INFINITE GAMES
FROM AN INTUITIONISTIC POINT OF VIEW

Abstract. In this paper, we consider determinacy in Brouwerian intuitionistic mathematics. We give some examples of games such that the character of this mathematical setting—the lack of the law of excluded middle and the adoption of continuity principle—makes the behavior of determinacy drastically different from that on the classical setting.

1. Introduction

Games on $\mathbb{N}^{\mathbb{N}}$ have been of great interest in mathematical logic for a long time. On one hand, determinacy of games has been used as a strong tool to investigate Baire space $\mathbb{N}^{\mathbb{N}}$ or Cantor space $\{0,1\}^{\mathbb{N}}$. On the other hand, as has been known, determinacy statements are quite sensitive to the mathematical setting: For example, with the axiom of choice, full determinacy is inconsistent; determinacy of analytic games are beyond ZFC.

The ultimate purpose of the author is to know how Baire space and Cantor space vary depending on settings other than usual ones. As the first step toward this, she has been investigating the promising tool, determinacy, on these settings. Among these are subsystems of second order arithmetic, much weaker ones than ZFC (cf. (Nemoto, et al. 2007), (Nemoto 2008)).

This paper treats another setting, Brouwerian intuitionistic mathematics. It denies the law of excluded middle (*LEM*) and adopts the *continuity principle*, asserting that all the functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ or to $\mathbb{N}$ are continuous (for detail, see Section 2). We give some examples of games, which show that the continuity principle and the lack of LEM make the behavior of determinacy drastically different from that on the classical setting. To explicate the role of classical principles in determinacy, we treat predeterminacy—a formalization of determinacy in the intuitionistic mathematics—also in the classical mathematics.

2. Axioms of the intuitionistic mathematics

In this section, we clarify the mathematical setting of this paper.

The logical constants have their constructive meanings and the rules of the intuitionistic logic are employed. In particular, a disjunctive statement $A \vee B$ means there exists a proof of A or one of B, and an existential statement $\exists x \in V[A(x)]$ means there exist an element a of V and an proof of $A(a)$. A statement A is *decidable* if $A \vee \neg A$ holds. A set $X \subseteq V$ is decidable if the statement $a \in X$ is decidable for each $a \in V$.

An infinite sequence α of natural numbers $\alpha(0), \alpha(1), \alpha(2), ...$ may be determined by some finitely described algorithm, i.e., the n-th element $\alpha(n)$ of α is the result of the algorithm for input n. Sometimes, however, such an infinite sequence may be constructed step by step by choosing its elements one by one. In this case, the construction of the sequence is never finished: At any point in time, only finitely many elements have been chosen, and so we can only know a finite part of the sequence.

The latter construction is not permitted in the constructive mathematics, and so this point divides the intuitionistic mathematics from the constructive mathematics.

Note that every infinite sequence, even if it is given by an algorithm, can be regarded as a result of step-by-step-construction. This is the reason we do not distinguish infinite sequences of natural number by their manners of construction.

Let $\mathbb{N}$ be the set of natural numbers. $X^{\mathbb{N}}$ is the set of infinite sequences from X. In particular $\mathbb{N}^{\mathbb{N}}$ is called *Baire space* and $2^{\mathbb{N}}$ is called *Cantor space*. X^n is the set of sequences from X of length n and $X^{<\mathbb{N}}$ denotes $\bigcup_{n\in\mathbb{N}} X^n$. Small letters, such as $m, n, ...$, are reserved for variables over $\mathbb{N}$ and Greek letters, such as $\alpha, \beta, ...$, are for those over $\mathbb{N}^{\mathbb{N}}$. $\alpha(m)$ is the m-th element of α. We fix a bijective coding from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}$. $\langle\alpha(0), \alpha(1), ..., \alpha(m-1)\rangle$ is denoted by $\overline{\alpha}m$. $s * t$ is the concatenation of sequences s and t. $lh(s)$ is the length of a sequence s.

The following axiom is employed in the mathematical setting of this paper, which is not accepted in the classical setting.

The continuity principle

If $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ is a binary relation such that, for every α, there is m with $\alpha R m$, then, for every α, there exist m and n such that, for every β, $\overline{\alpha}n = \overline{\beta}n$ implies $\beta R m$.

This axiom is adopted for the following reason. If, for every α, we can find a suitable m, then m must be determined by a finite part of α, because α must be step-by-step-constructed and so we can only know a finite part of α at each moment of time.

