\alpha(x) = 0) \vee \neg\neg\exists x(\alpha(x) = 0)$ (cf. [3, Ch1.1]), a kind of law of excluded middle, which does not hold constructive mathematics. Similarly, if $\{0, x\}$ were subfinite for each natural number x, then we would have WLPO. A subfinite set is neither finite nor finitely enumerable in general. A set $\mathcal{X}_P$ of the form $\{x \in \mathbb{N} : x = 0 \wedge P\}$ is a subset of the finite set $\{0\}$, and so it is subfinite. On the other hand, if $\mathcal{X}_P$ were finitely enumerable for all P, then we would have the law of excluded middle $P \vee \neg P$, which does not hold.

Now, we focus on sets of natural numbers. First, note that we have $\neg x = y \vee \neg\neg x = y$ for all natural numbers x and y. Hence, the notions of finite and finitely enumerable sets coincide for sets of natural numbers, which can be described as follows:

FIN1**:** There are natural numbers k, m_0,...,m_{k-1} such that $\mathcal{A} = \{m_0, ..., m_{k-1}\}$.

A set $\mathcal{A} \subset \mathbb{N}$ has an upper bound m if and only if $\mathcal{A}$ is a subset of $\{x : x < m\}$. Therefore the following corresponds to the notion of subfinite sets.

FIN2**:** $\mathcal{A}$ is has an upper bound, i.e., $\exists y \forall x(x \in \mathcal{A} \to x < y)$.

FIN2 is defined by concerning about the upper bound of the sizes of elements. It is useful for sets of natural numbers or ordinals. However, it is not always defined.

More general notion of finiteness is concerning about the upper bound of the sizes of its subsets:

FIN3**:** There is an upper bound $m \in \mathbb{N}$ of the cardinality of $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{A}$, i.e., $\forall m \forall \mathcal{B}(\mathcal{B} \subseteq \mathcal{A} \to |\mathcal{B}| < m)$.

These three notions are still not equivalent even for sets of natural numbers in constructive mathematics.

Recall the sets $\mathcal{X}_P$ of the form $\{x \in \mathbb{N} : x = 0 \wedge P\}$. As we have seen before, $\mathcal{X}_P$ is finite in the sense of FIN2 but not FIN1 in general.

If all the sets $\mathcal{Y}_\alpha$ of the form $\{x \in \mathbb{N} : x \text{ is the least natural number such that } \alpha(x) = 0\}$ are finite in the sense of FIN2, where α is an infinite sequence of natural numbers, then we can determine $\exists n(\alpha(n) = 0)$ or $\neg\exists n(\alpha(n) = 0)$ by checking some finite initial segment of α. Hence we have LPO $\exists n(\alpha(n) = 0) \vee \neg\exists n(\alpha(n) = 0)$, which is again not provable in general. On the other hand, the cardinality of $\mathcal{B}$ is less than 2 for any $\mathcal{B} \subseteq \mathcal{Y}_\alpha$. This $\mathcal{Y}_\alpha$ also fails to be finite in the sense of FIN1, since we cannot determine n such that there exists a surjection $f : \{x : x < n\} \to \mathcal{Y}_\alpha$.

Once it turned out that the above inequivalence, we cannot expect equivalence among notions of infiniteness. One of the most important aspects of infiniteness is that it is not finite. Therefore, by negating FIN1-FIN3, we obtain the following notions of infiniteness:

INF1: There are no natural numbers k, m_0,...,m_{k-1} such that $\mathcal{A} = \{m_0, ..., m_{k-1}\}$.

INF2: There is no upper bound $y \in \mathbb{N}$ of $\mathcal{A}$.

INF3: There is no upper bound $m \in \mathbb{N}$ of the cardinality of $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{A}$.

We can also define infiniteness positively, concerning the size of its elements and the size of subsets, respectively:

INF4: For each y, there exists $x > y$ such that $x \in \mathcal{A}$.

INF5: For each m, there exists $\mathcal{B} \subseteq \mathcal{A}$ of cardinality m, i.e., $\forall m \exists \mathcal{B}(\mathcal{B} \subseteq \mathcal{A} \wedge |\mathcal{B}| = m)$.

INF4 implies INF5. For the converse, i.e., to derive INF4 from INF5, we need some induction principles in general.

Similarly to INF1–INF3, we have notions of finiteness corresponding to INF4–INF5, which reflect the aspect of finiteness that they are not infinite:

FIN4: It is not the case that $\forall y \exists x > y(x \in \mathcal{A})$.

FIN5: It is not the case that, for each m, there exists $\mathcal{B} \subseteq \mathcal{A}$ of cardinality m, i.e., $\neg \forall m \exists \mathcal{B}(\mathcal{B} \subseteq \mathcal{A} \wedge |\mathcal{B}| = m)$.

Now we have five notions of finiteness and infiniteness. In classical mathematics, FIN1-FIN5 and INF1-INF5 are pairwise equivalent, respectively, and each of them are used in practice. For example, in [8], we use the notion FIN2 in Exercise 4.20 and the notion FIN3 in the argument immediately after 2.4 Definition in Section II.2. Then it is natural to ask how much they are different in terms of constructive mathematics, or what kinds of fragments of classical mathematics prove their equivalence. In this paper, we answer this question with the method of *constructive reverse mathematics*. We formalize the above notions of finiteness and infiniteness in the language of first order arithmetic, using a formula $A(x)$ with a distinct number variable x to describe the set $\{x \in \mathbb{N} : A(x)\}$. Then, for each class of formulae Γ, we consider a schema consists of all axioms of the form

> If A is finite (infinite) in the sense of FIN$*$ (INF$*$), then A is finite in the sense of FIN- (INF-),

where A in Γ and where $*$, - $\in \{1, ..., 5\}$. We show what non-constructive principles are necessary and sufficient to prove the above schemata on a base theory. See [6] for more information about constructive reverse mathematics in general.

An important example of finite sets is $\{x : A(x) \wedge x < m\}$. Since it has an upper bound m, it is finite in the sense of FIN2. It is also FIN1, if we admit the principle of *bounded comprehension*, which gives $k, m_0, ..., m_{k-1}$ such that $\{m_0, ..., m_{k-1}\} = \{x : A(x) \wedge x < m\}$ for a given m and formula A. This shows that the bounded comprehension implies FIN2 $\to$ FIN1 for B of a certain form. We will see that fragments of the bounded comprehension actually characterize many schemata we mentioned above.

Since the bounded comprehension can be characterized by some combinations of induction principles and non constructive principle such as the law of excluded middle (cf. [7, Subsection 3.1]), this situation tells us that we have more detailed picture on the relationship among the notions of infiniteness and finiteness, if we work over a theory with restricted induction principles. Therefore, as a base theory, we use $\mathbf{i\Sigma}_1$, a subsystem of **HA** with induction schema restricted to Σ_1 formulae.

In section 2, we prepare our formal system. In section 3, we overview the characterizations of bounded comprehension and its variants with induction principles and non-constructive principles. In section 4, we see relationships among the notions of finiteness corresponding to FIN1–FIN5. In section 5, we see relationships among the notions of infiniteness corresponding to INF1–INF5. In section 6, we focus on special class of sets, recursive, Σ_n and Π_n sets and show the relationship among its finiteness and infiniteness.

2 Preliminary

L_1 is a one-sorted first order language with equality $=$ consisting of constant symbols 0 and 1, binary function symbols $+$ and $\cdot$ and a binary predicate $<$. The formulae in L_1 defined as usual. We abbreviate $\exists x(x < t \wedge A(x))$ as $(\exists x < t)A(x)$ and $\forall x(x < t \to A(x))$ as $(\forall x < t)A(x)$, respectively, and such occurrences of quantifiers are called *bounded*.

Let Γ and Γ' be classes of formulae. For $\circ \in \{\wedge, \vee, \to\}$, the class $\Gamma \circ \Gamma'$ of formulae consists of all $A \circ B$ with A from Γ and B from Γ'. The class $\neg\Gamma$ is $\Gamma \to \{\bot\}$. For $Q \in \{\exists, \forall\}$, the class $Q\Gamma$ of formulae consists of all QxA with A from Γ. The class $\Delta(\Gamma)$ is the smallest class of formulae containing Γ and closed under $\wedge$, $\vee$, $\to$, $\exists x < t$, $\forall x < t$. A formula A is said to be Γ if it is in Γ. For the class Φ of atomic formulae, the class $\Delta(\Phi)$ is called Δ_0, Σ_0 or Π_0. The class Σ_{n+1} and Π_{n+1} are defined by $\exists\Pi_n$ and $\forall\Sigma_n$, respectively.

Definition 1. For a class Γ of formulae, the schema Γ-IND is defined as follows:

$$A(0) \wedge \forall x(A(x) \to A(x+1)) \to \forall x A(x), \text{ for any } \Gamma \text{ formula } A.$$

Definition 2. The L_1 theory $\mathbf{i\Sigma}_1$ is based on intuitionistic logic with equality and consists of Σ_1-IND and the following axioms for basic arithmetic:

$$x + 0 = x; \quad x + (y+1) = (x+y) + 1; \quad x \cdot 0 = 0; \quad x \cdot (y+1) = x \cdot y + x;$$
$$\neg(x < x); \quad x < y \wedge y < z \to x < z; \quad x < x+1; \quad x < y \to (x+1 < y) \vee (x+1 = y).$$

Remark 1. As presented above, we give axioms and schemata in open form but not universal closure of them. Therefore, if we use deduction system like natural deduction, we have to distinguish free variables in axioms and other assumptions (cf. [9, Ch.2.1.7])

In what follows, we only consider classes Γ of formulae with the following properties.

(I) For any Γ formula A and any Δ_0 formula B, there are formulae in Γ equivalent to $A \wedge B$, $B \wedge A$, $A \vee B$, $B \vee A$ and $B \to A$ in $\mathbf{i\Sigma}_1$.

(II) For any Γ formula $A(y)$, any Δ_0 formula $B(\vec{x}, y)$ and any term $t(\vec{x})$, there exists a Γ formula C such that $\mathbf{i\Sigma}_1$ proves $\forall \vec{x} \exists ! y{<}t(\vec{x}) B(\vec{x}, y) \to (\exists y{<}t(\vec{x})(B(\vec{x}, y) \wedge A(y)) \leftrightarrow C)$.

For a standard n, a function $f : \mathbb{N}^n \to \mathbb{N}$ is *bounded* Δ_0 *definable* in $\mathbf{i\Sigma}_1$ when $f(\vec{x}) = y$ if and only if $\mathbb{N} \models \exists y < t(\vec{x}) B(\vec{x}, y)$ for some Δ_0 formula $B(\vec{x}, y)$ and some term $t(\vec{x})$ such that $\mathbf{i\Sigma}_1 \vdash \forall x \exists ! y < t(\vec{x}) B(\vec{x}, y)$. For such f, we use the abbreviation $A(f(\vec{x}))$ for $\exists y < t(\vec{x})(B(\vec{x}, y) \wedge A(y))$. The above (II) means that Γ is closed under substitutions by $f(\vec{x})$ for any bounded Δ_0 definable f. Note that $\mathbf{i\Sigma}_1$ proves that

$$\forall \vec{x} (\exists ! y < t(\vec{x})) B(\vec{x}, y) \to (\exists y < t(\vec{x})(B(\vec{x}, y) \wedge A(y)) \leftrightarrow \forall y < t(\vec{x})(B(\vec{x}, y) \to A(y))).$$

We fix a bounded Δ_0 definable bijective pairing (-,-) : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and projections $(\text{-})_i$ satisfying $(x)_i \leq x$. We have a Δ_0 predicate $x \in \mathbb{N}^{<\mathbb{N}}$ in L_1, intended to express x is a code of finite sequence of natural numbers, and bounded Δ_0 definable functions length $x \mapsto |x|$, i-th element $[x, i] \mapsto x(i)$, 1-length sequence $x \mapsto \langle x \rangle$, concatenation $[x, y] \mapsto x * y$ and restriction $[x, i] \mapsto \overline{x}i$ such that $\mathbf{i\Sigma}_1$ proves the following (cf. [5, Ch.V.3]):

- $x \in \mathbb{N}^{<\mathbb{N}} \wedge y \in \mathbb{N}^{<\mathbb{N}} \wedge |x| = |y| \wedge (\forall i < |x|)(x(i) = y(i)) \to x = y$.
- $\exists x (x < \mathbb{N}^{<\mathbb{N}} \wedge |x| = 0)$.
- $|\langle x \rangle| = 1 \wedge (\langle x \rangle)(0) = x$.
- $x \in \mathbb{N}^{<\mathbb{N}} \wedge y \in \mathbb{N}^{<\mathbb{N}} \to x * y \in \mathbb{N}^{<\mathbb{N}} \wedge |x * y| = |x| + |y| \wedge (\forall i < |x| + |y|)((i < |x| \to (x * y)(i) = x(i)) \wedge (|x| \leq i < |x + y| \to (x * y)(i) = y(i - |x|)))$.
- $x \in \mathbb{N}^{<\mathbb{N}} \wedge i \leq |x| \to \overline{x}i \in \mathbb{N}^{<\mathbb{N}} \wedge |\overline{x}i| = i \wedge \forall j < i((\overline{x}i)(j) = x(j))$.
- $x \in \mathbb{N}^{<\mathbb{N}} \to |x| < x \wedge (\forall i < |x|)(x(i) < x \wedge \overline{x}i < x)$.
- $x \in \mathbb{N}^{<\mathbb{N}} \wedge y \in \mathbb{N} \wedge |x| \leq |y| \wedge \forall i < |x|(x(i) < y(i)) \to x < y$.

We use abbreviations $\exists x \in \mathbb{N}^{<\mathbb{N}} A$ and $\forall x \in \mathbb{N}^{<\mathbb{N}} A$ for $\exists x (x \in \mathbb{N}^{<\mathbb{N}} \wedge A)$ and $\forall x (x \in \mathbb{N}^{<\mathbb{N}} \to A)$, respectively. For each i and m, the sequence j^m is a sequence s of length m and $s(i) = j$ for each $i < m$. We write $s \in \{0,1\}^m$ for $|s| = m \wedge (\forall i < m)(s(i) < 2)$, $\exists s \in \{0,1\}^m A$ for $\exists s (s \in \{0,1\} \wedge A)$, and $\forall s \in \{0,1\}^m A$ for $\forall s (s \in \{0,1\}^m \to A)$. Note that the function $[i, m] \mapsto i^m$ is not bounded Δ_0 definable, and so the quantifiers $\exists s \in \{0,1\}^m$ and $\forall s \in \{0,1\}^m$ are not bounded.

We often use the following equivalence. First two equivalences generally hold in intuitionistic logic.

Lemma 1. $\mathbf{i\Sigma}_1$ *proves the following:*

- $(A \to B) \to (\neg\neg B \to \neg\neg A)$;

- $(\neg\neg A \to \neg\neg B) \leftrightarrow (A \to \neg\neg B)$;
- $(A \lor B) \leftrightarrow \exists x((x = 0 \to A) \land (x \neq 0 \to B)) \leftrightarrow (\exists x < 2)((x = 0 \to A) \land (x \neq 0 \to B))$.

Now we list several principles and its basic property which we need in the following sections.

Lemma 2. *1.* Γ-IND *and* $\neg\neg\Gamma$-IND *are equivalent over* $\mathbf{i\Sigma}_1 + \Gamma$-DNE.

2. $\neg\Gamma$-IND *is equivalent to* $\neg\neg\Gamma$-IND *over* $\mathbf{i\Sigma}_1$.

Proof. 1. First, we show $\neg\neg\Gamma$-IND implies Γ-IND over $\mathbf{i\Sigma}_1 + \Gamma$-DNE. Assume $A(0)$ and $\forall x(A(x) \to A(x+1))$ for a formula A in Γ. Then $\neg\neg A(0)$ and $\forall x(\neg\neg A(x) \to \neg\neg A(x+1))$. By $\neg\neg\Gamma$-IND, $\forall x \neg\neg A(x)$ hold. By Γ-DNE, we have $\forall x A(x)$. The converse can be proved in a similar way.

2. It is enough to show that $\neg\Gamma$-IND implies $\neg\neg\Gamma$-IND, since it shows $\neg\neg\Gamma$-IND implies $\neg\neg\neg\Gamma$-IND, which is equivalent to $\neg\Gamma$-IND. Let $A(x)$ be Γ. Assume $\neg\neg A(0)$, $\forall x(\neg\neg A(x) \to \neg\neg A(x+1))$ and $\neg A(k)$. For $B(x) \equiv x \leq k \to \neg A(k - x)$, we have $B(0)$ by $\neg A(k)$. If $B(x)$ and $x + 1 \leq k$, then we have $B(x+1)$ by $\forall x(\neg\neg A(x) \to \neg\neg A(x+1))$. Then we have $(\forall x \leq k)B(x)$ by $\neg\Gamma$-IND. In particular, we have $B(k)$ and so $\neg A(0)$, contradicting to $A(0)$. Therefore $\forall x \neg\neg A(x)$ holds. $\square$

Remark 2. Note that $\neg\Sigma_1$-IND is equivalent to Π_1-IND. Wehmeier [10, Theorem 4] essentially proved the above 1 by showing that $\neg\neg\Sigma_1$-IND is equivalent to Π_1-IND. Buss [4] proved that Σ_1-IND does not imply Π_1-IND.

A class Γ of formulae is *closed under bounded universal (existential) quantifier in* $\mathbf{T}$ if for each Γ formula $A(\vec{y})$ such that all free variables are displayed, there exists a Γ formula $B(\vec{y}, z)$ such that $\mathbf{T} \vdash \forall \vec{y} \forall z((\forall x < z)A(\vec{y}) \leftrightarrow B(\vec{y}, z))$ ($\mathbf{T} \vdash \forall \vec{y} \forall z((\exists x < z)A(\vec{y}) \leftrightarrow B(\vec{y}, z))$).

