

Determinacy and Π_1^1 transfinite recursion along ω

Takako Nemoto
 University of Bern
 Bern, Switzerland
 nemototakako@gmail.com

Abstract

It is known that almost all important set existence axioms in second order arithmetic can be characterized by determinacy schemes of infinite games. In this article, we sketch the proof of the equivalence between Π_1^1 transfinite recursion along ω and determinacy scheme.

1 Introduction

It is shown in [4], [6], [7], [10], [11], [12] and [13] that almost all important set existence axioms in reverse mathematics program can be characterized by determinacy schemes of infinite games, as summarized in the table, where the characterizing set existence axioms in the left row are equivalent to determinacy schemes in the same line over suitable base theory, e. g., RCA_0 , RCA_0^* .

Despite these “accordance” in weak systems of reverse mathematics, recent studies are unveiling some “discordance” between stronger set existence axioms and determinacy schemes in second order arithmetic. For example, in [5]¹, it is proved $\Pi_{n+2}^1\text{-CA}_0 \vdash (\Sigma_3^0)_n\text{-Det}$ and $\Delta_{n+2}^1\text{-CA}_0 \not\vdash (\Sigma_3^0)_n\text{-Det}$, whereas, in [3], it is proved $\Delta_1^1\text{-Det} \not\vdash \Delta_2^1\text{-comprehension}$. In [5], it is also proved $\text{Z}_2 \not\vdash (\Sigma_3^0)_\omega\text{-Det}$. In other words, the series of determinacy schemes and that of comprehension schemes diverge beyond some point. Since $\Pi_1^1\text{-TR}_0$, $\Sigma_1^1\text{-ID}_0$ etc. have somehow different flavor from comprehension schemes, the diverging point seems to be located around but above $\Pi_1^1\text{-CA}_0$.

Therefore, in order to see the relationship between the two series, it seems quite meaningful to investigate this region, i. e., the relationship between determinacy scheme and restrictions of Π_1^1 transfinite recursion. In this paper, we focus on the first important instance of the restrictions, namely, Π_1^1 transfinite recursion along ω .

Main Theorem. $\Pi_1^1\text{-CA}_0^+$ and $(\Sigma_1^0)_\omega\text{-Det}$ are equivalent over RCA_0 .

This is a refinement of the result in [12] and the Π_1^1 analogue of the result in [8]:

Fact. ACA_0^+ and $(\Sigma_1^0)_\omega\text{-Det}^*$ are equivalent over RCA_0 .

In this article, we sketch the proof of the main theorem. We also mention some prospective generalizations of this theorem as a conjecture at the end.

¹ $(\Sigma_3^0)_n\text{-Det}$ and $(\Sigma_3^0)_\omega\text{-Det}$ in our notation are denoted by $n\text{-}\Pi_3^0\text{-DET}$ and $n\text{-}\Pi_3^0\text{-DET}$ respectively, in [5].

Subsystem of second order arithmetic	Determinacy in $2^{\mathbb{N}}$ (Det*)	Determinacy in $\mathbb{N}^{\mathbb{N}}$ (Det)
WKL_0^*	Δ_1^0 Σ_1^0	-
WKL_0	$\text{Bisep}(\Delta_1^0, \Sigma_1^0)$	-
ACA_0	$(\Sigma_1^0)_2$	-
ATR_0	Δ_2^0 Σ_2^0	Δ_1^0 Σ_1^0
$\text{ATR}_0 + \Sigma_1^1$ induction	$\text{Bisep}(\Delta_1^0, \Sigma_2^0)$	
$\Pi_1^1\text{-CA}_0$	$\text{Bisep}(\Sigma_1^0, \Sigma_2^0)$ \vdots $\text{Sep}(\Sigma_1^0, \Sigma_2^0)$	$\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ $(\Sigma_1^0)_2$
$\Pi_1^1\text{-TR}_0$	$\text{Bisep}(\Delta_2^0, \Sigma_2^0)$	Δ_2^0
$\Sigma_1^1\text{-ID}_0$	$(\Sigma_2^0)_2$	Σ_2^0
\vdots	\vdots	\vdots
$[\Sigma_1^1]^k\text{-ID}_0$	$(\Sigma_2^0)_{k+1}$	$(\Sigma_2^0)_k$
\vdots	\vdots	\vdots
$[\Sigma_1^1]^{\text{TR}}\text{-ID}_0$	Δ_3^0	Δ_3^0