$T \subseteq \mathbb{N}$ is a *spread-law* if such that $0 \in T$ and, for each s, $s \in T$ if and only if there is n with $s * \langle n\rangle \in T$. For a spread-law T, $[T]$ denotes the set of all α such that $\overline{\alpha}m \in T$ for all m. $S \subseteq \mathbb{N}^{\mathbb{N}}$ is a *spread* if $S = [T]$ for some spread-law T. (In the classical mathematics, a spread-law is often called an *tree* without leaves and spread $[T]$ is the set of paths of T.) For a spread $S \subseteq \mathbb{N}^{\mathbb{N}}$, η is a *code of a continuous function* if, for any $\alpha \in S$ and m, there is n with $\eta(\langle m\rangle * \overline{\alpha}n) \neq 0$. For a code η of a continuous function, $\eta|\alpha$ is $\beta \in \mathbb{N}^{\mathbb{N}}$ such that, for all m, $\beta(m) = \eta(\langle m\rangle * \overline{\alpha}p) - 1$, where p is the least n with $\eta(\langle m\rangle * \overline{\alpha}n) \neq 0$.

Since, in the intuitionistic logic, $\forall x \in V[A(x)]$ means that there is a function $\mathcal{F}$ such that, for all $a \in V$, $\mathcal{F}(a)$ is a proof of $A(a)$, the continuity principle leads the following.

The second axiom of continuous choice

Let S be a spread on $\mathbb{N}$ and $R \subseteq S \times \mathbb{N}^{\mathbb{N}}$ a relation. If, for all α in S, there is β with $\alpha R\beta$, then there is a code η of a continuous function such that $\alpha R(\eta|\alpha)$ for all α.

A spread-law T is a *fan-law* if, for each s in T, there are only finitely many n with $s * \langle n\rangle \in T$. (Classically a fan-law is often called a finitary branching tree.) A spread S is a *fan* if $S = [T]$ for some fan-law T. $B \subseteq \mathbb{N}$ is a *bar* in a spread S if, for every sequence α in S, there is n with $\overline{\alpha}n \in B$. A bar B is *bounded* if there is n such that $lh(b) < n$ for each b in B.

The following is the intuitionistic counterpart of König's lemma. Although it is called a theorem, it is treated as an axiom here.

The strict fan theorem

For a fan S and a decidable bar B in S, there is a bounded sub-bar $B' \subseteq B$ in S.

While König's lemma and the strict fan theorem are equivalent in the classical mathematics, they are not in the intuitionistic mathematics. Actually we can construct a "so-called" intuitionistic counterexample, i.e., a fan T which has sequences of any finite length such that we cannot prove that T has an infinite path, i.e., $\alpha\mathbb{N} \to \mathbb{N}$ such that $\overline{\alpha}n \in T$ for all n. Let $i^n \in \{0,1\}^n$ be such that $i^n(k) = i$ for all $k < n$ and let $i^{\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ be such that $i^{\mathbb{N}}(n) = i$ for all n. Define $T \subseteq \{i^n : i < 2, n \in \mathbb{N}\}$ by

$0^n \in T \leftrightarrow$there is no $k < n$ such that $p_{k+i} = 9$ for all $i < 99$, or if the least such k is even,
$1^n \in T \leftrightarrow$there is no $k < n$ such that $p_{k+i} = 9$ for all $i < 99$, then the least such k is odd,

where p_k denotes the k-th digit of the decimal expansion of π. We can easily see that T is a fan which has sequences of any finite length and that if T has an infinite path α, then $\alpha = 0^{\mathbb{N}}$ or $\alpha = 1^{\mathbb{N}}$. Assume that T has an infinite path α. If $\alpha(0) = 0$ (or 1), then we must have a proof of the statement "if there is uninterrupted occurrences of 9 of length 99 in the decimal expansion of π, the least such one starts at an even (resp. odd) digit." Up to now, we do not have any proof of such statements, and so there is no infinite path in T. (If we have a proof in future, we can find another so-called counterexample using another unsolved problem in a similar way.)

3. Determinacy in intuitionistic mathematics

In this section, we introduce the notion of determinacy and variants.