In what follows, we only consider $\mathbf{T}$ containing $\mathbf{i\Sigma}_1$.

Lemma 3. *1. If* Γ *is closed under bounded universal quantifier in* $\mathbf{T}$*, and if* $\mathbf{T}$ *proves* $\exists\Gamma$-IND*, then* $\exists\Gamma$ *is closed under bounded universal quantifier in* $\mathbf{T}$*.*

2. If Γ *is closed under bounded universal quantifier in* $\mathbf{T}$*, and if* $\mathbf{T}$ *proves* $\neg\neg\exists\Gamma$-IND*, then* $\neg\neg\Gamma$ *is closed under bounded universal quantifier in* $\mathbf{T}$*.*

Proof. Let $B(x, y, \vec{v})$ be a Γ formula and set $C(x, y, \vec{v}) \equiv y \in \mathbb{N}^{<\mathbb{N}} \land |y| = x \land (\forall i < x)B(i, y(i), \vec{v})$. If Γ is closed under bounded universal quantifier in $\mathbf{T}$, $C(x, y, \vec{v})$ is equivalent to a Γ formula in $\mathbf{T}$. Then $\exists y C(x, y, \vec{v}) \to (\forall i < x)\exists y B(x, y, \vec{v})$.

1. It is enough to show that $\exists\Gamma$-IND implies $(\forall i < x)\exists y B(x, y, \vec{v}) \to \exists y C(x, y, \vec{v})$. Assume that Γ and $\mathbf{T}$ satisfy the conditions in the premise. Assume $(\forall i < x)\exists y B(x, y, \vec{v})$. Since $C(0, \langle\rangle, \vec{v})$, we have $0 \leq x \to \exists y C(0, y, \vec{v})$. By $(\forall i < x)\exists y B(x, y, \vec{v})$, $\exists y C(j, y, \vec{v})$ implies $j + 1 \leq x \to \exists y C(j+1, y, \vec{v})$. Then we have $(j \leq x \to \exists y C(j, y, \vec{v})) \to (j + 1 \leq x \to \exists y C(j+1, y, \vec{v}))$. By $\exists\Gamma$-IND on j, we have $(\forall j \leq x)\exists y C(j, y, \vec{v})$. In particular, we have $\exists y C(x, y, \vec{v})$.

2. We can prove $(\forall i < x)\neg\neg\exists y B(x, y, \vec{v}) \to \neg\neg\exists y C(x, y, \vec{v})$ in a similar way to 1 by $\neg\neg\exists\Gamma$-IND. $\square$

Corollary 1. *1.* $\exists\Delta(\Gamma)$ *is closed under bounded universal quantifier in* $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND*.*

2. $\neg\neg\exists\Delta(\Gamma)$ *is closed under bounded universal quantifier in* $\mathbf{i\Sigma}_1 + \neg\neg\exists\Delta(\Gamma)$-IND*.*

Proof. Since $\Delta(\Gamma)$ is closed under bounded universal quantifier, it follows from Lemma 3. □

For non-constructive principles, we consider the following.

Definition 3. We define the following schemata:

Γ-LEM**:** $A \lor \neg A$, for any Γ formula A.

Γ-DNE**:** $\neg\neg A \to A$, for any Γ formula A.

Γ-GDM**:** $\neg(\forall x < m)A(x) \to (\exists x < m)\neg A(x)$, for any Γ formula A.

Γ-DNS**:** $\forall x\neg\neg A(x) \to \neg\neg\forall x A(x)$, for any Γ formula.

Lemma 4. *1.* $\mathbf{i\Sigma}_1 + \Gamma$-LEM *proves* Γ-DNE*.*

2. $\mathbf{i\Sigma}_1 + \Delta(\Gamma)$-DNE *proves* $\Delta(\Gamma)$-LEM*.*

3. $\mathbf{i\Sigma}_1 + \Gamma$-DNE *proves* Γ-DNS*.*

Proof. 1. Easy.

2. For any $\Delta(\Gamma)$ formula A, set $B(i) \equiv (i = 0 \to A) \land (i \neq 0 \to \neg A)$. Then $A \lor \neg A$ is equivalent to $(\exists i < 2)B(i)$. Since $\neg(\forall i < 2)\neg B(i)$, we have $A \lor \neg A$ by $\Delta(\Gamma)$-DNE.

3. For any Γ formula $A(x)$, $\forall x\neg\neg A(x)$ implies $\forall x A(x)$ by Γ-DNE, and so $\neg\neg\forall x A(x)$. □

3 Bounded comprehension and equivalence schemata

In this section, we see that bounded comprehension axioms and its variants can be characterized by several combinations of inductions and logical principles. In the following sections, we will see bounded comprehension axioms actually characterize the relationships among the notions of finiteness and infiniteness.

Definition 4. Γ *bounded comprehension* Γ-BCA, Γ *weak bounded comprehension* Γ-WBCA, Γ *least number principle* Γ-LNP, and Γ *weak least number principle* Γ-WLNP are defined as follows:

Γ-BCA**:** $\exists s \in \{0,1\}^m \forall i < m(s(i) = 0 \leftrightarrow A(i))$, for any Γ formula A with no free s.

Γ-WBCA**:** $\neg\neg\exists s \in \{0,1\}^m \forall i < k(s(i) = 0 \leftrightarrow A(i))$, for any Γ formula A with no free s.

Γ-LNP**:** $A(x) \to \exists y \leq x(A(y) \land (\forall z < y)\neg A(y))$, for any Γ formula A.

Γ-WLNP**:** $A(x) \to \neg\neg(\exists y \leq x)(A(y) \land \forall z < y\neg A(z))$, for any Γ formula A.

Lemma 5. *1. The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) $\Delta(\Gamma)$-IND $+$ $\Delta(\Gamma)$-LEM*;*

   (b) $\Delta(\Gamma)$-LNP*;*

   (c) $\Delta(\Gamma)$-BCA.

2. The following are pairwise equivalent over $\mathbf{i\Sigma}_1$*:*

   (a) $\neg\neg\Delta(\Gamma)$-IND*;*

   (b) $\Delta(\Gamma)$-WLNP*;*

   (c) $\Delta(\Gamma)$-WBCA.

Proof. 1. Although 1 is proved in [7, Lemma 3.2], we give a proof for later use in 2.

(a)⇒(b): Let $C(x)$ be a $\Delta(\Gamma)$ formula. Define

$$D(x) \equiv (\forall u \leq x)[C(u) \to \exists y \leq u(C(y) \land \forall z < y \neg C(z))].$$

It is enough to show $\forall x D(x)$. We trivially have $D(0)$. Assume $D(x)$ and $C(x+1)$. If $(\exists u < x+1)C(u)$, we have $(\exists y \leq x+1)(C(y) \land \forall z < y \neg C(z))$ by $D(x)$. If $(\forall u < x+1)\neg C(u)$, then $C(x+1) \land (\forall z < x+1)\neg C(z)$, which implies $(\exists y \leq v+1)(C(y) \land \forall z < y \neg C(z))$. Hence, we have

$$((\exists u < x+1)C(u) \lor (\forall u < x+1)\neg C(u)) \to (D(x) \to D(x+1)). \tag{1}$$

By $\Delta(\Gamma)$-LEM, we have $D(x) \to D(x+1)$. Then $\Delta(\Gamma)$-IND yields $\forall x D(x)$.

(b)⇒(c): Let $B(x)$ be a $\Delta(\Gamma)$ formula. For a given m, define $C(x, m)$ as follows:

$$C(x, m) \equiv x \in \{0,1\}^m \land \forall i < m(x(i) = 0 \to B(i)).$$

Assume $C(x, m) \land (\forall y < x)\neg C(y, m)$. Then x satisfies $(\forall i < m)(x(i) = 0 \to B(i))$. Recall that our coding of finite sequences satisfies $|u| \leq |v| \land \forall i < |u|(u(i) < v(i)) \to u < v$. If $B(j)$ and $x(j) = 1$ for some $j < m$, then $x' \in \{0,1\}^m$ defined by $x'(i) = 0$ for $i = j$ and $x'(i) = x(i)$ for $i \neq j$ satisfies $(\forall i < k)(s(i) = 0 \to B(i))$ and $x' < x$. This contradicts to $C(x, m) \land \forall y < x \neg C(y, m)$. Hence we have $(\forall i < m)(x(i) = 0 \leftrightarrow B(i))$. Therefore we have

$$\exists x(C(x, m) \land (\forall y < x)\neg C(y, m)) \to \exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow B(i)). \tag{2}$$

Since we have $C(1^m, m)$, there exists x such that $C(x, m) \land (\forall y < x)\neg C(y, m)$ by $\Delta(\Gamma)$-LNP. Therefore we have $\exists s \in \{0,1\}^m(\forall i < m)(s(i) \leftrightarrow B(i))$.

(c)⇒(a): First, we prove $\Delta(\Gamma)$-LEM. Let B be a $\Delta(\Gamma)$ formula. For a variable i which does not occur freely in B, we have $s \in \{0,1\}^1$ such that $(\forall i < 1)(s(i) = 0 \leftrightarrow B)$ by $\Delta(\Gamma)$-BCA. Then, we have B if $s(0) = 0$ and $\neg B$ otherwise.

Now we prove $\Delta(\Gamma)$-IND. Let $C(x)$ be a $\Delta(\Gamma)$ formula. Assume $C(0)$ and $\forall x(C(x) \to C(x+1))$ and suppose, for given m, there is $s \in \{0,1\}^{m+1}$ such that $(\forall i < m+1)(s(i) = 0 \leftrightarrow$

$C(i))$. Then, for $B(x) \equiv x \leq m \to s(x) = 0$, we have $B(0)$ by $C(0)$ and $\forall x(B(x) \to B(x+1))$ by $\forall x(C(x) \to C(x+1))$. By Δ_0-IND, we have $\forall x B(x)$ and so $C(m)$. Therefore, we have the following:

$$\exists s \in \{0,1\}^{m+1}(\forall i < m+1)(s(i) = 0 \leftrightarrow C(i)) \to [C(0) \land \forall x(C(x) \to C(x+1)) \to C(m)].$$

By $\Delta(\Gamma)$-BCA, we have $\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow C(i))$, and so $C(m)$.

2. (a)⇒(b): Let $C(x)$ be a $\Delta(\Gamma)$ formula. Define $D(x)$ as in the proof of (a)⇒(b) in 1. By (1), we have the following for each v:

$$\neg\neg((\exists u < v+1)C(u) \lor (\forall u < v+1)\neg C(u)) \to (\neg\neg D(v) \to \neg\neg D(v+1)).$$

Since $\neg\neg((\exists u < v+1)C(u) \lor (\forall u < v+1)\neg C(u))$ holds, the above implies $\neg\neg D(v) \to \neg\neg D(v+1)$. Since $D(0)$ holds, by $\Delta(\Gamma)$-IND, we have $\forall x \neg\neg D(x)$ and so $\forall x(C(x) \to \neg\neg(\exists y \leq x)(C(y) \land (\forall z < y)\neg C(z)))$.

(b)⇒(c): Let $B(x)$ be a $\Delta(\Gamma)$ formula. For a given m, define $C(x, m)$ as in the proof of (b)⇒(c) of 1. Then the following holds by (2):

$$\neg\neg\exists x(C(x,m) \land (\forall y < x)\neg C(y,m)) \to \neg\neg\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow B(i)).$$

Since $C(1^m, m)$, we have $\neg\neg\exists x(C(x,m) \land (\forall y < x)\neg C(y,m))$ by $\Delta(\Gamma)$-WLNP. Therefore, we have $\neg\neg\exists x \in \{0,1\}^m(\forall i < n)(s(i) = 0 \leftrightarrow B(i))$.

(c)⇒(a): Let $C(x)$ be a $\Delta(\Gamma)$ formula. Assume $\neg\neg C(0)$ and $\forall x(\neg\neg C(x) \to \neg\neg C(x+1))$. Suppose $\neg\neg C(0)$ and $\forall x(\neg\neg C(x) \to \neg\neg C(x+1))$, and suppose, for given m, there is $s \in \{0,1\}^{m+1}$ such that $(\forall i < m+1)(s(i) = 0 \leftrightarrow C(i))$. Then, for $B(x) \equiv x \leq m \to s(x) = 0$, we have $s(0) = 0$ by $\neg\neg C(0)$, and so $B(0)$, and $\forall x(B(x) \to B(x))$ by $\forall x(\neg\neg C(x) \to \neg\neg C(x+1))$. By Δ_0-IND, we have $\forall x B(x)$. Hence, we have the following:

$$\exists s \in \{0,1\}^{m+1}\forall i < m(s(i) = 0 \leftrightarrow C(x)) \to [\neg\neg C(0) \land \forall x(\neg\neg C(x) \to \neg\neg C(x+1)) \to C(m)].$$

By $\Delta(\Gamma)$-WBCA, we have $\neg\neg\exists s \in \{0,1\}^{m+1}(\forall i < m+1)(s(i) = 0 \leftrightarrow C(i))$ and so $\neg\neg C(m)$.

□

Lemma 6. *1. The following are equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) Γ-BCA*;*

   (b) $\Delta(\Gamma)$-BCA.

2. The following are equivalent over $\mathbf{i\Sigma}_1$*:*

   (a) Γ-WBCA*;*

   (b) $\Delta(\Gamma)$-WBCA.

Proof. Although 1 is proved in [7, Lemma 3.2], we prove it together with 2.

For both 1 and 2, it is enough to prove (a)⇒(b), which are proved by the induction on the construction of $\Delta(\Gamma)$ formulae. It is clear for the base case. For the case $C(x) \equiv (Qy < t(x))D(x, y)$, where $Q \in \{\exists, \forall\}$, take $l_m = \max\{(x, y) : x < m \wedge y < t(x)\} + 1$. If $(\forall i < l_m)(s(i) = 0 \leftrightarrow D((i)_0, (i)_1))$ for some $s \in \{0,1\}^{l_m}$, then we have $(\forall i < m)(Qy < t(i)(s((i, y)) = 0) \leftrightarrow Qy < t(i)D(i, y))$, since $t(i) < l_m$ for all $i < x$. Hence we have $(\forall i < m)(s'(i) = 0 \leftrightarrow Qy < t(i)D(i, y))$ for $s' \in \{0,1\}^m$ defined by $(\forall i < m)(s'(i) = 0 \leftrightarrow Qy < t(i)(s((i, y)) = 0))$. Therefore the following holds:

$$\exists s \in \{0,1\}^{l_m}(\forall i < l_m)(s(i) = 0 \leftrightarrow D((i)_0, (i)_1)) \rightarrow \\ \exists s \in \{0,1\}^{l_m}(\forall i < m)(s(i) = 0 \leftrightarrow (Qy < t(i))D(i, y)).$$

This is enough for 1. For 2, the above yields

$$\neg\neg\exists s \in \{0,1\}^{l_m}(\forall i < l_m)(s(i) = 0 \leftrightarrow D((i)_0, (i)_1)) \rightarrow \\ \neg\neg\exists s \in \{0,1\}^{l_m}(\forall i < m)(s(i) = 0 \leftrightarrow (Qy < t(i))D(i, y)).$$

The cases of logical connectives are easier. □

Lemma 7. $\mathbf{i\Sigma}_1 + \Delta(\Gamma)$-IND + Γ-LEM *implies* Γ-BCA.

Proof. By Σ_1-IND, we can prove that $\forall m \exists t \in \mathbb{N}^{<\mathbb{N}}(t = 1^m)$. Since our coding of finite sequences satisfies $|u| \leq |v| \wedge \forall i < |u|(u(i) < v(i)) \rightarrow u < v$, for each m and $s \in \{0,1\}^l$ with $l \leq m$, we have $s \leq 1^m$. Then $\exists s \in \{0,1\}^l(\forall i < l)(s(i) = 0 \leftrightarrow A(i))$ is equivalent to $\exists s \leq 1^m[s \in \{0,1\}^m \wedge (\forall i < l)(s(i) = 0 \leftrightarrow A(i))]$ for each Γ formula A and for each $l \leq m$. Now, for each Γ formula A, we can prove $\forall l \leq m[\exists s \leq 1^m(s \in \{0,1\}^m \wedge (\forall i < l)(s(i) = 0 \leftrightarrow A(i)))]$ by $\Delta(\Gamma)$-IND on l, and so we have $\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow A(i))$. □

Corollary 2. *1. The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

- *(a)* Γ-BCA*;*
- *(b)* $\Delta(\Gamma)$-IND + $\Delta(\Gamma)$-LEM*;*
- *(c)* $\Delta(\Gamma)$-IND + Γ-LEM*;*
- *(d)* $\Delta(\Gamma)$-LNP*;*
- *(e)* $\Delta(\Gamma)$-BCA.

2. The following are pairwise equivalent over $\mathbf{i\Sigma}_1$*:*

- *(a)* Γ-WBCA*;*
- *(b)* $\neg\neg\Delta(\Gamma)$-IND*;*
- *(c)* $\Delta(\Gamma)$-WLNP*;*
- *(d)* $\Delta(\Gamma)$-WBCA.

4 "Finite" sets of natural numbers

In this section, we consider equivalences among various notions of "finiteness" of sets of natural numbers, induction principles and semi-constructive principles.