2 Preliminaries

2.1 Subsystems of second order arithmetic

We work in an usual language L_2 of second order arithmetic. Most notations and definitions in this paper are fairly standard. The readers can refer to [10].

Definition 2.1. For any formula θ , let $H_\theta((Y, \prec_Y), Z)$ be the following formula:

$$\forall j \forall k (\langle k, j \rangle \in Z \leftrightarrow \theta(k, (Z)^j)),$$

where $(Z)^j = \{\langle m, i \rangle \in Z : i \prec_Y j\}$. For any class Γ of formulae, Γ *transfinite recursion*, denoted by Γ -TR, is the following scheme:

$$\forall(Y, \prec_Y)(\text{WO}(Y, \prec_Y) \rightarrow \exists Z H_\theta((Y, \prec_Y), Z)),$$

where $\text{WO}(Y, \prec_Y)$ is a Π_1^1 formula stating the wellorderedness of (Y, \prec_Y) , and where $\theta(x, Y)$ is a Γ formula in which Z does not occur freely. The system $\Pi_1^1\text{-TR}_0$ consists of RCA_0 plus the schema of Π_1^1 transfinite recursion.

For a wellordering (W, \prec) , the schema of Γ transfinite recursion along (W, \prec) , denoted by $\Gamma\text{-TR}(W, \prec)$ is as follows:

$$\exists Z H_\theta((W, \prec), Z),$$

where θ is any Γ formula in which Z does not occur freely. In particular, $\Gamma\text{-TR}(\mathbb{N}, <)$ is denoted by $\Gamma\text{-TR}\omega$. The system $\Pi_1^1\text{-CA}_0^+$ consists of RCA_0 plus $\Pi_1^1\text{-TR}\omega$.

This is Π_1^1 analogue of ACA_0^+ (cf. [10, X.3] and [9]). $\Pi_1^1\text{-CA}_0^+$ is between $\Pi_1^1\text{-CA}_0$ and $\Pi_1^1\text{-TR}_0$. *Remark 2.2.* Over RCA_0 , $\Pi_1^1\text{-TR}$ is equivalent to $\Sigma_1^1\text{-TR}$ and $\Pi_1^1\text{-TR}\omega$ is equivalent to $\Sigma_1^1\text{-TR}\omega$.

2.2 Games and determinacy

Let X be either \mathbb{N} or $\{0, 1\}$. A *game in $X^\mathbb{N}$* is given by a formula $\varphi(f)$ with a distinguished function variable $f \in X^\mathbb{N}$. It is played as follows: Two players, say player I and player II, alternately choose $x \in X$ to form $f \in X^\mathbb{N}$. Player I wins if and only if $\varphi(f)$ holds and Player II wins otherwise. In this article, we assume that player I is male and that player II is female.

A *strategy* for player I (or II) is a function which assigns an element of X to each even-length (resp. odd-length) finite sequence from X . A strategy for a player is *winning* in a game $\psi(f)$ if the player wins $\psi(f)$ as long as he or she plays following it. A game is *determinate* if one of the players has a winning strategy. We regard a class Γ of formulae with a distinguished function variable as a class of games. Γ *determinacy* asserts that every game in Γ is determinate. In this article, Γ determinacy in $\mathbb{N}^\mathbb{N}$ is denoted by $\Gamma\text{-Det}$ and Γ determinacy in $2^\mathbb{N}$ is denoted by $\Gamma\text{-Det}^*$.