For $A \subseteq \mathbb{N}^{\mathbb{N}}$, the *game* $G(A)$ in $\mathbb{N}^{\mathbb{N}}$ is defined as follows. Two players, called players I and II, starting with player I, alternately choose a natural number to construct $\alpha \in \mathbb{N}^{\mathbb{N}}$. Player I wins if and only if the resulting play α is in A. Player II wins if and only if player I does not win. A strategy for player I (resp. II) is a function which assigns a natural number to each even-(resp. odd-)length sequence in $\mathbb{N}^{<\mathbb{N}}$.

For a strategy σ for player I (resp. II), $\alpha \in_I \sigma$ (resp. $\alpha \in_{II} \sigma$) denotes that α is a play in which player I (resp. II) follows σ at all his turns. Note that if $\alpha \in_I \sigma$ and $\alpha \in_{II} \tau$, then α is uniquely determined. A winning strategy for player I (resp. II) in $G(A)$ is a strategy for player I (resp. II) such that player I (resp. II) wins if he/she follows it.

In the classical mathematics, we say that $G(A)$ is determinate if one of the players has a winning strategy in $G(A)$. Note that the classical determinacy is a disjunctive statement.

There are many variations of game.

Game $G_{X^{\mathbb{N}}}$ in $X^{\mathbb{N}}$: For $A \subseteq X^{\mathbb{N}}$, players alternately choose an element of X to construct $\alpha \in X^{\mathbb{N}}$. Player I wins if $\alpha \in A$ and player II wins if player I does not win.

γ-length game G_γ: For a given ordinal γ and $A \subseteq \mathbb{N}^\gamma$, players alternately choose a natural number to construct $\alpha \in \mathbb{N}^\gamma$ Player I wins if $\alpha \in A$ and player II wins if player I does not win.

Game $G_{[S]}$ in a spread $[S]$: For a given spread-law S and $A \subseteq [S]$, players alternately choose a natural number to construct α so that $\overline{\alpha}n \in S$ for every n. Player I wins if $\alpha \in A$ and player II wins if player I does not win.

(Veldman 2004) gave three formalizations of determinacy in the intuitionistic mathematics.

$G(A)$ is *strongly determinate* if, in $G(A)$, *either* player I *or* player II has a winning strategy. This is the simplest formalization, but almost no game is strongly determinate.

$G(A)$ is *determinate from the view point of player I* if, *if* for every strategy τ of player II, there is $\alpha \in_{II} \tau$ with $\alpha \in A$, *then* player I has a winning strategy in $G(A)$. This statement corresponds to the classical statement "if player II has no winning strategy, then player I has one in $G(A)$," which is classically equivalent to "$G(A)$ is determinate."

To describe the last, we need a new notion. An *anti-strategy for player I in* $G(A)$ is a function η which assigns $\alpha \in_{II} \tau$ to each strategy τ for player II in $G(A)$. An anti-strategy η for player I *secures* A if, for any strategy τ for player II, $\eta(\tau) \in A$. $G(A)$ is *predeterminate from the viewpoint of player I* if, *if* he has an anti-strategy securing A, *then* he has a winning strategy in $G(A)$.

Note that $G(A)$ is predeterminate from the viewpoint of player I, if $G(A)$ is determinate from his viewpoint.

Moreover, in a game $\mathcal{G}(X)$ in $\mathbb{N}^{\mathbb{N}}$ (or spread $[S]$), the second axiom of continuous choice yields the converse, i.e., predeterminacy implies determinacy, since a strategy for a player can be regarded as a function from $\mathbb{N}$ to $\mathbb{N}$ and since if there is $\alpha \in_{II} \tau$ with $\alpha \in X$ for all strategy τ for player II, then by the second axiom of continuous choice an anti-strategy for player I securing X is given by a code η of a continuous function.

The intuitionistic determinacy theorem (Veldman 2004, Theorem 3.5) If $[S]$ is a II-finitary branching spread, i.e., S is a spread-law such that, for every odd-length $s \in S$, there are at most finitely many n with $s * \langle n \rangle \in T$, then $G_{[S]}(A)$ is predeterminate from the viewpoint of player I for every $A \subseteq [S]$.

In particular, if $A \subseteq \{0,1\}^{\mathbb{N}}$, $G_{\{0,1\}^{\mathbb{N}}}(A)$ is predeterminate from the viewpoint of player I. (Veldman 2004) also gave $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that $G(A)$ is not predeterminate from the viewpoint of player I.

Remark The notion of predeterminacy can be formalized from the viewpoint of player II and we can obtain similar results to the last theorem.