Now we consider the notions corresponding FIN1-FIN5 in the introduction on $\mathbf{i\Sigma}_1$. Let A be a formula with a distinct number variable x. Informally, A can be regarded as a set $\{x \in \mathbb{N} : A(x)\}$. Similarly, a finite sequence s can be regarded as a code of set $\{s(i) : i < |s|\}$. We often write $x \in A$, $s = A$, $A \subseteq s$ and $s \subseteq A$ instead of $A(x)$, $\forall x((\exists i < |s|)s(i) = x \leftrightarrow A(x))$, $\forall x(A(x) \to (\exists i < |s|)(s(i) = x))$ and $\forall x((\exists i < |s|)(s(i) = x) \to A(x))$, respectively. Let $s \in [\mathbb{N}]^{<\mathbb{N}} \equiv s \in \mathbb{N}^{<\mathbb{N}} \wedge (\forall i, j < |s|)(i < j \to s(i) < s(j))$. Then $s \in [\mathbb{N}]^{<\mathbb{N}}$ gives a non-redundant code of the set $\{s(i) : i < |s|\}$. We use abbreviations $\forall s \in [\mathbb{N}]^{<\mathbb{N}} B$ and $\exists s \in [\mathbb{N}]^{<\mathbb{N}} B$ for $\forall s(s \in [\mathbb{N}]^{<\mathbb{N}} \to B)$ and $\exists s(s \in [\mathbb{N}]^{<\mathbb{N}} \wedge B)$, respectively.

We consider the following schemata $\mathrm{fin}_*(A)$:

1. $\mathrm{fin}_1(A) \equiv \exists s \in [\mathbb{N}]^{<\mathbb{N}}(A = s)$;
2. $\mathrm{fin}_2(A) \equiv \exists y \forall x(x \in A \to x < y)$;
3. $\mathrm{fin}_3(A) \equiv \exists m \forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m)$;
4. $\mathrm{fin}_4(A) \equiv \neg \forall y(\exists x > y)(x \in A)$;
5. $\mathrm{fin}_5(A) \equiv \neg \forall m \exists s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \wedge |s| = m)$.

Remark 3. 1. $\mathrm{fin}_1(A)$ is equivalent to $\exists s \in \mathbb{N}^{<\mathbb{N}}(A = s)$, which corresponds to $\exists s : \{i : i < |s|\} \to A$($s$ is a surjection).

2. $\mathrm{fin}_2(A)$ is equivalent to $\exists s \in [\mathbb{N}]^{<\mathbb{N}}(A \subseteq s)$ and $s \in \mathbb{N}^{<\mathbb{N}}(A \subseteq s)$ over $\mathbf{i\Sigma}_1$.

3. In $\mathrm{fin}_3(A)$ and $\mathrm{fin}_5(A)$, we cannot replace $[\mathbb{N}]^{<\mathbb{N}}$ with $\mathbb{N}^{<\mathbb{N}}$.

The following is easy to prove.

Lemma 8. $\mathbf{i\Sigma}_1$ *proves the following for any* A*;*

1. $\mathrm{fin}_1(A) \to \mathrm{fin}_2(A)$;
2. $\mathrm{fin}_2(A) \to \mathrm{fin}_3(A)$;
3. $\mathrm{fin}_3(A) \to \mathrm{fin}_4(A)$;
4. $\mathrm{fin}_4(A) \to \mathrm{fin}_5(A)$;

For a class Γ of formulae and $*, - \in \{1, ..., 5\}$, $(\mathrm{fin}_* \to \mathrm{fin}_-)(\Gamma)$ is the following schema:

$$\mathrm{fin}_*(A) \to \mathrm{fin}_-(A), \text{ for any } \Gamma \text{ formula } A.$$

Theorem 1. *The following are equivalent over* $\mathbf{i\Sigma}_1$*:*

1. Γ-BCA;

2. $(\mathrm{fin}_2 \to \mathrm{fin}_1)(\Gamma)$.

Proof. For "1⇒2", assume Γ-BCA. Let $A(x)$ be a Γ formula. Assume $\mathrm{fin}_2(A)$ and let y be such that $\forall x(x \in A \to x < y)$. Take $t \in \{0,1\}^y$ by Γ-BCA and $s \in [\mathbb{N}]^{<\mathbb{N}}$ such that

$$\forall i < y(t(i) = 0 \leftrightarrow A(i)) \qquad \text{and} \qquad \forall x(x \in s \leftrightarrow x < y \wedge t(x) = 0).$$

Then we have $A = s$.

For the converse, assume $(\mathrm{fin}_2 \to \mathrm{fin}_1)(\Gamma)$. Let $B(x)$ be a Γ formula. For a given m, the formula $A(x) \equiv B(x) \wedge x < m$ is equivalent to a Γ formula and satisfies $\forall x(x \in A \to x < m)$. Therefore $\mathrm{fin}_2(A)$ and so $\mathrm{fin}_1(A)$ by $(\mathrm{fin}_2 \to \mathrm{fin}_1)(\Gamma)$. Let t be such that $A = t$. Define $s \in \{0,1\}^m$ by $(\forall i < m)(s(i) = 0 \leftrightarrow i \in t)$. Then we have $(\forall i < m)(s(i) = 0 \leftrightarrow B(i))$. □

Lemma 9. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. $\mathbf{T}$ *proves* $\exists\Gamma$-BCA;

2. Γ *is closed under bounded universal quantifier in* $\mathbf{T}$,

then $\mathbf{T}$ *proves* $(\mathrm{fin}_3 \to \mathrm{fin}_1)(\exists\Gamma)$.

Proof. Assume $\mathbf{T}$ and Γ satisfy the conditions in the premise. Let $A(x)$ be a $\exists\Gamma$ formula. We can write $A(x)$ as $\exists y B(x,y)$ where $B(x,y)$ is Γ. Set $C(x,z)$ as follows

$$C(x,z) \equiv z \in [\mathbb{N}]^{<\mathbb{N}} \wedge |z| = x + 1 \wedge (\forall j \in z)\exists y B(j,y).$$

By Lemma 3.1 and Lemma 2.1, $\exists\Gamma$ is closed under bounded universal quantifier. Hence both $C(x,z)$ and $\exists z C(x,z)$ are equivalent to some formulae in $\exists\Gamma$.

Assume $\mathrm{fin}_3(A)$, and let m be such that $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m)$. Then there is $t \in \{0,1\}^m$ such that $(\forall i < m)(t(i) = 0 \leftrightarrow \exists z C(i,z))$ by $\exists\Gamma$-BCA. If there is no $i < m$ such that $t(i) = 0$, then $A = \langle\rangle$. Otherwise, take the maximum $i < m$ with $t(i) = 0$ and u with $C(i,u)$. Then $\forall j(j \in u \to \exists y B(j,y))$. If $\exists y B(j,y)$ for some $j \notin u$, then $u' \in [\mathbb{N}]^{<\mathbb{N}}$ defined by $\forall v(v \in u' \leftrightarrow v \in u \vee v = j)$ satisfies $u' \subseteq A$, and so $|u'| < m$ by the assumption $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m)$. Hence we have $C(i+1, u')$, which contradicts to the maximality of i. Therefore $\forall x(x \in u \leftrightarrow \exists z B(x,z))$. Thus we have $\mathrm{fin}_1(A)$. □

Lemma 10. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-BCA *proves* $(\mathrm{fin}_3 \to \mathrm{fin}_1)(\exists\Delta(\Gamma))$.

Proof. Since $\Delta(\Gamma)$ is closed under bounded universal quantifier, it follows from Lemma 9. □

Corollary 3. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-BCA *proves* $(\mathrm{fin}_3 \to \mathrm{fin}_2)(\exists\Delta(\Gamma))$.

Proof. This follows from Lemma 10 and Lemma 8. □

A class Γ of formulae is *closed under uniformization* in $\mathbf{T}$ if, for each formula $C(x,\vec{y})$ in Γ such that all free variables are displayed, there is a formula $C'(x,\vec{y})$ in Γ with the same free variables as $C(x,\vec{y})$ such that $\mathbf{T}$ proves

$$\forall\vec{y}(\exists x C(x,\vec{y}) \leftrightarrow \exists x C'(x,\vec{y})), \qquad \text{and} \qquad \forall\vec{y}(\forall x\forall x'(C'(x,\vec{y}) \wedge C'(x',\vec{y}) \to x = x')).$$

Lemma 11. *If $\mathbf{T}$ is a theory and Γ is a class of formulae such that*

1. *$\mathbf{T}$ proves Γ-BCA and $(\mathrm{fin}_3 \to \mathrm{fin}_2)(\Gamma)$;*
2. *Γ is closed under uniformization and bounded existential quantifier in $\mathbf{T}$,*

then $\mathbf{T}$ proves $\exists\Gamma$-BCA.

Proof. Assume that $\mathbf{T}$ and Γ satisfy the conditions in the premise. Let $B(x) \equiv \exists z C(x,z)$ be a $\exists\Gamma$ formula, where $C(x,z)$ is Γ. Then there is a formula $C'(x,z)$ in Γ such that, for any x,

$$\exists z C(x,z) \leftrightarrow \exists z C'(x,z) \qquad \text{and} \qquad \forall z\forall z'(C'(x,z) \wedge C'(x,z') \to z = z').$$

For a given m, let $A(x) \equiv (x)_0 < m \wedge C'((x)_0,(x)_1)$. Then $A(x)$ is equivalent to a Γ formula. Since $\exists z A((i,z))$ implies $i < m$, and since $A((i,z)) \wedge A((i,z')) \to z = z'$ for each $i < m$, we have $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m+1)$, i.e., $\mathrm{fin}_3(A)$. By $\mathrm{fin}_3(A) \to \mathrm{fin}_2(A)$, we have $\mathrm{fin}_2(A)$. Take y such that $\forall x(x \in A \to x < y)$. Then the following holds:

$$\forall i < m[\exists z C(i,z) \leftrightarrow \exists z C'(i,z) \leftrightarrow \exists z A((i,z)) \leftrightarrow (\exists z < y)A((i,z))]. \tag{3}$$

Since Γ is closed under bounded existential quantifier, $(\exists z < y)A((i,z))$ is equivalent to some Γ formula, and hence we have $s \in \{0,1\}^m$ such that $(\forall i < m)(s(i) = 0 \leftrightarrow (\exists z < y)A((i,z)))$ by Γ-BCA. Then it is easy to see that $\forall i < m(s(i) = 0 \leftrightarrow \exists z C(i,z))$ by (3). □

Lemma 12. *If $\mathbf{T}$ proves $\Delta(\Gamma)$-LNP, then $\Delta(\Gamma)$ is closed under uniformization in $\mathbf{T}$.*

Proof. Let $C(x,\vec{v})$ be a $\Delta(\Gamma)$ formulas such that all free variables are displayed. Then $C'(x,\vec{v}) \equiv C(x,\vec{v}) \wedge (\forall y < x)\neg C(y,\vec{v})$ is still $\Delta(\Gamma)$. By $\Delta(\Gamma)$-LNP, we have the following:

$$\forall\vec{v}(\exists x C(x,\vec{v}) \leftrightarrow \exists C'(x,\vec{v})) \qquad \text{and} \qquad \forall\vec{v}(\forall x\forall x'(C'(x,\vec{v}) \wedge C'(x',\vec{v}) \to x = x')).$$

□

Corollary 4. $\mathbf{i\Sigma}_1 + \Gamma$-BCA $+ (\mathrm{fin}_3 \to \mathrm{fin}_2)(\Delta(\Gamma))$ *proves* $\exists\Delta(\Gamma)$-BCA.

Proof. Note that $\Delta(\Gamma)$ is closed under uniformization and bounded existential quantifier in $\mathbf{T} = \mathbf{i\Sigma}_1 + \Delta(\Gamma)$-LNP $+ (\mathrm{fin}_3 \to \mathrm{fin}_2)(\Delta(\Gamma))$, by Lemma 12. Then $\mathbf{T}$ proves $\exists\Delta(\Gamma)$-BCA, by Lemma 11. The theory $\mathbf{T}$ is equivalent to $\mathbf{i\Sigma}_1 + \Gamma$-BCA $+ (\mathrm{fin}_3 \to \mathrm{fin}_2)(\Delta(\Gamma))$, by Corollary 2.1. □

Theorem 2. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

1. $\exists\Delta(\Gamma)$-BCA*;*

2. $(\mathrm{fin}_3{\to}\mathrm{fin}_2)(\exists\Delta(\Gamma)) + \Gamma$-BCA*;*

3. $(\mathrm{fin}_3{\to}\mathrm{fin}_2)(\Delta(\Gamma)) + \Gamma$-BCA.

Corollary 5. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

1. $\exists\Delta(\Gamma)$-BCA*;*

2. $(\mathrm{fin}_3{\to}\mathrm{fin}_1)(\exists\Delta(\Gamma))$*;*

3. $(\mathrm{fin}_3{\to}\mathrm{fin}_1)(\Delta(\Gamma))$.

Proof. "1⇒2" follows from Lemma 10. "2⇒3" is clear. For "3⇒1", note that $(\mathrm{fin}_3{\to}\mathrm{fin}_1)(\Delta(\Gamma))$ implies $(\mathrm{fin}_2{\to}\mathrm{fin}_1)(\Delta(\Gamma))$, by Theorem 8, and so $\Delta(\Gamma)$-BCA, by Theorem 1. Then, by Lemma 4, it implies $\exists\Delta(\Gamma)$-BCA. □

Lemma 13. $\mathbf{i\Sigma}_1 + \exists\forall\neg\Gamma$-DNE $+ \exists\Gamma$-DNS *proves* $(\mathrm{fin}_4{\to}\mathrm{fin}_2)(\exists\Gamma)$.

Proof. Let $A(x)$ be a $\exists\Gamma$ formula. We can write $A(x)$ as $\exists z B(x,z)$ where $B(x,z)$ is Γ. Assume $\mathrm{fin}_4(A)$. Since $\neg(B(x,z)\wedge x>y) \leftrightarrow (B(x,z)\to x\le y)$, we have the following:

$$
\begin{aligned}
&\neg\forall y\exists x(A(x)\wedge x>y) \\
\leftrightarrow&\neg\forall y\exists x\exists z(B(x,z)\wedge x>y) \\
\leftrightarrow&\neg\forall y\neg\neg\exists x\exists z(B(x,z)\wedge x>y) && \text{(by }\exists\Gamma\text{-DNS)} \\
\leftrightarrow&\neg\neg\exists y\forall x\forall z\neg(B(x,z)\wedge x>y) \\
\leftrightarrow&\exists y\forall x\forall z\neg(B(x,z)\wedge x>y) && \text{(by }\exists\forall\neg\Gamma\text{-DNE)} \\
\leftrightarrow&\exists y\forall x\forall z(B(x,z)\to x\le y) \\
\leftrightarrow&\exists y\forall x(A(x)\to x\le y)
\end{aligned}
$$

The last line is equivalent to $\mathrm{fin}_2(A)$. □

Lemma 14. *If* $\mathbf{T}$ *is a theory, and* Ψ *and* Ψ' *are classes of formulae such that*

1. for each formula B *in* Ψ*, we can find a formula* C *in* Ψ' *such that* $\mathbf{T}\vdash\neg B\leftrightarrow C$*;*

2. $\mathbf{T}$ *proves* $\exists\Psi'$-IND*,* $\Delta(\Psi')$-LNP *and* $(\mathrm{fin}_4{\to}\mathrm{fin}_2)(\Psi')$*;*

3. Ψ' *is closed under uniformization and bounded universal quantifier in* $\mathbf{T}$*,*

then $\mathbf{T}$ *proves* $\exists\forall\Psi$-DNE*.*

Proof. Assume that $\mathbf{T}$, Ψ and Ψ' satisfy the conditions in the premise. Let $B(y,z)$ be a formula in Ψ. Suppose $\neg\neg\exists y\forall z B(y,z)$. Then there is a formula $C(x,y)$ in Ψ' such that

$$\forall x(\exists y\neg B(x,y)\leftrightarrow\exists y C(x,y)), \qquad \text{and} \qquad \forall x\forall y\forall y'(C(x,y)\wedge C(x,y)\to y=y').$$

Define $A(x) \equiv x \in \mathbb{N}^{<\mathbb{N}} \land (\forall i < |x|)C(i, x(i))$. Then $A(x)$ is equivalent to some formula in Ψ', since Ψ' is closed under bounded universal quantifier.

We show $\mathrm{fin}_4(A)$. For contradiction, assume $\forall y(\exists x > y)A(x)$. If $A(x) \land |x| = m$, then take x' such that $x' > x \land A(x')$. Note that we have $A(v) \land A(v') \land |v| = |v'| \to v = v'$ and $A(v) \land v' \in \mathbb{N}^{<\mathbb{N}} \land \exists i \leq |v|(v' = \overline{v}i) \to A(v')$. Then $|x'| \geq m + 1$ and hence $y = \overline{x'}(m+1)$ satisfies $A(y) \land |y| = m + 1$. Therefore $\forall m(\exists x(A(x) \land |x| = m) \to \exists x(A(x) \land |x| = m+1))$. Since $A(\langle\rangle) \land |\langle\rangle| = 0$, we have $\forall m \exists x(A(x) \land |x| = m)$ by $\exists\Psi'$-IND on m. This implies $\forall y \exists z \neg B(y, z)$, which contradicts to the assumption $\neg\neg\exists y \forall z B(y, z)$. Therefore $\mathrm{fin}_4(A)$ holds.

By $(\mathrm{fin}_4 \to \mathrm{fin}_2)(\Psi)$, we have $\mathrm{fin}_2(A)$. Take y such that $\forall x(x \in A \to x < y)$. Note that we have $\neg(\exists x < y)(A(x) \land |x| = y)$, since our cording of sequences of natural numbers satisfies $|s| < s$ for any s with $s \in \mathbb{N}^{<\mathbb{N}}$. By $\Delta(\Psi')$-LNP, we have m such that $\neg(\exists x < y)(A(x) \land |x| = m) \land (\forall l < m)\neg\neg(\exists x < y)(A(x) \land |x| = l)$. Since $\Delta(\Psi')$-DNE is implied by $\Delta(\Psi')$-LNP by Corollary 2.1 and Lemma 4.1, we have $(\forall l < m)(\exists x < y)(A(x) \land |x| = l)$. Since $\langle\rangle \in s$ holds by $A(\langle\rangle) \land |\langle\rangle| = 0$ and by $\forall x(x \in A \to x < y)$, we have $m > 0$. Then $\neg(\exists x < y)(A(x) \land |x| = m) \land (\exists x < y)(A(x) \land |x| = m - 1)$, which implies $\neg\exists z \neg B(m-1, z)$. Therefore we have $\exists y \forall z B(y, z)$. □

Lemma 15. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \Delta(\Gamma)$-LEM $+ (\mathrm{fin}_4 \to \mathrm{fin}_2)(\Delta(\Gamma))$ *proves* $\exists\forall\Delta(\Gamma)$-DNE.

Proof. Let $\mathbf{T} \equiv \mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \Gamma$-LNP $+ (\mathrm{fin}_4 \to \mathrm{fin}_2)(\Delta(\Gamma))$ and $\Psi \equiv \Psi' \equiv \Delta(\Gamma)$. Then $\mathbf{T}$, Ψ and Ψ' satisfy the conditions of Lemma 14, and hence $\mathbf{T}$ proves $\exists\forall\Delta(\Gamma)$-IND. Note that $\mathbf{T}$ is equivalent to $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \Gamma$-LEM $+ (\mathrm{fin}_4 \to \mathrm{fin}_2)(\Delta(\Gamma))$, by Corollary 2.1. □

Theorem 3. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND*:*

1. $\exists\forall\Delta(\Gamma)$-DNE*;*
2. $(\mathrm{fin}_4 \to \mathrm{fin}_2)(\exists\Delta(\Gamma)) + \Delta(\Gamma)$-LEM*;*
3. $(\mathrm{fin}_4 \to \mathrm{fin}_2)(\Delta(\Gamma)) + \Delta(\Gamma)$-LEM.

Proof. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND$+\exists\forall\Delta(\Gamma)$-DNE implies $\exists\forall\neg\Delta(\Gamma)$-DNE, $\exists\Delta(\Gamma)$-DNS and $\Delta(\Gamma)$-LEM, by Lemma 4.2. Therefore we have "1⇒2" by Lemma 13. "2⇒3" is trivial. "3⇒1" follows from Lemma 15. □

Lemma 16. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. $\mathbf{T}$ *proves* $\exists\forall\neg\Gamma$-DNE *and* $\exists\Gamma$-DNS*;*
2. $\exists\Gamma$ *is closed under bounded universal quantifier in* $\mathbf{T}$*,*

then $\mathbf{T}$ *proves* $(\mathrm{fin}_5 \to \mathrm{fin}_3)(\exists\Gamma)$*.*

Proof. Assume that $\mathbf{T}$ and Γ satisfy the conditions in the premise. Let A be a $\exists\Gamma$ formula. We can write $A(x)$ as $\exists z B(x, z)$ where $B(x, z)$ is Γ. Since $\exists\Gamma$ is closed under bounded universal quantifier in $\mathbf{T}$, we have $B'(x, s)$ in Γ such that $\forall s((\forall x \in s)\exists z B(x, z) \leftrightarrow \exists z B'(s, z))$. Assume

$\mathrm{fin}_5(A)$. Note that $\mathrm{fin}_5(A)$ is equivalent to $\neg\forall m\exists s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \wedge |s| \geq m)$. Then we have the following.

$$\begin{aligned}
&\neg\forall m\exists s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \wedge |s| \geq m) \\
\leftrightarrow&\neg\forall m\exists s \in [\mathbb{N}]^{<\mathbb{N}}(\exists z B'(s,z) \wedge |s| \geq m) \\
\leftrightarrow&\neg\forall m\exists s \in [\mathbb{N}]^{<\mathbb{N}}\exists z(B'(s,z) \wedge |s| \geq m) \\
\leftrightarrow&\neg\forall m\neg\neg\exists s \in [\mathbb{N}]^{<\mathbb{N}}\exists z(B'(s,z) \wedge |s| \geq m) && \text{(by } \exists\Gamma\text{-DNS)} \\
\leftrightarrow&\neg\neg\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}\neg(\exists z B'(s,z) \wedge |s| \geq m) \\
\leftrightarrow&\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}\neg(\exists z B'(s,z) \wedge |s| \geq m) && \text{(by } \exists\forall\neg\Gamma\text{-DNE)} \\
\leftrightarrow&\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}(\exists z B'(s,z) \rightarrow |s| < m)
\end{aligned}$$

The last line is equivalent to $\mathrm{fin}_3(A)$. □

Lemma 17. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \exists\forall\Delta(\Gamma)$-DNE *proves* $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\exists\Delta(\Gamma))$.

Proof. Let $\mathbf{T} = \mathbf{i\Sigma}_1 + \exists\forall\Delta(\Gamma)$-DNE. Note that $\exists\forall\Delta(\Gamma)$-DNE implies $\exists\forall\neg\Delta(\Gamma)$-DNE and $\exists\Delta(\Gamma)$-DNS, by Lemma 4.3, and that $\Delta(\Gamma)$ is closed under bounded universal quantifier. Then $\exists\Delta(\Gamma)$ is closed under bounded universal quantifier in $\mathbf{T}$ by Corollary 1. By Lemma 16, $\mathbf{T}$ proves $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\Delta(\Gamma))$. □

Lemma 18. *If* $\mathbf{T}$ *is a theory, and* Ψ *and* Ψ' *are classes of formulae such that*

1. *for each formula* B *in* Ψ*, we can find a formula* C *in* Ψ' *such that* $\mathbf{T} \vdash \neg B \leftrightarrow C$*;*
2. $\mathbf{T}$ *proves* $\exists\Psi'$-IND*,* Ψ-DNE*,* $\exists\Psi'$-GDM *and* $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\Psi')$*;*
3. Ψ' *is closed under uniformization and bounded universal quantifier in* $\mathbf{T}$*.*

then $\mathbf{T}$ *proves* $\exists\forall\Psi$-DNE*.*

Proof. Assume that $\mathbf{T}$, Ψ and Ψ' satisfy the conditions in the premise. For a formula $B(y,z)$ in Ψ, suppose $\neg\neg\exists y\forall z B(y,z)$. Since $\neg B(x,y)$ is equivalent to some formula in Ψ', and since Ψ' is close under uniformization, there is a formula $C(x,y)$ in Ψ' such that

$$\forall x(\exists y\neg B(x,y) \leftrightarrow \exists y C(x,y)) \qquad \text{and} \qquad \forall x\forall y\forall y'(C(x,y) \wedge C(x,y) \rightarrow y = y').$$

Let $A(x) \equiv x \in \mathbb{N}^{<\mathbb{N}} \wedge (\forall i < |x|)C(i, x(i))$. Then we have $\mathrm{fin}_4(A)$ as in the proof of Lemma 14 and so $\mathrm{fin}_5(A)$ by Lemma 8.4.

By $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\Psi')$, we have $\mathrm{fin}_3(A)$. Take m such that $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \rightarrow |s| < m)$. If $\forall i < m\exists y C(i,y)$, then $\exists\Psi'$-IND on l yields $\forall l \leq m\exists x(A(x) \wedge |x| = l)$. Then A has at least $m+1$ elements, contradicting to $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \rightarrow |s| < m)$. Therefore $\neg(\forall i < m)\exists z\neg B(i,z)$. By $\exists\Psi'$-GDM, we have $(\exists i < m)\forall z\neg\neg B(i,z)$, and so $\exists y\forall z B(y,z)$ by Ψ-DNE. □

Lemma 19. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND$+\Gamma$-LEM$+\exists\Delta(\Gamma)$-GDM$+(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\Delta(\Gamma))$ *proves* $\exists\forall\Delta(\Gamma)$-DNE.

Proof. Let $\mathbf{T} = \mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)\text{-IND} + \Delta(\Gamma)\text{-DNE} + \exists\Delta(\Gamma)\text{-GDM} + (\text{fin}_5 \to \text{fin}_3)(\Delta(\Gamma))$. Note that $\neg\Delta(\Gamma) \subseteq \Delta(\Gamma)$ and $\mathbf{T}$ proves $\Delta(\Gamma)$-LEM, by Lemma 4.2, and $\Delta(\Gamma)$-LNP by Corollary 2.1. Hence $\Delta(\Gamma)$ is closed under uniformization, by Lemma 12. The class $\Delta(\Gamma)$ is closed under bounded universal quantifier in $\mathbf{T}$. Hence $\mathbf{T}$ proves $\exists\forall\Delta(\Gamma)$-DNE by Lemma 18. By Lemma 2.1., $\mathbf{T}$ is equivalent to $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)\text{-IND} + \Gamma\text{-LEM} + \exists\Delta(\Gamma)\text{-GDM} + (\text{fin}_5 \to \text{fin}_3)(\Delta(\Gamma))$. □

Theorem 4. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)\text{-IND}$*:*

1. $\exists\forall\Delta(\Gamma)$-DNE*;*
2. $(\text{fin}_5 \to \text{fin}_3)(\exists\Gamma) + \Delta(\Gamma)\text{-LEM} + \exists\Delta(\Gamma)\text{-GDM}$.
3. $(\text{fin}_5 \to \text{fin}_3)(\Gamma) + \Delta(\Gamma)\text{-LEM} + \exists\Delta(\Gamma)\text{-GDM}$.

Proof. $\mathbf{i\Sigma}_1 + \exists\forall\Delta(\Gamma)$-DNE proves $\exists\Delta(\Gamma)$-GDM as follows: Let $B(i)$ be a $\exists\Delta(\Gamma)$ formula. We can write $B(i)$ as $\exists z C(i,z)$ for some $\Delta(\Gamma)$ formula. Then $\neg(\forall i < y)B(i)$ is equivalent to $\neg(\forall i < y)\neg\neg\exists z C(i,z)$, $\neg\neg(\exists i < y)\forall z \neg C(i,z)$, and to $\exists i < y \neg B(i)$ by $\exists\forall\Delta(\Gamma)$-DNE. Hence "1⇒2" holds, by Lemma 17. "2⇒3" is trivial. "3⇒1" follows from Lemma 19. □

Lemma 20. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. $\mathbf{T}$ *proves* $\exists\Gamma$-DNS *and* $\neg\neg\exists\Gamma$-IND*;*
2. Γ *is closed under bounded universal quantifier in* $\mathbf{T}$*,*

then $\mathbf{T}$ *proves* $(\text{fin}_5 \to \text{fin}_4)(\exists\Gamma)$*.*

Proof. Assume that $\mathbf{T}$ and Γ satisfies the conditions in the premise. Let $A(x)$ be a $\exists\Gamma$ formula. Assume $\text{fin}_5(A)$ and $\forall y \exists x > y A(x)$, and set

$$B(z,s) \equiv s \in [\mathbb{N}]^{<\mathbb{N}} \wedge |s| = z \wedge (\forall i < |s|)\neg\neg A(s(i))$$

By Lemma 3.2, $\neg\neg\exists\Gamma$ is closed under bounded universal quantifier in $\mathbf{T}$. Hence $\neg\neg B(z,s)$ is equivalent to $\neg\neg\exists w C(z,s,w)$ for some Γ formula $C(z,s,w)$. Then $\neg\neg\exists s B(z,s)$ is still equivalent to a $\neg\neg\exists\Gamma$ formula, since it is equivalent to $\neg\neg\exists s \neg\neg\exists w C(z,s,w)$ and $\neg\neg\exists s \exists w C(z,s,w)$. By $B(0, \langle\rangle)$, we have $\neg\neg\exists s B(0,s)$. If $\exists s B(z,s)$, then we have $B(z+1, t * \langle v \rangle)$ for t such that $B(z,t)$ and v such that $v > t \wedge A(t)$, and so $\exists s B(z+1,s)$. Hence we have $\forall z(\exists s B(z,s) \to \exists s B(z+1,s))$. By $\neg\neg\exists\Gamma$-IND, we have $\forall z \neg\neg\exists s B(z,s)$. Then we have

$$\begin{aligned}
\forall z \neg\neg\exists s B(z,s) &\leftrightarrow \forall z \neg\neg\exists s(s \in [\mathbb{N}]^{<\mathbb{N}} \wedge |s| = z \wedge (\forall i < |s|)\neg\neg A(s(i))) \\
&\leftrightarrow \forall z \neg\neg\exists s \neg\neg(s \in [\mathbb{N}]^{<\mathbb{N}} \wedge |s| = z \wedge (\forall i < |s|)A(s(i))) \quad \text{(by } \exists\Gamma\text{-DNS)} \\
&\leftrightarrow \forall z \neg\neg\exists s(s \in [\mathbb{N}]^{<\mathbb{N}} \wedge |s| = z \wedge (\forall i < |s|)A(s(i))).
\end{aligned}$$

By $\exists\Gamma$-DNS, we have $\neg\neg\forall z \exists s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \wedge |s| = z)$, which contradicts to $\text{fin}_5(A)$. Therefore we have $\text{fin}_4(A)$, i.e., $\neg\forall y \exists x > y A(x)$. □

A class Γ is *weakly closed under uniformization* in $\mathbf{T}$ if for each formula $C(x,\vec{y})$ in Γ such that all free variables are displayed, there is a formula $C'(x,\vec{y})$ in Γ with the same free variables as $C(x,\vec{y})$ such that $\mathbf{T}$ proves

$$\forall\vec{y}(\neg\neg\exists x C(x,\vec{y}) \leftrightarrow \neg\neg\exists x C'(x,\vec{y})), \quad \text{and} \quad \forall\vec{y}(\forall x\forall x'(C'(x,\vec{y}) \wedge C'(x',\vec{y}) \to x = x')).$$

A class Γ is *weakly closed under bounded existential quantifier in* $\mathbf{T}$ if, for each Γ formula $A(\vec{y})$ such that all free variables are displayed, there exists a Γ formula $B(\vec{y},z)$ such that $\mathbf{T} \vdash \forall\vec{y}\forall z(\neg\neg(\exists x < z)A(\vec{y}) \leftrightarrow \neg\neg B(\vec{y},z))$.

Lemma 21. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. $\mathbf{T}$ *proves* $\exists\Gamma$-DNS, $\neg\neg\Gamma$-IND *and* $(\mathrm{fin}_5{\to}\mathrm{fin}_4)(\Gamma)$*;*
2. Γ *is weakly closed under uniformization and bounded existential quantifier in* $\mathbf{T}$*,*

then $\mathbf{T}$ *proves* $\neg\neg\exists\Gamma$-IND.

Proof. Assume that $\mathbf{T}$ and Γ satisfies the conditions in the premise. Let $B(x)$ be a $\exists\Gamma$ formula. We can write $B(x)$ as $\exists z C(x,z)$ where $C(x,z)$ is Γ. Assume $\neg\neg\exists z C(0,z)$ and $\forall x(\neg\neg\exists z C(x,z) \to \neg\neg\exists z C(x+1,z))$. Since Γ is weakly closed under uniformization in $\mathbf{T}$, we have a formula $C'(x,z)$ in Γ satisfying the following:

$$\forall x(\neg\neg\exists z C(x,z) \leftrightarrow \neg\neg\exists z C'(x,z)) \quad \text{and} \quad \forall x\forall z\forall z'(C(x,z) \wedge C(x,z') \to z = z'). \tag{4}$$

For a given m, let $A(x) \equiv C'((x)_0,(x)_1) \wedge (x)_0 \leq m$. Then we have $\mathrm{fin}_5(A)$ and so $\mathrm{fin}_4(A)$ by $(\mathrm{fin}_5{\to}\mathrm{fin}_4)(\Gamma)$.

Assume $\forall x(A(x) \to x < y)$ for some y. For a given m, $\neg\neg(x \leq m \to \exists z < yA((x,z)))$ is equivalent to $x \leq m \to \neg\neg\exists z < yA((x,z))$, which is still equivalent to some $\neg\neg\Gamma$ formula, since Γ is weakly closed under bounded existential quantifier. By $\neg\neg\exists z C(0,z)$, $\neg\neg\exists z < yA((0,z))$. If $\exists z < yA((x,z))$ and $x+1 \leq m$, then $\exists z C'(x,z)$, and so $\exists z C(x,z)$, which implies $\neg\neg\exists z C(x+1,z)$ and $\neg\neg\exists z < yA((x+1,z))$. By $\neg\neg$-IND, we have $\forall x \leq m\neg\neg\exists z < yA((x,z))$. In particular, we have $\neg\neg\exists z < yA(m,z)$. Therefore the following hold.

$$\exists y\forall x(A(x) \to x < y) \to \exists z A((m,z)). \tag{5}$$

By $\mathrm{fin}_4(A)$, we have $\neg\forall y\exists x \geq yA(x)$. Then, by $\exists\Gamma$-DNS, we have the following

$$\begin{aligned} \neg\forall y\exists x \geq yA(x) &\leftrightarrow \neg\forall y\neg\neg\exists x(A(x) \wedge x \geq y) && \text{(by } \exists\Gamma\text{-DNS)} \\ &\leftrightarrow \neg\forall y\neg\forall x(A(x) \to x < y) \\ &\leftrightarrow \neg\neg\exists y\forall x(A(x) \to x < y). \end{aligned}$$

By (5), we have $\neg\neg\exists z A(m,z)$, which implies $\neg\neg\exists z C(m,z)$. □

Lemma 22. *If* $\mathbf{T}$ *proves* $\Delta(\Gamma)$-WLNP*, then* $\Delta(\Gamma)$ *is weakly closed under uniformization in* $\mathbf{T}$.