4. Variations of games and predeterminacy

In this section, we consider other variations of games in the intuitionistic mathematics. For these games, we can define the three formalizations of determinacy in the same way.

4.1. 2-length games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$

This subsection treats one of the simplest cases in which less strategies are allowed than in the classical context. $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ denotes the product topological space of Cantor space and discrete space $\{0,1\}$.

For given $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, the game $\mathcal{G}_1(A)$ is defined as follows:

- Player I chooses $\alpha \in \{0,1\}^{\mathbb{N}}$.
- Player II chooses $i \in \{0,1\}$.
- Player I wins if $(\alpha, i) \in A$ and player II wins if player I does not win.

Although $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ is homeomorphic to Cantor space topologically, we must be sensitive to the ordertype of the indexing set for the sequences.

In this game, a strategy for player I is his initial move α, and a strategy for player II is a function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}$. The continuity principle forces all the strategies for player II to be continuous, and so we may regard a strategy τ for player II as a code of a continuous function such that $(\tau|\alpha)(0) \in \{0,1\}$ for all $\alpha \in \{0,1\}^{\mathbb{N}}$. $B = \{s \in \{0,1\}^{<\mathbb{N}} | \tau(s) > 0\}$ is a decidable bar in the fan $\{0,1\}^{\mathbb{N}}$. Then, by the strict fan theorem, there is a bounded sub-bar $B' \subseteq B$. Take n such that $lh(s) < n$ for every $s \in B'$. Then, $\{0,1\}^n$ is also a bar in $\{0,1\}^{\mathbb{N}}$, and, for every $\alpha, \beta \in \{0,1\}^{\mathbb{N}}$, $\overline{\alpha}n = \overline{\beta}n$ implies $\tau|\alpha(0) = \tau|\beta(0)$. Thus we can regard τ as a function from $\{0,1\}^{n_\tau}$ to $\{0,1\}$, which can be coded by a natural number. Because an anti-strategy η for player I is a function from the set of all strategies for player II to the set of plays in this game, it can be regarded as a function from $\mathbb{N}$ with the discrete topology to $\{0,1\}^{\mathbb{N}} \times \{0,1\}$.

The following examples shows that even simpler sets, such as open or closed sets, are not predeterminate from the viewpoint of player I.

Example 1 *An open game $\mathcal{G}_1(A)$ which is not predeterminate from the view point of player I:* Define $A_i = \{0^n * \langle 1, i\rangle : n \in \mathbb{N}\}$ and $A = \{(\alpha, i) : \exists n[\overline{\alpha}n \in A_i]\}$. Then A is open. Let η be the anti-strategy for player I which assigns $(0^n_\tau * \langle 1, \tau(0^n_\tau)\rangle * 0^{\mathbb{N}}, \tau(0^n_\tau))$ to each strategy τ for player II. Then $\eta(\tau) \in A$ for each strategy τ for player II, and so η is an anti-strategy for player I securing A. On the other hand, it is clear that player I has no winning strategy in $\mathcal{G}_1(A)$.

Example 2 *A closed game $\mathcal{G}_1(B)$ which is not predeterminate from the viewpoint of player I:* Let T be an intuitionistic counterexample to König's lemma, i.e., an unbounded binary tree without infinite paths. Let $T_i = \{t * i^n | t \in T \wedge n \in \mathbb{N}\}$. Then $B = \{(\alpha, i) | \forall n[\overline{\alpha}n \in T_i]\}$ is a closed set. If player I had a winning strategy α in $\mathcal{G}_1(B)$, α would be an infinite path of T. Thus player I cannot have a winning strategy in $\mathcal{G}_1(B)$. On the other hand, player I has an anti-strategy securing B. Fix an enumeration of T and let t_n be the minimum $s \in T$ such that $lh(s) = n$ with respect to this enumeration. Let η be the anti-strategy for player I which assigns $(t_n * (\tau(t_n))^{\mathbb{N}}, \tau(t_n))$ to each strategy $\tau : \{0,1\}^n \to \{0,1\}$ for player II. Clearly η secures B.

4.2. $\omega + 1$ length games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$

In this subsection, we consider another kind of games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$.

For given $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, the game $\mathcal{G}_2(A)$ is defined as follows.

- Player I and player II alternately choose $i \in \{0,1\}$ to form $\alpha \in \{0,1\}^{\mathbb{N}}$.
- After α is formed, player I chooses $i \in \{0,1\}$.
- Player I wins $\mathcal{G}_2(A)$ if and only if $(\alpha, i) \in A$.