Proof. For each $\Delta(\Gamma)$ formula $C(x,\vec{v})$ such that all free variables are displayed, let $C'(x,\vec{v}) \equiv C(x,\vec{v}) \wedge (\forall y < x)\neg C(y,\vec{v})$. Then, by $\Delta(\Gamma)$-WLNP, we have the following:

$$\forall \vec{v}(\neg\neg\exists x C(x,\vec{v}) \leftrightarrow \neg\neg\exists x C'(x,\vec{v})) \quad \text{and} \quad \forall \vec{v}(\forall x \forall x'(C(x,\vec{v}) \wedge C(x',\vec{v}) \rightarrow x = x')).$$

□

Theorem 5. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Gamma)\text{-IND} + \exists\Delta(\Gamma)\text{-DNS}$.

1. $\neg\neg\exists\Delta(\Gamma)$-IND;
2. $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\exists\Delta(\Gamma))$;
3. $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\Delta(\Gamma))$.

Proof. First, note that $\Delta(\Gamma)$ is closed under bounded quantifiers. "1⇒2" follows from Lemma 20. "2⇒3" is trivial. By Corollary 2.2 and Lemma 22, $\Delta(\Gamma)$ is closed under uniformization in $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Gamma)\text{-IND} + \exists\Delta(\Gamma)\text{-DNS} + (\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\Delta(\Gamma))$. Then "3⇒1" follows from Lemma 21. □

5 "Infinite" set of natural numbers

In this section, we consider equivalences among notions of "infiniteness" of the sets of natural numbers, induction principles and semi-constructive principles.

We can formalize INF1-INF5 in the introduction as follows:

1. $\mathrm{inf}_1(A) \equiv \neg\exists s \in [\mathbb{N}]^{<\mathbb{N}}(A = s)$;
2. $\mathrm{inf}_2(A) \equiv \neg\exists y \forall x(x \in A \rightarrow x < y)$;
3. $\mathrm{inf}_3(A) \equiv \neg\exists m \forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \rightarrow |s| < m)$;
4. $\mathrm{inf}_4(A) \equiv \forall y(\exists x > y)A(x)$;
5. $\mathrm{inf}_5(A) \equiv \forall m \exists s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \wedge |s| = m)$.

For a class Γ of formulae and $*, - \in \{1, ..., 5\}$, $(\mathrm{inf}_* \rightarrow \mathrm{inf}_-)(\Gamma)$ is the following schema:

$$\mathrm{inf}_*(A) \rightarrow \mathrm{inf}_-(A), \text{ for any } \Gamma \text{ formula } A.$$

The following is easy to prove.

Lemma 23. $\mathbf{i\Sigma}_1$ *proves the following:*

1. $(\mathrm{inf}_2 \rightarrow \mathrm{inf}_1)(\Gamma)$;
2. $(\mathrm{inf}_3 \rightarrow \mathrm{inf}_2)(\Gamma)$;

3. $(\mathrm{inf}_4 \to \mathrm{inf}_3)(\Gamma)$*;*

4. $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Gamma)$.

Lemma 24. *The following are equivalent over* $\mathbf{i\Sigma}_1$*:*

1. Γ-WBCA*;*

2. $(\mathrm{inf}_1 \to \mathrm{inf}_2)(\Gamma)$.

Proof. First we prove 1⇒2. Let A be a Γ formula. For a given m, if $(\forall i < m)(s(i) = 0 \leftrightarrow A(i))$ for some $s \in \{0,1\}^m$ and if $\forall x(A(x) \to x < m)$, then $t \in [\mathbb{N}]^{<\mathbb{N}}$ defined by $\forall x(x \in t \leftrightarrow x < m \wedge s(x) = 0)$ satisfies $A = t$. Therefore we have the following:

$$\forall m(\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow A(i)) \to (\forall x(A(x) \to x < m) \to \mathrm{fin}_1(A))).$$

Since $(P \to Q)$ implies $\neg\neg Q \to \neg\neg P$ and since $\neg\neg(R \to S)$ is equivalent to $\neg S \to \neg R$ in intuitionistic logic, we have

$$\forall m(\neg\neg\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow A(i)) \to (\neg\mathrm{fin}_1(A) \to \neg\forall x(A(x) \to x < m))).$$

Since $\forall x(U \to V)$ implies $\forall x U \to \forall x V$, the above implies

$$\forall m\neg\neg\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow A(i)) \to (\neg\mathrm{fin}_1(A) \to \forall l\neg\forall x(A(x) \to x < l))$$

Since $\mathrm{inf}_1(A) \equiv \neg\mathrm{fin}_1(A)$ and $\mathrm{inf}_2(A) \equiv \neg\mathrm{fin}_2(A)$, the above shows Γ-WBCA implies $\mathrm{inf}_1(A) \to \mathrm{inf}_2(A)$.

Next we prove 2⇒1. Let $B(x)$ be a Γ formula. Then, for a given m, $A(x) \equiv x < m \wedge B(x)$ is equivalent to a Γ formula. It is clear that $\forall x(x \in A \to x < m)$, and so $\mathrm{fin}_2(A)$. If $\mathrm{fin}_2(A) \to \mathrm{fin}_1(A)$, then $\mathrm{fin}_1(A)$ and so there is $t \in [\mathbb{N}]^{<\mathbb{N}}$such that $A = t$. Then $s \in \{0,1\}^m$ defined by $(\forall i < m)(s(i) = 0 \leftrightarrow (\exists j < |t|)(t(j) = i))$ satisfies $(\forall i < m)(s(i) \leftrightarrow B(i))$. Therefore we have

$$(\mathrm{fin}_2(A) \to \mathrm{fin}_1(A)) \to (\exists s \in \{0,1\}^m)(\forall i < m)(s(i) = 0 \leftrightarrow B(i)).$$

Since $\neg\neg\mathrm{fin}_2(A) \to \neg\neg\mathrm{fin}_1(A)$ is equivalent to $\mathrm{inf}_1(A) \to \mathrm{inf}_2(A)$, we have

$$(\mathrm{inf}_1(A) \to \mathrm{inf}_2(A)) \to \neg\neg(\exists s \in \{0,1\}^m)(\forall i < m)(s(i) = 0 \leftrightarrow B(i)).$$

Therefore $(\mathrm{fin}_1 \to \mathrm{fin}_2)(\Gamma)$ implies Γ-WBCA. □

Lemma 25. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. $\mathbf{T}$ *proves* $\neg\neg\exists\Gamma$-IND*;*

2. Γ *is weakly closed under uniformization and closed under bounded universal quantifier in* $\mathbf{T}$*,* *then* $\mathbf{T}$ *proves* $(\text{inf}_2 \to \text{inf}_3)(\exists\Gamma)$.

Proof. Assume that $\mathbf{T}$ and Γ satisfy the conditions in the premise. Let $A(x) \equiv \exists y B(x,y)$ be a $\exists\Gamma$ formula, where $B(x,y)$ is Γ. Since Γ is weakly closed under uniformization in $\mathbf{T}$, there is $B'(x,y)$ in Γ such that, for any x,

$$\neg\neg\exists y B(x,y) \leftrightarrow \neg\neg\exists y B'(x,y), \qquad \forall y \forall y'(B'(x,y) \land B'(x,y') \to y = y').$$

Let $A'(x) \equiv B'((x)_0, (x)_1)$. Then $\neg\exists y A'((x,y)) \to \neg A(x)$.

If $\forall x(A'(x) \to x < z)$ for some z, then $\forall v > z \neg\exists y A'((v,y))$ and so $\forall v > z \neg A(v)$. Therefore we have $\text{fin}_2(A') \to \text{fin}_2(A)$.

Assume $\text{inf}_2(A)$. Since $\text{inf}_2(A) \equiv \neg\text{fin}_2(A)$, we have $\text{inf}_2(A')$. Let $C(m,s) \equiv s \in [\mathbb{N}]^{<\mathbb{N}} \land s \subseteq A' \land |s| \geq m$. Since Γ is closed under bounded universal quantifier, $\neg\neg\exists s C(m,s)$ is equivalent to some formula in $\neg\neg\exists\Gamma$. If $C(m,t)$ for some $t \in [\mathbb{N}]^{<\mathbb{N}}$ and $\neg\exists s C(m+1,s)$, then we have $\neg\exists x(A'(x) \land x \geq t)$ and so $\forall x(A'(x) \to x < t)$, which contradicts to $\text{inf}_2(A')$. Hence $\exists s C(m,s) \to \neg\neg\exists s C(m+1,s)$. Since we have $C(0,\langle\rangle)$ and so $\neg\neg\exists s C(0,s)$, by $\neg\neg\exists\Gamma$-IND, we have $\forall m \neg\neg\exists s C(m,s)$, and so $\text{inf}_3(A')$.

Note that $A'((u,v)) \land A'((u,v')) \to v = v'$ and $\exists v A'((u,v)) \to A(u)$ for each u. Then $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m)$ implies $\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A' \to |s| < m)$. Therefore we have $\text{fin}_3(A) \to \text{fin}_3(A')$. Since $\text{inf}_3(A') \equiv \neg\text{fin}_3(A')$, we have $\text{inf}_3(A)$. □

Lemma 26. $\mathbf{i\Sigma}_1 + \neg\neg\exists\Delta(\Gamma)$-IND *proves* $(\text{inf}_2 \to \text{inf}_3)(\exists\Delta(\Gamma))$.

Proof. Note that $\Delta(\Gamma)$ is closed under bounded universal quantifier in $\mathbf{i\Sigma}_1$ and weakly closed under uniformization in $\mathbf{i\Sigma}_1 + \neg\neg\exists\Delta(\Gamma)$-IND by Corollary 2.2 and Lemma 22. Then, by Lemma 25, $(\text{inf}_2 \to \text{inf}_3)(\exists\Delta(\Gamma))$ holds in $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND. □

Lemma 27. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formulae such that*

1. Γ *is weakly closed under uniformization and bounded existential quantifier in* $\mathbf{T}$*;*
2. Γ *is closed under bounded universal quantifier in* $\mathbf{T}$*;*
3. $\mathbf{T}$ *proves* $\neg\neg\Gamma$-IND *and* $(\text{inf}_2 \to \text{inf}_3)(\Gamma)$,

then $\mathbf{T}$ *proves* $\neg\neg\exists\Gamma$-IND.

Proof. Assume that $\mathbf{T}$ and Γ satisfy the conditions in the premise.

Suppose $\neg\neg\exists z B(0,z)$ and $\forall y(\neg\neg\exists z B(y,z) \to \neg\neg\exists z B(y+1,z))$ for a formula $B(y,z)$ in Γ. Since Γ is weakly closed under uniformization in $\mathbf{T}$, there is $B'(x,y)$ in Γ such that

$$\forall y(\neg\neg\exists z B(y,z) \leftrightarrow \neg\neg\exists z B'(y,z)) \qquad \text{and} \qquad \forall z \forall z'(B'(y,z) \land B'(y,z) \to z = z').$$

Set $A(x) \equiv x \in \mathbb{N}^{<\mathbb{N}} \land (\forall i < |x|) B'(i, x(i))$. Since Γ is closed under bounded universal quantifier, $A(x)$ is equivalent to a formula in Γ.

First, we show $\text{inf}_2(A)$. For contradiction, assume that $\forall x(x \in A \to x < y)$ for some y. Let $C(m) \equiv \exists x < y(A(x) \land |x| = m+1)$. Since Γ is weakly closed under bounded existential quantifier, we have a formula in $\neg\neg\Gamma$ equivalent to $\neg\neg C(m)$. Since $\neg\neg\exists z B(0,z)$, we have $\neg\neg\exists x(A(x) \wedge |x| = 1)$. By the assumption $\forall x(x \in A \to x < y)$, we have $\neg\neg C(0)$. If $C(m)$, then $\neg\neg\exists z B'(m,z)$. This implies $\neg\neg\exists z B(m,z)$, and so $\neg\neg\exists z B(m+1,z)$. Then we have $\neg\neg\exists z B'(m+1,z)$, which implies $\neg\neg\exists x(A(x) \wedge |x| = m+2)$, i.e., $\neg\neg C(m+1)$. Therefore we have $\neg\neg C(m) \to \neg\neg C(m+1)$. By $\neg\neg\Gamma$-IND, we have $\forall m \neg\neg C(m)$. This contradicts to $\forall x(x \in A \to x < y)$, which implies $\neg(\exists x < y)(|x| = y)$. Therefore we have $\neg\exists y \forall x(x \in A \to x < y)$ and so $\mathrm{inf}_2(A)$.

By $\mathrm{inf}_2(A) \to \mathrm{inf}_3(A)$, we have $\mathrm{inf}_3(A)$. Assume $\neg\exists z B(m,z)$ for some m. Then $\neg\exists x(A(x) \wedge |x| = m+1)$. This implies $(\forall l \geq m+1)\neg\exists x(A(x) \wedge |x| = l)$ and so $\forall x(A(x) \to |x| < m+1)$. Since $A(x) \wedge A(x') \wedge |x| = |x'|$ implies $x = x'$, then A satisfies $\forall s \in \mathbb{N}^{<\mathbb{N}}(s \subseteq A \to |s| < m)$, which contradicts to $\mathrm{inf}_3(A)$. Therefore we have $\forall x \neg\neg\exists z B(x,z)$. □

Lemma 28. $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Gamma)$-IND $+ (\mathrm{inf}_2 \to \mathrm{inf}_3)(\Delta(\Gamma))$ *proves* $\neg\neg\exists\Delta(\Gamma)$-IND.

Proof. By Lemma 22, the class $\Delta(\Gamma)$ is weakly closed under uniformization in $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Gamma)$-IND$+$ $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Delta(\Gamma))$. Note that $\Delta(\Gamma)$ is closed under bounded universal and existential quantifiers in $\mathbf{i\Sigma}_1$. Therefore $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Gamma)$-IND $+ (\mathrm{inf}_2 \to \mathrm{inf}_3)(\Delta(\Gamma))$ proves $\neg\neg\exists\Delta(\Gamma)$-IND by Lemma 27. □

Theorem 6. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

1. $\neg\neg\exists\Delta(\Gamma)$-IND;
2. $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\exists\Delta(\Gamma)) + \neg\neg\Delta(\Gamma)$-IND;
3. $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Delta(\Gamma)) + \neg\neg\Delta(\Gamma)$-IND.

Lemma 29. *If* $\mathbf{T}$ *proves* $\exists\Gamma$-DNE*, then* $\mathbf{T}$ *proves* $(\mathrm{inf}_2 \to \mathrm{inf}_4)(\exists\Gamma)$.

Proof. Assume $\mathbf{T}$ satisfies the condition in the premise. Let $A(x) \equiv \exists z B(x,z)$ is a $\exists\Gamma$ formula, where $B(x,y)$ is Γ. Assume $\mathrm{inf}_2(A)$. Then we have the following:

$$\begin{aligned}
\neg\exists y \forall x(x \in A \to x < y) &\leftrightarrow \neg\exists y \forall x(\exists z B(x,z) \to x < y)\\
&\leftrightarrow \neg\exists y \forall x \forall z(B(x,z) \to x < y)\\
&\leftrightarrow \neg\exists y \forall x \forall z \neg(B(x,z) \wedge x \geq y)\\
&\leftrightarrow \forall y \neg\neg\exists x \exists z(B(x,z) \wedge x \geq y)\\
&\leftrightarrow \forall y \exists x \exists z(B(x,z) \wedge x \geq y) \qquad \text{(by } \exists\Gamma\text{-DNE)}\\
&\leftrightarrow \forall y \exists x(\exists z B(x,z) \wedge x \geq y).
\end{aligned}$$

The last is equivalent to $\mathrm{inf}_4(A)$. □

Lemma 30. $\mathbf{i\Sigma}_1 + (\mathrm{inf}_2 \to \mathrm{inf}_4)(\Gamma)$ *proves* $\exists\Gamma$-DNE.