In this game, a strategy σ for player I is a pair (σ_0, σ_1) of functions $\sigma_0 : \bigcup_{n\in\mathbb{N}}\{0,1\}^{2n} \to \{0,1\}$ and $\sigma_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. By the strict fan theorem, we can regard, as well as in the last subsection, σ_1 as a function from $\{0,1\}^n$ to $\{0,1\}$ for some $n \in \mathbb{N}$.

A strategy for player II is a function $\tau : \bigcup_{n\in\mathbb{N}}\{0,1\}^{2n+1} \to \{0,1\}$, which can be regarded as an element of $\{0,1\}^{\mathbb{N}}$. Then an anti-strategy η for player I is a function from

$\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}$, which can be regarded a pair (η_0, η_1) of codes of continuous functions such that, for any strategy τ for player II, $(\eta_0|\tau, (\eta_1|\tau)(0)) \in_{II} \tau$. By the strict fan theorem, there is n such that for any strategies τ and τ', $\overline{\tau}n = \overline{\tau'}n$ implies $(\eta_1|\tau)(0) = (\eta_1|\tau')(0)$, and so we can regard η_1 as a function from $\{0,1\}^n$ to $\{0,1\}$.

Theorem 1 *For any $C \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_2(C)$ is predeterminate from the viewpoint of player I.*

Proof. For $i < \{0,1\}$, set $C_i = \{\alpha : (\alpha, i) \in C\}$. Assume that $\eta = (\eta_0, \eta_1)$ is an anti-strategy for player I securing C and η_1 can be regarded as a function from $\{0,1\}^n$ to $\{0,1\}$ for some n. Note that, in $G_{\{0,1\}^{\mathbb{N}}}(C_0 \cup C_1)$, η_0 is an anti-strategy for player I securing $C_0 \cup C_1$. Let σ_0 be a winning strategy for player I constructed in the proof of The intuitionistic determinacy theorem in $G_{\{0,1\}^{\mathbb{N}}}(C_0 \cup C_1)$. Set $P_{\sigma_0} = \{\alpha : \alpha \in_I \sigma_0\}$. Note that P_{σ_0} is a spread. By the proof of The intuitionistic determinacy theorem, for any $\alpha \in P_{\sigma_0}$, there exists a strategy δ for player II with $\eta_0|\delta = \alpha$. By the second axiom of continuous choice, there exists a code of continuous function ζ such that, for any strategy $\alpha \in P_{\sigma_0}$, $\zeta|\alpha$ is a strategy for player II with $\eta_0|(\zeta|\alpha) = \alpha$. By the strict fan theorem, there exists a natural number N such that, for any α and β in P_{σ_0}, $\overline{\alpha}N = \overline{\beta}N$ implies $\overline{(\zeta|\alpha)}n = \overline{(\zeta|\beta)}n$. Then we can define $\sigma_1 : P_{\sigma_0} \to \{0,1\}$ by $\sigma_1(\alpha) = \eta_1(\overline{(\zeta|\alpha)}n)$, since $\sigma_1(\alpha)$ is determined by $\overline{\alpha}N$. Define a new strategy $\sigma = (\sigma_0, \sigma_1)$ for player I in $\mathcal{G}_2(C)$. Then, for any $(\alpha, i) \in_I \sigma$, a strategy $\delta = \zeta|\alpha$ for player II satisfies $(\alpha, i) = (\eta_0|\delta, (\eta_1|\delta)(0))$, and so σ is a winning strategy for player I in $\mathcal{G}_2(C)$. $\square$

Comparing this theorem with the examples in the last subsection, we can conclude that predeterminacy depends *how* players construct the sequence rather than *what* sequence they do.

4.3. $\omega + 2$-length game in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^2$

Next we consider slightly longer games.

For a given set $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^2$, consider the following game $\mathcal{G}_3(A)$.

- First, player I and player II alternately choose $n \in \{0,1\}$ to form $\alpha \in \{0,1\}^{\mathbb{N}}$.
- After α is formed, player I chooses $i \in \{0,1\}$ and player II chooses $j \in \{0,1\}$.
- Player I wins if $(\alpha, \langle i, j\rangle) \in A$ and player II wins if player I does not win.

Similarly to the previous subsection, a strategy σ for player I is a pair (σ_0, σ_1), where σ_0 is a function $\bigcup_{n\in\mathbb{N}}\{0,1\}^{2n}$ to $\{0,1\}$ and where σ_1 is a function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}$. We can regard σ_1 as a function from $\{0,1\}^n$ to $\{0,1\}$ for some $n \in \mathbb{N}$.

A strategy τ for player II is a pair (τ_0, τ_1), where τ_0 is a function from $\bigcup_{n\in\mathbb{N}}\{0,1\}^{2n+1}$ to $\{0,1\}$ and where τ_1 is a function from $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ to $\{0,1\}$. Note that since τ_1 is continuous, its restriction $\tau_{1,i}$ to $\{0,1\}^{\mathbb{N}} \times \{i\}$ is also continuous and so we can regard τ_1 as a pair (τ_{10}, τ_{11}) of functions $\{0,1\}^{n_i}$ to $\{0,1\}$ for some n_i's.

Hence, the set of strategies for player II can be regarded as $\{0,1\}^{\mathbb{N}} \times \mathbb{N}$, and so an anti-strategy for player I can be regarded as a function η from $\{0,1\}^{\mathbb{N}} \times \mathbb{N}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}^2$ such that $\eta(\tau) \in_{II} \tau$ for each strategy τ for player II.

As in the case of $\mathcal{G}_1(X)$, we have the following examples. For any $s \in \{0,1\}^{<\mathbb{N}}$, let s' be the sequence $\langle s(0), s(2), ..., s(2n)\rangle$, where n is the maximal m with $2m < lh(s)$.

Example 3 Recall A_i defined in Example 1. Then the open game $\mathcal{G}_3(A')$ defined by $A' = \{(\alpha, \langle i, j\rangle) : \exists n[(\overline{\alpha}n)' \in A_j]\}$ is not predeterminate from the viewpoint of player I.

Example 4 Recall T_i defined in Example 2. Then the closed game $\mathcal{G}_3(B')$ defined by $B' = \{(\alpha, \langle i, j\rangle) : \forall n(\overline{\alpha}n)' \in T_j\}$ is not predeterminate from the viewpoint of player I.

5. Predeterminacy in the classical mathematics

In this section, we consider predeterminacy in the classical mathematics in order to investigate the role of classical principles in predeterminacy. Note that all the definitions and statements in this section are made in the classical mathematics which includes the countable axiom of choice.

Recall that, in the intuitionistic mathematics, an anti-strategy is a function η such that $\eta(\tau) \in_{II} \tau$ for each strategy τ for player II. We translate this definition into the classical mathematics, noticing that every function on $\mathbb{N}^{\mathbb{N}}$ is continuous in the intuitionistic mathematics:

Let $\mathcal{G}(X)$ be any of games treated in the previous sections. An *anti-strategy for player I in* $\mathcal{G}(X)$ is a continuous function which assigns $\alpha \in_{II} \tau$ to every *continuous* strategy τ for player II in $\mathcal{G}(X)$. An anti-strategy η for player I in $\mathcal{G}(X)$ *secures* X if $\eta(\tau) \in X$ for all continuous strategies τ for player II. $\mathcal{G}(X)$ is *predeterminate* from the viewpoint of player I if,

> if player I has an anti-strategy η securing X then player I has a winning strategy in $\mathcal{G}(X)$.

Note that the ordinary definition of determinacy statement can be seen as "if there is a function η such that $\eta(\tau) \in_{II} \tau$ and $\eta(\tau) \in X$ for *all* strategies τ for player II, then player I has a winning strategy in $\mathcal{G}(X)$."

For $X \subseteq \mathbb{N}^{\mathbb{N}}$, strategies for players in the game $G(X)$ can be regarded as functions $\mathbb{N}$ to $\mathbb{N}$, and so all the strategies are continuous. Therefore the condition "continuous" for strategies has no effect in games $G(X)$, but it does in the games $\mathcal{G}_1(X)$, $\mathcal{G}_2(X)$ and $\mathcal{G}_3(X)$. Moreover the continuity in the definition of anti-strategy is essential in the following discussion.

As mentioned in (Veldman 2004, 1.1), The intuitionistic determinacy theorem holds also in the classical mathematics. In particular, for all $A \subseteq \{0,1\}^{\mathbb{N}}$, $G_{\{0,1\}^{\mathbb{N}}}(A)$ is predeterminate from the viewpoint of player I in the classical mathematics.