Proof. Let $B(z)$ be a formula in Γ and assume that $\neg\neg\exists z B(z)$. Let $A(x) \equiv B((x)_0)$. Then $B(x)$ implies $A((x,y))$ for all y. If $\exists y \forall x(x \in A \to x < y)$, then $\neg\exists z B(z)$, a contradiction. Therefore $\neg\exists y \forall x(x \in A \to x < y)$. By $\mathrm{inf}_2(A) \to \mathrm{inf}_4(A)$, we have $\forall y(\exists x > y)A(x)$. Take x such that $x > 0 \wedge A(x)$. Then $B((x)_0)$ holds, and so $\exists z B(z)$. □

Theorem 7. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

1. $\exists\Delta(\Gamma)$-DNE*;*
2. $(\mathrm{inf}_2{\to}\mathrm{inf}_4)(\exists\Delta(\Gamma))$*;*
3. $(\mathrm{inf}_2{\to}\mathrm{inf}_4)(\Delta(\Gamma))$*.*

Proof. By Lemma 29, we have 1⇒2. 2⇒3 is trivial. "3⇒1" follows from Lemma 30. □

Lemma 31. *If* $\mathbf{T}$ *is a theory and* Γ *is a class of formula such that*

1. $\mathbf{T}$ *proves* $\exists\Gamma$-DNE*;*
2. $\exists\Gamma$ *is closed under bounded universal quantifier in* $\mathbf{T}$*,*

then $\mathbf{T}$ *proves* $(\mathrm{inf}_3{\to}\mathrm{inf}_5)(\exists\Gamma)$*.*

Proof. Assume that $\mathbf{T}$ and Γ satisfy the conditions in the premise. Let A be a $\exists\Gamma$ formula. We can write $A(x)$ as $\exists zB(x,z)$ where $B(x,z)$ is Γ. Since $\exists\Gamma$ is closed under bounded universal quantifier in $\mathbf{T}$, we have $B'(x,s)$ such that $\forall s((\forall x \in s)\exists zB(x,z) \leftrightarrow \exists zB'(s,z))$. Assume $\mathrm{inf}_3(A)$. Then we have the following:

$$\begin{aligned}
\neg\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}(s \subseteq A \to |s| < m) &\leftrightarrow \neg\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}(\exists zB'(z,s) \to |s| < m)\\
&\leftrightarrow \neg\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}\forall z(B'(z,s) \to |s| < m)\\
&\leftrightarrow \neg\exists m\forall s \in [\mathbb{N}]^{<\mathbb{N}}\forall z\neg(B'(z,s) \wedge |s| \geq m)\\
&\leftrightarrow \forall m\neg\neg\exists s \in [\mathbb{N}]^{<\mathbb{N}}\exists z(B'(z,s) \wedge |s| \geq m)\\
&\leftrightarrow \forall m\exists s \in [\mathbb{N}]^{<\mathbb{N}}\exists z(B'(z,s) \wedge |s| \geq m) \qquad \text{(by } \exists\Gamma\text{-DNE)}.
\end{aligned}$$

The last line is equivalent to $\mathrm{inf}_5(A)$. □

Lemma 32. $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \exists\Delta(\Gamma)$-DNE *proves* $(\mathrm{inf}_3{\to}\mathrm{inf}_5)(\exists\Delta(\Gamma))$*.*

Proof. Let $\mathbf{T} = \mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND $+ \exists\Delta(\Gamma)$-DNE. By Corollary 1, $\exists\Delta(\Gamma)$ is closed under bounded universal quantifier in $\mathbf{T}$. Hence $\mathbf{T}$ proves $(\mathrm{inf}_3{\to}\mathrm{inf}_5)(\exists\Delta(\Gamma))$ by Lemma 31. □

Lemma 33. $\mathbf{i\Sigma}_1 + (\mathrm{inf}_3{\to}\mathrm{inf}_5)(\Gamma)$ *proves* $\exists\Gamma$-DNE*.*

Proof. This can be proved in an almost same way as Lemma 30. □

Theorem 8. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \exists\Delta(\Gamma)$-IND*:*

1. $\exists\Delta(\Gamma)$-DNE*;*
2. $(\mathrm{inf}_3{\to}\mathrm{inf}_5)(\exists\Delta(\Gamma))$*;*
3. $(\mathrm{inf}_3{\to}\mathrm{inf}_5)(\Delta(\Gamma))$*.*

Proof. By Lemma 31, we have "1⇒2". "2⇒3"is trivial and "3⇒1" follows from Lemma Lemma 33. □

Lemma 34. *If* **T** *is a theory and* Γ *is a class of formulae such that*

1. **T** *proves* $\exists\Gamma$-IND*;*

2. Γ *is closed under bounded universal quantifier in* **T***,*

then **T** *proves* $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\exists\Gamma)$.

Proof. Assume that **T** and Γ satisfy the conditions in the premise. Let A be a $\exists\Gamma$ formula. Assume $\mathrm{inf}_4(A)$. Take $B(z,s)$ as in the proof of Lemma 20. Then, in a similar way to it, we can prove $\exists sB(0,s)$ and $\forall z(\exists sB(z,s) \to \exists sB(z+1,s))$. By $\exists\Gamma$-IND, we have $\forall m\exists sB(m,s)$, and so $\mathrm{inf}_5(A)$. □

Lemma 35. *If* **T** *is a theory and* Γ *is a class of formulae such that*

1. **T** *proves* $\Delta(\Gamma)$-LNP *and* $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Gamma)$*;*

2. Γ *is closed under uniformization and bounded universal quantifier in* **T***,*

then **T** *proves* $\exists\Gamma$-IND.

Proof. Assume that **T** and Γ satisfies the conditions in the premise. Let $B(x)$ be a $\exists\Gamma$ formula. We can write $B(x)$ as $\exists zC(x,z)$ where $C(x,z)$ is $\Delta(\Gamma)$. Assume $\exists zC(0,z)$ and $\forall x(\exists zC(x,z) \to \exists zC(x+1,z))$. Since Γ is closed under uniformization in **T**, we have a formula $C'(x,z)$ in Γ satisfying the following:

$$\forall x(\exists zC(x,z) \leftrightarrow \exists zC'(x,z)) \qquad \text{and} \qquad \forall x\forall z\forall z'(C(x,z) \wedge C(x,z') \to z = z'). \tag{6}$$

Let $A(x) \equiv x \in \mathbb{N}^{<\mathbb{N}} \wedge (\forall i < |x|)C'(x, x(i))$. Since Γ is closed under bounded universal quantifier, $A(x)$ is equivalent to a formula in Γ. Note that $A(x) \wedge A(x') \wedge |x| = |x'|$ implies $x = x'$ and A is closed under initial segment, i.e., $A(x) \wedge x' \in \mathbb{N}^{<\mathbb{N}} \wedge \exists i < |x|(\overline{x}i = x') \to A(x')$.

First, we prove $\forall y\exists x > yA(x)$. For a given y, take z such that $C'(0,z)$. Then $A(\langle z\rangle)$ holds. If $\langle z\rangle > y$, then we have $\exists x > yA(x)$. Otherwise, $\langle z\rangle \leq y$ and so we can take the maximal $u \leq y$ such that $A(u)$ by $\Delta(\Gamma)$-LNP. Since A is closed under initial segment, $|u| = \max\{|v| : v \leq y \wedge A(v)\}$. By the definition of A, we have $C'(|u|-1, u(|u|-1))$. By the assumption that $\forall x(B(x) \to B(x+1))$, we have z such that $C'(|u|, z)$. By the maximality of u, we have $u * \langle z\rangle > y$. Therefore, we have $\forall y\exists x > yA(x)$, and so $\mathrm{inf}_4(A)$.

By $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Gamma)$, we have $\mathrm{inf}_5(A)$. For given x, take $s \subseteq A$ such that $|s| = x+2$. Then there is $i < x+2$ such that $|s(i)| > x$, which satisfies $C'(x, (s(i))(x))$. Therefore we have $\forall xB(x)$. □

Corollary 6. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Delta(\Gamma)$-LNP*:*

1. $\exists\Delta(\Gamma)$-IND*;*

2. $(\mathrm{inf}_5 \to \mathrm{inf}_4)(\exists\Delta(\Gamma))$*;*

3. $(\mathrm{inf}_5 \to \mathrm{inf}_4)(\Delta(\Gamma))$.

6 Some special classes

6.1 Recursive sets

In [9, Ch.3.7], elementary recursion theory is formalized in **HA**. Actually this does not need full induction scheme but Σ_1 induction, which means it can be done in $\mathbf{i\Sigma}_1$. In particular, we have Kleene's T predicate, the result extracting function U both in Δ_0 and their basic properties in $\mathbf{i\Sigma}_1$. The following is an example of them.

Lemma 36. $\mathbf{i\Sigma}_1$ *proves the following:*

$$\forall e \forall x \forall y \forall y' (Texy \wedge Texy' \rightarrow y = y'). \tag{7}$$

Proof. See [9, Ch.3.3.6 Theorem]. □

Then we can treat recursive sets in our language. For a formula A, $\mathrm{Rec}(A)$ is an abbreviation of $\exists e[\forall x(\exists y Texy) \wedge \forall x(\exists y(Texy \wedge U(y) = 0) \leftrightarrow A(x))]$. The intuitive meaning of $\mathrm{Rec}(A)$ is "the set $\{x \in \mathbb{N} : A(x)\}$ is recursive".

Remark 4. For any formula A, there is a Σ_1 formula $B(x)$ such that

$$\mathbf{i\Sigma}_1 \vdash \mathrm{Rec}(A) \rightarrow \forall x(A(x) \leftrightarrow B(x)).$$

The following shows that we have the law of the excluded middle, the least number principle and bounded comprehension for a formula with $\mathrm{Rec}(A)$ in $\mathbf{i\Sigma}_1$.

Theorem 9. $\mathbf{i\Sigma}_1$ *proves the following:*

1. $\mathrm{Rec}(A) \rightarrow A(x) \vee \neg A(x)$;
2. $\mathrm{Rec}(A) \rightarrow (A(x) \rightarrow (\exists y \leq x)(A(y) \wedge (\forall z < y)\neg A(z)))$;
3. $\mathrm{Rec}(A) \rightarrow \forall m \exists s \in \{0,1\}^m (\forall i < m)(s(i) = 0 \leftrightarrow A(i))$.

Proof. Assume $\mathrm{Rec}(A)$, and let e be such that $\forall x \exists y Texy \wedge \forall x(\exists y(Texy \wedge U(y) = 0) \leftrightarrow A(x))$.

For 1, we have $(U(y) = 0 \rightarrow A(x)) \wedge (U(y) \neq 0 \rightarrow \neg A(x))$ for y such that $Texy$ by Lemma 36.

For 2 and 3, note that $\exists t(t \in \mathbb{N}^{<\mathbb{N}} \wedge |t| = x \wedge (\forall y < x)Teyt(y)) \leftrightarrow (\forall y < x)\exists z Teyz$ for any x by the proof of Lemma 3.1. Since $\forall x \exists z Texz$, we have $\forall x \exists t(\forall y < x)Teyt(y)$.

For 2, assume $A(x)$. Let t be such that $|t| = x + 1 \wedge (\forall y < x + 1)Teyt(y)$. By Lemma 36, we have $U(t(x)) = 0$. We can take the least $y \leq x$ such that $U(t(y)) = 0$, which satisfies $y \leq x \wedge A(y) \wedge (\forall z < y)\neg A(z)$ by Lemma 36.

For 3, let t be such that $|t| = m \wedge (\forall i < m)Teit(i)$. Then $s \in \{0,1\}^m$ defined by $(\forall i < m)(s(i) = 0 \leftrightarrow U(t(i)) = 0)$ satisfies $(\forall i < m)(s(i) = 0 \leftrightarrow A(i))$ by Lemma 36. □

Theorem 10. *1.* $\mathbf{i\Sigma}_1$ *proves the following:*

(a) $\mathrm{Rec}(A) \rightarrow (\mathrm{fin}_1(A) \leftrightarrow \mathrm{fin}_2(A))$ *for any formula A;*

(b) $\mathrm{Rec}(A) \to (\mathrm{fin}_2(A) \to \mathrm{fin}_3(A))$ *for any formula A;*

(c) $\mathrm{Rec}(A) \to (\mathrm{fin}_3(A) \to \mathrm{fin}_4(A))$ *for any formula A;*

(d) $\mathrm{Rec}(A) \to (\mathrm{fin}_4(A) \leftrightarrow \mathrm{fin}_5(A))$ *for any formula A.*

2. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$

   (a) Σ_1-BCA*;*

   (b) $\mathrm{Rec}(A) \to (\mathrm{fin}_3(A) \to \mathrm{fin}_1(A))$ *for any formula A;*

   (c) $\mathrm{Rec}(A) \to (\mathrm{fin}_3(A) \to \mathrm{fin}_2(A))$ *for any formula A.*

3. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) Σ_2-DNE*;*

   (b) $\mathrm{Rec}(A) \to (\mathrm{fin}_4(A) \to \mathrm{fin}_2(A))$*;*

   (c) $\mathrm{Rec}(A) \to (\mathrm{fin}_5(A) \to \mathrm{fin}_3(A)) + \Sigma_1$-GDM.

4. $\mathbf{i\Sigma}_1$ *proves the following:*

   (a) $\mathrm{Rec}(A) \to (\mathrm{inf}_1(A) \leftrightarrow \mathrm{inf}_2(A))$*, for any formula A;*

   (b) $\mathrm{Rec}(A) \to (\mathrm{inf}_3(A) \to \mathrm{inf}_2(A))$*, for any formula A;*

   (c) $\mathrm{Rec}(A) \to (\mathrm{inf}_4(A) \to \mathrm{inf}_3(A))$*, for any formula A;*

   (d) $\mathrm{Rec}(A) \to (\mathrm{inf}_4(A) \leftrightarrow \mathrm{inf}_5(A))$*, for any formula A.*

5. *The following are equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) $\neg\neg\Sigma_1$-IND*;*

   (b) $\mathrm{Rec}(A) \to (\mathrm{inf}_2(A) \to \mathrm{inf}_3(A))$*, for any formula A.*

6. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) Σ_1-DNE*;*

   (b) $\mathrm{Rec}(A) \to (\mathrm{inf}_3(A) \to \mathrm{inf}_4(A))$*, for any formula A;*

   (c) $\mathrm{Rec}(A) \to (\mathrm{inf}_2(A) \to \mathrm{inf}_5(A))$*, for any formula A.*

Proof. 1. Since the rest follows from Lemma 8 and Lemma 5, it is enough to show that $\mathrm{Rec}(A) \to (\mathrm{fin}_2(A) \to \mathrm{fin}_1(A))$ in (a). Assume $\mathrm{Rec}(A)$ and $\mathrm{fin}_2(A)$. Then there are e and z such that $\forall x(x \in A \to x < z)$ and $\forall x \exists y Texy \wedge \forall x(\exists y(Texy \wedge U(y) = 0) \leftrightarrow A(x))$. As in the proof of Lemma 3.1, there exists t such that $t \in \mathbb{N}^{<\mathbb{N}} \wedge |t| = z \wedge (\forall x < z) Text(x)$. By Lemma 36, we have $A = s$ for $s \in [\mathbb{N}]^{<\mathbb{N}}$ defined by $\forall x(x \in s \leftrightarrow x < z \wedge U(t(x)) = 0)$.

2. Let Γ be the class of all atomic formulae. Then $\Delta(\Gamma)$ is Π_0 and $\exists\Delta(\Gamma)$ is Σ_1. If $\mathrm{Rec}(A)$, then A is equivalent to some formula in Σ_1. Therefore "(a)⇒(b)" holds by Corollary 5. It is clear that "(b)⇒(c)" by 1. For "(c)⇒(a)", note that $\mathrm{Rec}(A)$ holds for any Δ_0 formula A and Δ_0-BCA holds in $\mathbf{i\Sigma}_1$. Then this follows from Corollary 5.

3. Note that Δ_0-DNE and Δ_0-LEM hold in $\mathbf{i\Sigma}_1$. Then, similarly to the above, this follows from Theorem 3 and Theorem 4 by taking Γ as the class of atomic formulae.

4. This follows from Lemma 23, Lemma 6 and 1.

5. This is provable in a similar way to 2, using Theorem 6, by taking Γ as the class of atomic formulae.

6. This follows from Theorem 7 and Theorem 8 by taking Γ as the class of atomic formulae. $\square$

6.2 Σ_n and Π_n sets

First, we show that Corollary 2 can be enhanced more for the classes Σ_{n+1} or Π_{n+1}.

Lemma 37. *For $\Gamma \equiv \Sigma_n$ or Π_n, we have the following:*

1. Γ-BCA *is equivalent to* Γ-LNP *over* $\mathbf{i\Sigma}_1$.
2. Γ-WBCA *is equivalent to* Γ-WLNP *over* $\mathbf{i\Sigma}_1$.

Proof. 1. For $\Gamma \equiv \Sigma_{n+1}$, it is enough to show that Σ_{n+1}-LNP implies Σ_{n+1}-BCA, by Corollary 2.1. Let $B(x)$ be a Σ_{n+1} formula. We can write $B(x)$ as $\exists y C(x,y)$ where $C(x,y)$ is Π_n.

For a given m, let $D(x)$ be as follows:

$$D(x) \equiv x \in \{0,1\}^m \wedge \exists u(|u| = m \wedge (\forall i < m)(x(i) = 0 \to C(i, u(i)))).$$

Then $D(x)$ is equivalent to a Σ_{n+1} formula. Assume $D(x) \wedge (\forall y < x)\neg D(y)$. Then $(\forall i < m)(x(i) = 0 \to \exists j C(i,j))$. If $x(i) \neq 0$ and $C(i,j)$ for some $i < m$ and j, then we have $x' < x \wedge D(x)$ for $x' \in \{0,1\}^m$ defined by $x'(i) = 0$ and $x'(l) = x(l)$ for $l \neq i$, which contradicts to $D(x) \wedge (\forall y < x)\neg D(y)$. Therefore we have the following:

$$\exists x(D(x) \wedge (\forall y < x)\neg D(y)) \to \exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow B(x)). \tag{8}$$

Since $D(1^m)$, there is x such that $D(x) \wedge (\forall y < x)\neg D(y)$ by Σ_{n+1}-LNP. Therefore we have $\exists s \in \{0,1\}^m(\forall i < m)(s(i) = 0 \leftrightarrow B(x))$.

For $\Gamma \equiv \Pi_{n+1}$, we can be proved more easily. See [7, Lemma 3.2 (i) and Lemma 3.3], which is proved in the same way as [5, Ch.I.2.13 Lemma].

2. For $\Gamma \equiv \Sigma_{n+1}$, it is enough to show that Σ_{n+1}-WLNP implies Σ_n-WBCA, by Corollary 2.2. Let $B(x)$ be a Σ_n formula. We can write $B(x)$ as $\exists y C(x,y)$ where $C(x,y)$ is Π_n. For a given m, take $D(x)$ as in the proof of 1. Since $D(1^m)$ holds, $\neg\neg\exists x(D(x) \wedge (\forall y < x)\neg D(y))$ by Σ_n-WLNP. Therefore we have $\neg\neg\exists s \in \{0,1\}^m(s(i) = 0 \leftrightarrow B(i))$ by (8).

For $\Gamma \equiv \Pi_{n+1}$, we can be proved in a similar way. $\square$

Lemma 38. $\neg\neg\Sigma_{n+1}$-IND *is equivalent to* Σ_{n+1}-WLNP *over* $\mathbf{i\Sigma}_1$.

Proof. It is enough to show that $\mathbf{i\Sigma}_1 + \neg\neg\Sigma_{n+1}$-IND proves Σ_{n+1}-WLNP by Corollary 2.2 and Lemma 37.2. Let $A(x)$ be a Σ_{n+1} formula. We can write $A(x)$ as $\exists j B(x,j)$ where $B(x,j)$ is Π_n. For a given m, assume $A(m)$ and set $C(x,j)$ as follows:

$$C(x,j) \equiv |j| = x+1 \wedge (j(0))_0 = m \wedge (\forall i < x)((j(i+1))_0 < (j(i))_0 \wedge B((j(i))_0, (j(i))_1)).$$

Then $C(x,j)$ is equivalent to a Π_n formula.