Now we consider the predeterminacy of the games $\mathcal{G}_1(X)$, $\mathcal{G}_2(X)$ and $\mathcal{G}_3(X)$ which are defined in the last section, in the classical mathematics. Due to König's lemma, the classical counterpart of the strict fan theorem, also in the classical mathematics, a continuous function from $\{0,1\}^{\mathbb{N}} \to \{0,1\}$ or $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is given by its code η defined in Section 2. In particular, a strategy for player II in $\mathcal{G}_1(A)$ can be seen as a function $\tau : \{0,1\}^n \to \{0,1\}$ for some n and an anti-strategy for player I in $\mathcal{G}_2(A)$ can be seen as a pair (η_0, η_1) of a code η_0 of continuous function and $\eta_1 : \{0,1\}^m \to \{0,1\}$ for some m.

The game $\mathcal{G}_1(A)$ is not predeterminate from the viewpoint of player I, where A is defined in the proof of Example 1. For closed games, the situation differs: Whereas Example 2 is a closed game which is not predeterminate from the viewpoint of player I in the intuitionistic mathematics, we will show that there is no such closed game in the classical mathematics.

For $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$ and $s \in \{0,1\}^{<\mathbb{N}}$, η is an *anti-strategy for player I securing X above s* if η is an anti-strategy for player I such that, for each strategy τ for player II, $\eta(\tau) = (\alpha, i)$ satisfies $(s * \alpha, i) \in X$.

Note that, by the countable axiom of choice, player I has an anti-strategy securing X above s, if and only if, for any n and any strategy $\tau : \{0,1\}^n \to \{0,1\}$ for player II, there exists $\alpha \in \{0,1\}^{\mathbb{N}}$ such that $\langle s * \alpha, \tau(\overline{\alpha}n)\rangle \in X$.

Lemma 1 *If player I has an anti-strategy securing X above s, then player I has an anti-strategy securing X above either $s * \langle 0\rangle$ or $s * \langle 1\rangle$.*

Proof. Assume that, for contradiction, player I has an anti-strategy η securing X above s, but neither above $s * \langle 0\rangle$ nor above $s * \langle 1\rangle$. Then, for each $i < 2$, there exist strategies $\tau^i : \{0,1\}^{n_i} \to \{0,1\}$ such that, for every $\alpha \in \{0,1\}^{\mathbb{N}}$, $(s * \langle i\rangle * \alpha, \tau_i(\overline{\alpha}n_i)) \notin X$. Fix such τ^0 and τ^1. Take $n = \max\{n_0 + 1, n_1 + 1\}$. Then define $\tau : \{0,1\}^n \to \{0,1\}$ by

$$\tau(t) = \begin{cases} \tau^0(\langle t(1), ..., t(n_0)\rangle) & \text{if } t(0) = 0, \\ \tau^1(\langle t(1), ..., t(n_1)\rangle) & \text{otherwise.} \end{cases}$$

It is easy to see that $(s * \alpha, \tau(\overline{\alpha}n)) = (s * \langle k\rangle * \beta, \tau^k(\overline{\beta}n_k))$, where $k = \alpha(0)$ and where $\beta(m) = \alpha(m+1)$ for all m. By the assumption that η is an anti-strategy for player I securing X, $\eta(\tau) = (\alpha, \tau(\overline{\alpha}n))$ satisfies $(s * \alpha, \tau(\overline{\alpha}n)) \in X$. However, letting $\alpha = \langle i\rangle * \beta$, we can say $(s * \langle i\rangle * \beta, \tau^i(\overline{\beta}n_i)) = (s * \alpha, \tau(\overline{\alpha}n)) \in X$ which contradicts the choice of τ^i. $\square$

Theorem 2 *For closed $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_1(X)$ is predeterminate from the viewpoint of player I.*

Proof. Let $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$ be a closed set and $X_i = \{\alpha \in \{0,1\}^{\mathbb{N}} : (\alpha, i) \in X\}$. Then there exist $X'_i \subseteq \{0,1\}^{<\mathbb{N}}$ such that $\alpha \in X_i$ if and only if $\overline{\alpha}n \in X'_i$ for all n. Assume that player I has an anti-strategy securing X. Note that player I has an anti-strategy securing X above $\langle\rangle$. Define α by recursion as follows:

$$\alpha(n) = \begin{cases} 0 & \text{if player I has an anti-strategy securing } X \text{ above } \overline{\alpha}n, \\ 1 & \text{otherwise.} \end{cases}$$

By Lemma 1 and by induction, we can prove that player I has an anti-strategy securing X above $\overline{\alpha}n$ for all n. In particular, for all n, $\overline{\alpha}n \in X'_0 \cap X'_1$, and so α is a winning strategy for player I. $\square$

For the $\omega + 1$-length game $\mathcal{G}_2(X)$, we have the following theorem.