Assume $\neg(\exists y \le m)(A(y) \wedge (\forall z < y)\neg A(z))$. Then we have

$$(\forall y \le m)(A(y) \to \neg\neg(\exists z < y)A(z)). \tag{9}$$

If $C(x,j)$, then $(j(x))_0 \le m$ and $A((j(x))_0)$ hold. By (9), we have $\neg\neg(\exists z < (j(x))_0)A(z)$, and so $\neg\neg\exists l C(x+1,l)$. Therefore $\neg\neg\exists j C(x,j) \to \neg\neg\exists j C(x+1,j)$. By $A(m)$, we have $\exists j C(0,j)$. Therefore we have $\forall x \neg\neg\exists j C(x,j)$ by $\neg\neg\Sigma_n$-IND on x.

Assume $C(m+1,j)$. Then, by Δ_0-IND, we can prove $(j(i+1))_0 < m-i$ for all $i \le m$ and so $(j(m+1))_0 < 0$, which is a contradiction. Hence $\neg\exists j C(m+1,j)$. This contradicts to $\forall x \neg\neg\exists j C(x,j)$.

Therefore we have $A(m) \to \neg\neg(\exists y \le m)(A(y) \wedge (\forall z < y)\neg A(z))$. □

Lemma 39. $\mathbf{i\Sigma}_1 + \Sigma_n$-IND $+ \Sigma_{n+1}$-DNE *proves* Σ_n-BCA.

Proof. See [7, Lemma 3.4]. □

Corollary 7. *1. The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

- *(a)* Σ_n-BCA*;*
- *(b)* $\Delta(\Sigma_n)$-IND $+ \Delta(\Sigma_n)$-LEM*;*
- *(c)* $\Delta(\Sigma_n)$-IND $+ \Sigma_n$-LEM*;*
- *(d)* Σ_n-IND $+ \Delta(\Sigma_n)$-DNE*;*
- *(e)* $\Delta(\Sigma_n)$-LNP*;*
- *(f)* $\Delta(\Sigma_n)$-BCA*;*
- *(g)* Σ_n-LNP.

2. The following are pairwise equivalent over $\mathbf{i\Sigma}_1$*:*

- *(a)* Σ_n-WBCA*;*
- *(b)* $\neg\neg\Delta(\Sigma_n)$-IND*;*
- *(c)* $\Delta(\Sigma_n)$-WLNP*;*
- *(d)* $\Delta(\Sigma_n)$-WBCA*;*
- *(e)* Σ_n-WLNP*;*
- *(f)* $\neg\neg\Sigma_n$-IND.

Proof. It is enough to show "(d)⇒(a)" in 1. Let $A(x)$ be a Σ_n formula. By Lemma 2.1 and Lemma 38, Σ_n-IND + $\Delta(\Sigma_n)$-DNE proves the following for a given m:

$$\neg\neg\exists s \in \{0,1\}^m (\forall i < m)(s(i) = 0 \leftrightarrow A(i))) \tag{10}$$

By Σ_1-IND, we have $\forall m \exists t(t = 1^m)$. Then $\exists s \in \{0,1\}^m (\forall i < m)(s(i) = 0 \leftrightarrow A(i))$ is equivalent to $\exists s \leq 1^m[s \in \{0,1\}^m \wedge (\forall i < m)(s(i) = 0 \leftrightarrow A(i))]$, which is in $\Delta(\Sigma_n)$. Therefore we have $\exists s \in \{0,1\}^m (\forall i < m)(s(i) = 0 \leftrightarrow A(i))$ by $\Delta(\Sigma_n)$-DNE. □

Corollary 8. *1. The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

- *(a)* Π_n-BCA*;*
- *(b)* $\Delta(\Pi_n)$-IND + $\Delta(\Pi_n)$-LEM*;*
- *(c)* $\Delta(\Pi_n)$-IND + Π_n-LEM*;*
- *(d)* $\Delta(\Pi_n)$-LNP*;*
- *(e)* $\Delta(\Pi_n)$-BCA*;*
- *(f)* Π_n-LNP*.*

2. The following are pairwise equivalent over $\mathbf{i\Sigma}_1$*:*

- *(a)* Π_n-WBCA*;*
- *(b)* $\neg\neg\Delta(\Pi_n)$-IND*;*
- *(c)* $\Delta(\Pi_n)$-WLNP*;*
- *(d)* $\Delta(\Pi_n)$-WBCA*;*
- *(e)* Π_n-WLNP*.*

Remark 5. For 2 above we can add $\neg\neg\Pi_n$-IND if $n = 1$, since $\neg\neg\Pi_1$-IND is equivalent to $\neg\neg\neg\Sigma_1$-IND and $\neg\neg\Sigma_1$-IND by Lemma 2.1, which implies $\Delta(\Pi_1)$-IND by Lemma 7.2 and Lemma 40 below. We do not know yet if we can add $\neg\neg\Pi_n$-IND for $n \neq 1$.

Lemma 40. *1. For each* Π_{n+1} *formula* A*, there is a* Σ_{n+1} *formula* B *such that* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE $\vdash A \leftrightarrow \neg B$*.*

2. For each Σ_{n+1} *formula* A*, there is a* Π_{n+1} *formula* B *such that* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE $\vdash \neg A \leftrightarrow B$*.*

3. For each $\Delta(\Pi_{n+1})$ *formula* A*, there is a* $\Delta(\Sigma_{n+1})$ *formula* B *such that* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE $\vdash A \leftrightarrow B$*.*

Proof. 1 and 2 are proved by meta-induction on n. 3 is proved by the induction on the construction of $\Delta(\Pi_{n+1})$ formulae. For the base cases, use 1. □

Lemma 41. *1.* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-DNE *proves* Σ_n-LEM*.*

2. $\mathbf{i\Sigma}_1 + \Sigma_n$-LEM *proves* Π_n-LEM.

3. $\mathbf{i\Sigma}_1 + \Sigma_n$-LEM *proves* Π_n-DNE.

Proof. See [1, Theorem 3.1]. Although they were proved in **HA**, we need no induction stronger than Δ_0. □

Next we proves several closure properties for Σ_n and Π_n.

Lemma 42. *1. Σ_{n+1} is closed under bounded universal quantifier in $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND.*

2. $\neg\neg\Sigma_{n+1}$ is closed under bounded universal quantifier in $\mathbf{i\Sigma}_1 + \neg\neg\Sigma_{n+1}$-IND.

Proof. Since Π_n is closed under bounded universal quantifier, this follows from Lemma 3. □

Lemma 43. *1. Π_{n+1} is closed under bounded existential quantifier in $\mathbf{i\Sigma}_1 + \Delta(\Pi_{n+1})$-IND$+$ Σ_{n+1}-LEM;*

2. Π_{n+1} is weakly closed under bounded existential quantifier in $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Pi_{n+1})$-IND$+$ Σ_n-DNS.

Proof. Let $B(x, y)$ be a Σ_n formula. Then $C(x, z) \equiv (z \in \mathbb{N}^{<\mathbb{N}} \wedge |z| = x) \to (\exists i < x)B(i, z(i))$ is equivalent to some Σ_n formula. We prove the following:

1′. $\mathbf{i\Sigma}_1 + \Delta(\Pi_{n+1})$-IND $+ \Sigma_{n+1}$-LEM *proves* $\forall x((\exists i < x)\forall y B(x, y) \leftrightarrow \forall z C(x, z))$;

2′. $\mathbf{i\Sigma}_1 + \neg\neg\Delta(\Pi_{n+1})$-IND $+ \Sigma_n$-DNS *proves* $\forall x(\neg\neg(\exists i < x)\forall y B(x, y) \leftrightarrow \neg\neg\forall z C(x, z))$.

It is trivial $(\exists i < x)\forall y B(i, y) \to \forall z C(x, z)$. Let $D(x) \equiv \forall z C(x, z) \to (\exists i < x)\forall y B(i, y)$.

1′. It is enough to show $\forall x D(x)$. Assume $D(x)$ and $\forall z C(x + 1, z)$. If $\forall y B(x, y)$, then $(\exists i < x + 1)\forall y B(i, y)$. If there is v such that $\neg B(x, v)$, then, by $\forall z C(x + 1, z)$, we have $(\exists i < x)B(i, (w * \langle v\rangle)(i))$ for all w with $w \in \mathbb{N}^{<\mathbb{N}} \wedge |w| = x$, and so $\forall z C(x, z)$, which yields $(\exists i < x)\forall y B(i, y)$ by $D(x)$ and $(\exists i < x + 1)\forall y B(i, y)$. Hence we have the following:

$$\forall x[\forall y B(x, y) \vee \exists y \neg B(x, y) \to (D(x) \to D(x + 1))]. \tag{11}$$

Using Lemma 40.2, Σ_{n+1}-LEM yields $\forall y B(j, y) \vee \exists y \neg B(j, y)$. Therefore we have $\forall x(D(x) \to D(x + 1))$. Since $D(0)$ trivially holds, we have $\forall x D(x)$, by $\Delta(\Pi_{n+1})$-IND.

2′. It is enough to show that $\forall x \neg\neg D(x)$. By (12), we have

$$\forall x[\neg\neg(\forall y B(x, y) \vee \exists y \neg B(x, y)) \to (\neg\neg D(x) \to \neg\neg D(x + 1))]. \tag{12}$$

Since $\neg\neg(\forall y B(x, y) \vee \exists y \neg B(x, y))$ is equivalent to $\neg(\neg\forall y B(j, y) \wedge \neg\neg\forall y B(j, y))$ by Σ_n-DNS, we have $\forall x \neg\neg(D(x) \to D(x + 1))$. Since $\neg\neg D(0)$ trivially holds, we have $\forall x \neg\neg D(x)$ by $\neg\neg\Delta(\Pi_{n+1})$-IND. □

Lemma 44. *1. Π_n is closed under uniformization in $\mathbf{i\Sigma}_1 + \Delta(\Sigma_n)$-IND$+ \Sigma_n$-LEM.*

2. Π_n is weakly closed under uniformization in $\mathbf{i\Sigma}_1 + \Sigma_n$-IND$+ \Sigma_n$-DNE.

Proof. 1 and 2 are proved by simultaneous meta-induction on n. For $n = 0$, let $C(x)$ be a Π_0 formula. Then $C'(x) \equiv C(x) \wedge (\forall y < x)\neg C(y)$ satisfies the desired property.

Assume for $n = k$. For $n = k + 1$, we first reason in $\mathbf{i\Sigma}_1 + \Sigma_{k+1}$-IND $+ \Sigma_{k+1}$-DNE.

Let $C(x, \vec{v})$ be a Π_{k+1} formula such that all free variables are displayed. Write $C(x, \vec{v})$ as $\forall i D(x, i, \vec{v})$ where $D(x, i, \vec{v})$ is Σ_k. By Lemma 40.2, there is a formula in Π_k equivalent to $\neg D(x, i, \vec{v})$ for any $\vec{v}$. Since Σ_{k+1}-DNE implies Σ_k-LEM by Lemma 41.1, we can find, by the induction hypothesis, a Π_k formula $D'(x, i, \vec{v})$ such that

$$\forall x \forall \vec{v}(\exists i \neg D(x, i, \vec{v}) \leftrightarrow \exists i D'(x, i, \vec{v})) \text{ and } \forall x \forall \vec{v}(\forall i \forall i'(D'(x, i, \vec{v}) \wedge D'(x, i', \vec{v}) \rightarrow i = i')). \tag{13}$$

Then, for each x and $\vec{v}$, we have the following:

$$
\begin{aligned}
& C(x, \vec{v}) \wedge (\forall y < x)\neg C(y, \vec{v}) \\
\leftrightarrow & C(x, \vec{v}) \wedge (\forall y < x)\neg \forall j D(y, j, \vec{v}) \\
\leftrightarrow & C(x, \vec{v}) \wedge (\forall y < x)\neg \forall j \neg\neg D(y, j, \vec{v}) && \text{(by } \Sigma_{k+1}\text{-DNE)} \\
\leftrightarrow & C(x, \vec{v}) \wedge (\forall y < x)\neg\neg \exists j \neg D(y, j, \vec{v}) \\
\leftrightarrow & C(x, \vec{v}) \wedge (\forall y < x)\neg\neg \exists j D'(y, j, \vec{v}) && \text{(by (13))} \\
\leftrightarrow & C(x, \vec{v}) \wedge (\forall y < x)\exists j D'(y, j, \vec{v}) && \text{(by } \Sigma_{k+1}\text{-DNE)} \\
\leftrightarrow & C(x, \vec{v}) \wedge \exists s(s \in \mathbb{N}^{<\mathbb{N}} \wedge |s| = x \wedge (\forall y < x)D'(y, s(y), \vec{v})) && \text{(by } \Sigma_{k+1}\text{-IND and Lemma 42)} \\
\leftrightarrow & \exists s(C(x, \vec{v}) \wedge s \in \mathbb{N}^{<\mathbb{N}} \wedge |s| = x \wedge (\forall y < x)D'(y, s(y), \vec{v}))
\end{aligned}
$$

It is easy to see that there is a Π_{k+1} formula $C'(x, \vec{v})$ equivalent to $C((x)_0, \vec{v}) \wedge (x)_1 \in \mathbb{N}^{<\mathbb{N}} \wedge |(x)_1| = (x)_0 \wedge (\forall w < (x)_0)D'(w, (x)_1(w), \vec{v}))$. Then, for each x, we have $C(x, \vec{v}) \wedge (\forall y < x)\neg C(y, \vec{v}) \leftrightarrow \exists s C'((x, s), \vec{v})$, and so $\exists s C'((x, s), \vec{v}) \wedge \exists s C'((x', s), \vec{v}) \rightarrow x = x'$. Furthermore, we have $C'((x, s), \vec{v}) \wedge C'((x, s), \vec{v}) \rightarrow s = s'$ by (13), which implies $C'(x, \vec{v}) \wedge C'(x', \vec{v}) \rightarrow x = x'$.

For 1, we have Π^0_{k+1}-LNP in $\mathbf{i\Sigma}_1 + \Delta(\Sigma_{k+1})$-IND $+ \Sigma_{k+1}$-LEM by Lemma 7.1, Lemma 8.1 and Lemma 40.3. Therefore the following holds:

$$\exists x C(x, \vec{v}) \leftrightarrow \exists x(C(x, \vec{v}) \wedge (\forall y < x)\neg C(y, \vec{v})) \leftrightarrow \exists x C'(x, \vec{v}).$$

For 2, we reason in $\mathbf{i\Sigma}_1 + \Sigma_{k+1}$-IND $+ \Sigma_{k+1}$-DNE. Since we have $\neg\neg\Sigma_{k+1}$-IND, by Lemma 2, we have Π^0_{k+1}-WLNP, by Lemma 7.2, Lemma 8.2 and Lemma 40.3. Therefore the following holds:

$$\neg\neg\exists x C(x, \vec{v}) \leftrightarrow \neg\neg\exists x(C(x, \vec{v}) \wedge (\forall y < x)\neg C(y, \vec{v})) \leftrightarrow \neg\neg\exists x C'(x, \vec{v}).$$

□

Now we show the relationship among finiteness and infiniteness of Σ_n and Π_n.

Theorem 11. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

1. Σ_{n+1}-BCA;

2. $(\text{fin}_3{\rightarrow}\text{fin}_1)(\Sigma_{n+1})$;

3. $(\text{fin}_3{\rightarrow}\text{fin}_1)(\Pi_n)$;

4. $(\text{fin}_3{\rightarrow}\text{fin}_2)(\Sigma_{n+1})$.

5. $(\text{fin}_3{\rightarrow}\text{fin}_2)(\Pi_n)$.

Proof. This is proved by meta-induction on n.

For $n = 0$, let Γ be the class of all atomic formulae. Then $\Delta(\Gamma)$ is Π_0 and $\exists\Delta(\Gamma)$ is Σ_1, and we have "1⇒2", by Lemma 10 and "5⇒1", by Lemma 4. "2⇒3" and "4⇒5" hold trivially. "2⇒4" and "3⇒5" follow from Lemma 8.1.

Next we prove the case of $n = k+1$. For "1⇒2", let $\mathbf{T} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+2}$-BCA. By Corollary 7.1, $\mathbf{T}$ proves Σ_{k+2}-IND. The class Π_{k+1} is closed under under bounded universal quantifier. Therefore $\mathbf{T}$ proves $(\text{fin}_3{\rightarrow}\text{fin}_1)(\Sigma_{k+2})$ by Lemma 9.

"2⇒3" and "4⇒5" are trivial. "2⇒4" and "3⇒5" follow from Lemma 8.1.

For "5⇒1", let $\mathbf{S} = \mathbf{i\Sigma}_1 + (\text{fin}_3{\rightarrow}\text{fin}_2)(\Pi_{k+1})$. Then $\mathbf{S}$ proves Σ_{k+1}-BCA by the induction hypothesis, and so it proves $\Delta(\Sigma_{k+1})$-IND and Σ_{k+1}-LEM by Lemma 7.1. Hence the class Π_{k+1} is closed under uniformization in $\mathbf{S}$ by Lemma 44.1. Furthermore, $\mathbf{S}$ proves $\Delta(\Pi_{k+1})$-IND by Lemma 40.3, and so Π_{k+1} is closed under bounded existential quantifier in $\mathbf{S}$ by Lemma 43.1. Therefore $\mathbf{S}$ proves Σ_{k+2}-BCA by Lemma 11. □

Theorem 12. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND*:*

1. Σ_{n+2}-DNE;

2. $(\text{fin}_4{\rightarrow}\text{fin}_2)(\Sigma_{n+1})$;

3. $(\text{fin}_4{\rightarrow}\text{fin}_2)(\Pi_n)$.