Theorem 3 *For any $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_2(X)$ is predeterminate from the viewpoint of player I.*

Proof. This can be proved in a similar way to Theorem 1. $\square$

Let us turn to the game $\mathcal{G}_3(X)$. There is an open game $\mathcal{G}_3(X)$ which is not predeterminate from the viewpoint of player I ($\mathcal{G}_3(A')$ defined in Example 3 enjoys this property). How about closed game? The author has not yet solved it.

6. Further problems

Predeterminacy of closed game $\mathcal{G}_3(X)$ in the classical mathematics The first problem the author is interested in is whether the closed games $\mathcal{G}_3(X)$ are predeterminate or not in the classical mathematics. It will be solved by analyzing the property of continuous functions in Cantor space.

Classical investigation of predeterminacy We can consider various formalizations of predeterminacy in the classical mathematics other than defined in Section 5, e.g.,

> If player I has an anti-strategy such that $\eta(\tau) \in A$ for each continuous strategy τ for player II, then player I has a *continuous* winning strategy in $\mathcal{G}(A)$.

Note that the italicized part is newly added. Again, in game $G(X)$ in $\mathbb{N}^{\mathbb{N}}$, this modification has no effect. However, we can easily find $X \subseteq \{0,1\}^{\mathbb{N}}$ which is not predeterminate in this sense but which is predeterminate in the sense of Section 5. The author expects that the investigation on these variations explicates how continuity confines functions on Baire space or Cantor space.

Constructive reverse mathematical analysis of predeterminacy Constructive reverse mathematics is a study to measure the strength of mathematical statements by nonconstructive principles using *constructive mathematics* as a base theory. Constructive mathematics is a mathematics which is based on the intuitionistic logic, but which does not adopt axioms introduced in Section 2. Therefore it is included both in the classical mathematics and in the intuitionistic mathematics.

(1) The role of the second axiom of continuous choice for predeterminacy Under the second axiom of continuous choice, predeterminacy implies determinacy. This implication needs only a fragment of the second axiom of continuous choice, and it is natural to ask exactly how strong fragments are required. If we measure the strength of fragments by the complexity of R in the axiom, the difficulty is in the reduction of general formulas of the form $\forall\alpha\exists\beta R(\alpha,\beta)$ to the form $\forall\tau\exists\sigma\forall\alpha(\alpha \in_I \sigma \wedge \alpha \in_{II} \tau \to R'(\alpha))$.

(2) Equivalences between predeterminacy and intuitionistic axioms (Veldman 200x) proposed intuitionistic second order arithmetic and proved that the predeterminacy of open subsets of II-finitary branching spreads in $\mathbb{N}$ is equivalent to the strict fan theorem over the system BIM, which corresponds a popular classical base theory RCA_0 in the field called Friedman-Simpson's reverse mathematics (cf. (Simpson 1999)). The author of the present paper is now looking for similar equivalences beyond open sets. The first task in this direction is to find a suitable intuitionistic axiom to compare with. One of candidates is *almost-fan-theorem* proposed in (Veldman 2001).

(3) The role of *LEM* for predeterminacy In the proof of Theorem 2, we use the law of excluded middle. It seems impossible to prove it without this classical law, because we have B of Example 2 in the intuitionistic mathematics. The next natural question is what fragment of the classical law (such as the excluded middle or double negation elimination) is necessary and sufficient for determinacy or predeterminacy statements. (Akama, et al. 2004) discovered a hierarchy consisting of these fragments over Heyting arithmetic HA, which is the constructive counterpart to Peano arithmetic. The author of present paper tries to measure predeterminacy or determinacy statements along this hierarchy.

(4) Equivalences between predeterminacy and classical axioms Since we treat predeterminacy also in the classical mathematics, it is natural to consider Friedman-Simpson's reverse mathematical study of predeterminacy. Using constructive mathematics as a base theory, we can make a finer reverse mathematical study of predeterminacy.

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