4. $(\text{fin}_5{\rightarrow}\text{fin}_3)(\Sigma_{n+1}) + \Sigma_{n+1}$-GDM;

5. $(\text{fin}_5{\rightarrow}\text{fin}_3)(\Pi_n) + \Sigma_{n+1}$-GDM;

Proof. This is proved by meta-induction on n. For $n = 0$, it follows from Theorem 3 and Theorem 4 by taking Γ as the class of atomic formulae.

Now we prove for $n = k+1$. For "1⇒2", we reason in $\mathbf{i\Sigma}_1 + \Sigma_{k+2}$-IND $+ \Sigma_{k+3}$-DNE. By Lemma 40.1, for each Π_{k+1} formula A, there is a Σ_{k+1} formula B such that $\neg A \leftrightarrow B$. Hence Σ_{k+3}-DNE implies $\exists\forall\neg\Pi_{k+1}$-DNE. By Lemma 4.3, it also implies Σ_{k+2}-DNS. Then "1⇒2" follows Lemma 13. "2⇒3" is trivial.

For "3⇒1", let $\Psi \equiv \Sigma_{k+1}$, $\Psi' \equiv \Pi_{k+1}$ and $\mathbf{T} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+2}$-IND $+ (\text{fin}_4{\rightarrow}\text{fin}_2)(\Pi_{k+1})$. Note that $\mathbf{T}$ proves Σ_{k+2}-DNE by the induction hypothesis.

- For each formula B in Ψ, there exists a formula C in Ψ' such that $\mathbf{T}$ proves $\neg B \leftrightarrow C$;
- $\mathbf{T}$ proves $(\text{fin}_4{\rightarrow}\text{fin}_2)(\Psi')$ and $\Delta(\Psi)$-LNP, by Lemma 39 and Lemma 7.1;

- Ψ' is closed under uniformization and bounded universal quantifier in $\mathbf{T}$.

Therefore $\mathbf{T}$ proves Σ_{k+3}-DNE, by Lemma 13.

As we have seen in the proof of "1⇒2", $\exists\forall\neg\Pi_{k+1}$-DNE and Σ_{k+2}-IND are implied by Σ_{k+3}-DNE. Let Γ be Π_{k+1}. It is easy to see that Σ_{k+2}-GDM is implied by Σ_{k+3}-DNE. By Lemma 7.1, Σ_{k+2} is closed under bounded universal quantifier. Therefore "1⇒4" follows from by Lemma 16. "4⇒5" is trivial.

For "5⇒1", let $\Psi \equiv \Sigma_{k+1}$, $\Psi' \equiv \Pi_{k+1}$ and $\mathbf{S} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+2}$-IND $+ (\mathrm{fin}_4 \rightarrow \mathrm{fin}_3)(\Pi_{k+1}) +$ Σ_{k+2}-GDM. By the induction hypothesis, $\mathbf{S}$ implies Σ_{k+2}-DNE. Hence $\mathbf{S}$ proves $\Delta(\Sigma_{k+1})$-IND and Σ_{k+1}-LEM by Lemma 39 and Corollary 7.1. Then we have the following:

- For each formula B in Ψ, there exists a formula C in Ψ' such that $\mathbf{S}$ proves $\neg B \leftrightarrow C$ by Lemma 40.2;
- $\mathbf{S}$ proves $\exists\Psi'$-IND, Ψ-DNE, $\exists\Psi'$-GDM and $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_3)(\Psi')$.
- Π_{k+1} is closed under uniformization bounded universal quantifier in $\mathbf{S}$ by Lemma 44.1.

Therefore we have Σ_{k+3}-DNE by Lemma 18. □

Theorem 13. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE $+ \Sigma_{n+1}$-DNS*:*

1. $\neg\neg\Sigma_{n+1}$-IND*;*
2. $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\Sigma_{n+1})$*;*
3. $(\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\Pi_n)$.

Proof. This is proved by the meta-induction on n. For $k = 0$, it follows from Theorem 5. Now we prove for $n = k+1$. Since Π_n is closed under bounded universal quantifier, we have "1⇒2", by Lemma 20. "2⇒3" is trivial. Let $\mathbf{T} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+1}$-DNE $+ \Sigma_{k+2}$-DNS $+ (\mathrm{fin}_5 \rightarrow \mathrm{fin}_4)(\Pi_{k+1})$. Note that $\mathbf{T}$ proves $\neg\neg\Sigma_{k+1}$-IND by the induction hypothesis. Then Π_{k+1} is weakly closed under uniformization in $\mathbf{T}$, since, by Lemma 2.1, $\mathbf{T}$ proves Σ_{k+1}-IND. Since $\mathbf{T}$ proves $\neg\neg\Sigma_{k+1}$-IND, $\mathbf{T}$ proves $\neg\neg\Delta(\Sigma_{k+1})$-IND by Lemma 7.2, which implies $\neg\neg\Delta(\Pi_{k+1})$-IND by Lemma 40.3. Hence Π_n is weakly closed under bounded existential quantifier in $\mathbf{T}$ by Lemma 4.3 and Lemma 43.2. Therefore, by Lemma 21, $\mathbf{T}$ proves $\neg\neg\Sigma_{k+2}$-IND. □

Theorem 14. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE*:*

1. $\neg\neg\Sigma_{n+1}$-IND*;*
2. $(\mathrm{inf}_2 \rightarrow \mathrm{inf}_3)(\Sigma_{n+1})$*;*
3. $(\mathrm{inf}_2 \rightarrow \mathrm{inf}_3)(\Pi_n)$.

Proof. This is proved by meta-induction on n. For $n = 0$, it follows from Theorem 6 by taking Γ as the set of atomic formulae.

For $n = k + 1$, it is enough to show 1⇒2 and 3⇒1.

For "1⇒2", let $\mathbf{T} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+1}$-DNE + $\neg\neg\Sigma_{k+1}$-IND. Since $\mathbf{T}$ proves Σ_{k+1}-IND, by Lemma 2.1, the class Π_{k+1} is weakly closed under uniformization in $\mathbf{T}$ by Lemma 44.2. Since Π_{k+1} is closed under bounded universal quantifier in $\mathbf{T}$, we have $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Sigma_{k+2})$ by Lemma 25.

For "3⇒1", let $\mathbf{S} \equiv \mathbf{i\Sigma}_1 + \Sigma_{k+1}$-DNE + $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Pi_{k+1})$. Then $\neg\neg\Sigma_{k+1}$-IND holds in $\mathbf{S}$ by the induction hypothesis, and so Σ_{k+1}-IND holds by Lemma 2.1. Hence Π_{k+1} is weakly closed under uniformization in $\mathbf{S}$, by Lemma 44.2. By Lemma 4.3 and Lemma 43.2, Π_{k+1} is weakly closed under bounded existential quantifier in $\mathbf{S}$. Since Π_{k+1} is closed under bounded universal quantifier in $\mathbf{S}$, we have $\neg\neg\Sigma_{k+2}$-IND by Lemma 27, □

Theorem 15. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND*:*

1. Σ_{n+1}-DNE*;*
2. $(\mathrm{inf}_2 \to \mathrm{inf}_4)(\Sigma_{n+1})$*;*
3. $(\mathrm{inf}_2 \to \mathrm{inf}_4)(\Pi_n)$*.*
4. $(\mathrm{inf}_3 \to \mathrm{inf}_5)(\Sigma_{n+1})$*;*
5. $(\mathrm{inf}_3 \to \mathrm{inf}_5)(\Pi_n)$*;*

Proof. This is proved by meta-induction on n. For $n = 0$, it follows from Lemma 7 and Lemma 8 by taking Γ as the class of atomic formulae.

Now we prove for $n = k + 1$. "1⇒2" follows from Lemma 29. "2⇒3" is trivial. "3⇒1" follows from Lemma 30. "1⇒4" follows from Lemma 31 and Lemma 42.1. "4⇒5" is trivial. "5⇒1" follows from Lemma 33. □

Theorem 16. *The following are equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-LNP*.*

1. Σ_{n+1}-IND*;*
2. $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Sigma_{n+1})$*;*
3. $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Pi_n)$*.*

Proof. Since Σ_n-LNP is equivalent to $\Delta(\Sigma_n)$-LNP, this follows from Lemma 34 and Lemma 35. □

Corollary 9. 1. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1$*:*

   (a) Σ_{n+1}-BCA*;*
   (b) $(\mathrm{fin}_2 \to \mathrm{fin}_1)(\Sigma_{n+1})$*;*
   (c) $(\mathrm{fin}_3 \to \mathrm{fin}_1)(\Sigma_{n+1})$*;*

(d) $(\mathrm{fin}_3 \to \mathrm{fin}_2)(\Sigma_{n+1})$.

2. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND:

   (a) Σ_{n+2}-DNE;

   (b) $(\mathrm{fin}_4 \to \mathrm{fin}_2)(\Sigma_{n+1})$;

   (c) $(\mathrm{fin}_5 \to \mathrm{fin}_3)(\Sigma_{n+1}) + \Sigma_{n+1}$-GDM.

3. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-IND $+ \Sigma_n$-DNE $+ \Sigma_{n+1}$-DNS:

   (a) $\neg\neg\Sigma_{n+1}$-IND;

   (b) $(\mathrm{fin}_5 \to \mathrm{fin}_4)(\Sigma_{n+1})$.

4. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE:

   (a) $\neg\neg\Sigma_{n+1}$-IND;

   (b) $(\mathrm{inf}_1 \to \mathrm{inf}_2)(\Sigma_{n+1})$;

   (c) $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Sigma_{n+1})$.

5. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND:

   (a) Σ_{n+1}-DNE;

   (b) $(\mathrm{inf}_2 \to \mathrm{inf}_4)(\Sigma_{n+1})$;

   (c) $(\mathrm{inf}_3 \to \mathrm{inf}_5)(\Sigma_{n+1}) + \Sigma_{n+1}$-GDM.

6. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-LNP:

   (a) Σ_{n+1}-IND;

   (b) $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Sigma_{n+1})$;

   (c) $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Pi_n)$.

Corollary 10. 1. *The following are equivalent over* $\mathbf{i\Sigma}_1$:

   (a) Π_n-BCA;

   (b) $(\mathrm{fin}_2 \to \mathrm{fin}_1)(\Pi_n)$.

2. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE:

   (a) Σ_{n+1}-BCA;

   (b) $(\mathrm{fin}_3 \to \mathrm{fin}_1)(\Pi_n)$;

   (c) $(\mathrm{fin}_3 \to \mathrm{fin}_2)(\Pi_n)$.

3. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_{n+1}$-IND:

(a) Σ_{n+2}-DNE;

(b) $(\mathrm{fin}_4 \to \mathrm{fin}_2)(\Pi_n)$;

(c) $(\mathrm{fin}_5 \to \mathrm{fin}_3)(\Pi_n) + \Sigma_{n+1}$-GDM.

4. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-DNE $+ \Sigma_{n+1}$-DNS*:*

(a) Σ_{n+1}-IND;

(b) $(\mathrm{fin}_5 \to \mathrm{fin}_4)(\Pi_n)$.

5. *The following are equivalent over* $\mathbf{i\Sigma}_n$*:*

(a) Π_n-WBCA;

(b) $(\mathrm{inf}_1 \to \mathrm{inf}_2)(\Pi_n)$.

6. *The following are equivalent over* $\mathbf{i\Sigma}_n + \Sigma_n$-DNE*:*

(a) $\neg\neg\Sigma_{n+1}$-IND;

(b) $(\mathrm{inf}_2 \to \mathrm{inf}_3)(\Pi_n)$.

7. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-IND*:*

(a) Σ_{n+1}-DNE;

(b) $(\mathrm{inf}_2 \to \mathrm{inf}_4)(\Pi_n)$;

(c) $(\mathrm{inf}_3 \to \mathrm{inf}_5)(\Pi_n)$.

8. *The following are pairwise equivalent over* $\mathbf{i\Sigma}_1 + \Sigma_n$-LNP*:*

(a) Σ_{n+1}-IND;

(b) $(\mathrm{inf}_4 \to \mathrm{inf}_5)(\Pi_{n+1})$.

Throughout this paper, we used the class $\Delta(\Gamma)$. One may wonder what principles are needed to reform formulae in $\Delta(\Gamma)$ into prenex normal form. In [1, 2.3], Prenex Normal Form Theorem was proved in **HA** which has full induction scheme. It gave proper subclasses P_n, U_n and E_n of $\Delta(\Sigma_n \cup \Pi_n)$ and showed that, for each formula A in $P_n \cup U_n \cup E_n$, we can find formulae B in Σ_{n+1} and C in Π_{n+1} such that $\mathbf{HA} + \Sigma_n\text{-LEM} \vdash A \leftrightarrow B \leftrightarrow C$. We conclude this paper by prenex normal form theorem for $\Delta(\Sigma_n \cup \Pi_n)$ with restricted induction.

Theorem 17. *Let A be in $\Delta(\Sigma_n \cup \Pi_n)$. Then we can find formulae B in Σ_{n+1} and C in Π_{n+1} such that* $\mathbf{i\Sigma}_1 + \Delta(\Sigma_{n+1})\text{-IND} + \Sigma_{n+1}\text{-LEM}$ *proves $A \leftrightarrow B \leftrightarrow C$.*

Proof. This is proved by meta-induction on n. It is trivial for $n = 0$.

Assume for $n = k$. For the case of $n = k+1$ we prove by the induction on the construction of $\Delta(\Sigma_{k+1} \cup \Pi_{k+1})$ formulae.

For the base case, it is clear.

For induction step, assume that A_0 and A_1 are in $\Delta(\Sigma_{k+1} \cup \Pi_{k+1})$, that B_0 and B_1 are in Σ_{k+2} and that C_0 and C_1 are in Π_{k+2} such that, for each $i < 2$,

$$\mathbf{i\Sigma}_1 + \Delta(\Sigma_{k+1})\text{-IND} + \Sigma_{k+1}\text{-LEM} \vdash A_j \leftrightarrow B_j \leftrightarrow C_j.$$

Write B_j as $\exists x \forall y B'_j(x, y)$ and C_j as $\forall x \exists y C_j(x, y)$, where $B'_j(x, y)$ is Σ_k and $C'_j(x, y)$ is Π_k. Now we work in $\mathbf{i\Sigma}_1 + \Delta(\Sigma_{k+2})$-IND $+ \Sigma_{k+2}$-LEM, which proves $\Delta(\Pi_{k+2})$-IND by Lemma 40.3.

It is easy for the case $A \equiv A_0 \wedge A_1$.

If $A \equiv A_0 \vee A_1$, then $B_0 \vee B_1$ is equivalent to $\exists x \forall y[((x)_0 = 0 \to B'_0((x)_1, y)) \wedge ((x)_0 \neq 0 \to B'_1((x)_1, y))]$. By the induction hypothesis, there is a Π_{k+1} formula equivalent to $((x)_0 = 0 \to B'_0((x)_1, y)) \wedge ((x)_0 \neq 0 \to B'_1((x)_1, y))$. To see A is equivalent to a Π_{k+2} formula, note that $C_0 \vee C_1$ is equivalent to $(\exists i < 2)\forall x \exists y[(i = 0 \to C'_0(x, (y)_0)) \wedge (i = 1 \to C'_1(x, (y)_1))]$. By the induction hypothesis, we can find a Σ_{k+1} formula $D(i, x, y)$ equivalent to $(i = 0 \to C'_0(x, (y)_0)) \wedge (i = 1 \to C'_1(x, (y)_1))$. Since $\forall x \exists y D(i, x, y)$ is equivalent to a Π_{k+2} formula, we can find a Π_{k+2} formula equivalent to $(\exists i < 2)\forall x \exists y D(i, x, y)$ by Lemma 43.1.

If $A \equiv A_0 \to A_1$, then

$$A \leftrightarrow (\forall x \exists y C'_0(x, y) \to \exists u \forall v B_1(u, v)) \leftrightarrow (\exists x \forall y B'_0(x, y) \to \forall u \exists y C'_1(x, y)).$$

By Σ_{k+2}-LEM, we have

$$A \leftrightarrow \exists x \exists u \forall y \forall v (C'_0(x, y) \to B'_1(u, v)) \leftrightarrow \forall x \forall u \exists y \exists v (B'_0(x, y) \to C'_1(u, v)).$$

By the induction hypothesis, there are a Π_{k+1} formula $D(x, y, u, v)$ and a Σ_{k+1} formula $D'(x, y, u, v)$ such that $(C'_0(x, y) \to B'_1(u, v)) \leftrightarrow D(x, y, u, v)$ and $(B_0(x, y) \to C_1(u, v)) \leftrightarrow D'(x, y, u, v)$.

If $A \equiv (\forall i < t)A_0$, then $A \leftrightarrow (\forall i < t)B_0 \leftrightarrow (\forall i < t)C_0$. It is easy to see that $(\forall i < t)C_0$ is equivalent to a Π_{k+2} formula. For $(\forall i < x)B_0$, we can find an equivalent Σ_{k+2} formula by Lemma 42.

If $A \equiv (\exists i < t)A_0$, then $A \leftrightarrow (\exists i < t)B_0 \leftrightarrow (\exists i < t)C_0$. It is easy to see that $(\exists i < t)B_0$ is equivalent to a Σ_{k+2} formula. Therefore, for $(\exists i < t)C_0$, we can find an equivalent Π_{k+2} formula by Lemma 43.1. □

Acknowledgement

The author would like to express her gratitude to Prof. Hajime Ishihara for his support and encouragement. She also thanks to Dr. Kentaro Sato for insightful discussions with him. This work has been supported by JSPS Core-to-Core program (A. Advanced Research Networks).

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