For a game $\psi(f)$ and $s \in X^{<\mathbb{N}}$, a strategy σ for a player is *s-winning* in $\psi(f)$ if the player always wins $\psi(f)$ in any play f with $s \subset f$ in which he or she follows σ at all steps after $|s|$. A player *wins $\psi(f)$ at s* if he or she has a winning s -strategy in $\psi(f)$.

The following is a determinacy scheme to each class in the hierarchy between Σ_1^0 and Δ_2^0 , which is motivated by Hausdorff's difference hierarchy (cf. [1]).

Definition 2.3 ($(\Sigma_n^0)_{(Y, \prec)}$ determinacy). A sequence $\langle \varphi(n, f) : n \in \mathbb{N} \rangle$ with variable $f \in X^\mathbb{N}$ of formulae is called (Y, \prec) -increasing if $\varphi(n, f) \rightarrow \varphi(m, f)$ holds for all $n \prec m$ and $f \in X^\mathbb{N}$. A $(\Sigma_n^0)_{(Y, \prec)}$ game is defined by $(\exists \text{ even } x)(\varphi(x', f) \wedge \neg \varphi(x, f))$ with a (Y, \prec) -increasing sequence of Σ_n^0 formulae, where x' is \prec -successor of x .

The scheme of $(\Sigma_n^0)_{(Y, \prec)}$ determinacy in $X^\mathbb{N}$ asserts that every $(\Sigma_n^0)_{(Y, \prec)}$ game in $X^\mathbb{N}$ is determinate. In particular, we call $(\Sigma_n^0)_{(\mathbb{N}, <)}$ game as $(\Sigma_n^0)_\omega$.

2.3 Π_1^1 -TR₀ and Δ_2^0 determinacy

In [12], the following theorem is proved. Actually, it is easy to see that the base theory ACA₀ can be replaced to RCA₀.

Theorem 2.4. *Over ACA₀, Π_1^1 -TR and Δ_2^0 -Det are equivalent.*

The following lemma is equivalent to [12, Theorem 5.2], which plays an important role in the proof of this theorem.

Lemma 2.5. *For any Δ_2^0 game $\psi(f)$, we can find a Π_0^0 formula $\theta(x, i, y)$ and (Y, \prec) such that ACA₀ proves the following.*

- WO(Y, \prec);
- $(\forall f \in \mathbb{N}^{\mathbb{N}})((x, i) \prec^* (y, j) \wedge \exists n \theta(x, i, f[n])) \rightarrow \exists n \theta(y, j, f[n])$ (increasing along (Y, \prec));
- $(\forall f \in \mathbb{N}^{\mathbb{N}})(\psi(f) \leftrightarrow \exists x \in Y(\exists n \theta(x, 1, f[n]) \wedge \neg \exists n \theta(x, 0, f[n])))$,

where $(Y \times 2, \prec)$ is a wellordering defined by

$$(x, i) \prec^* (y, j) \leftrightarrow x \prec y \vee (x = y \wedge i < j).$$

From this lemma, each Δ_2^0 game can be expressed by an increasing sequence of Σ_1^0 formulae (coded by a single Σ_1^0 formula) along some wellordering. This is a formalization of [1, §30.VI. Theorem 1], which states that every Δ_2^0 set belongs to some level of difference hierarchy starting from Σ_1^0 . Then, it is natural to expect that Π_1^1 transfinite recursion along a certain wellordering (Y, \prec) corresponds to $(\Sigma_1^0)_{(Y, \prec)}$ -Det. In the following section, we show that this is the case for $(Y, \prec) = (\mathbb{N}, <)$.

3 Π_1^1 -CA₀⁺ and $(\Sigma_1^0)_\omega$ -Det

In this section, we sketch a proof of the equivalence between Π_1^1 -TR ω and $(\Sigma_1^0)_\omega$ -Det. All proofs are refinements of those in [12].

The following lemma is easy to prove.

Lemma 3.1. *For any Π_1^0 (or Σ_1^0) game $\psi(f)$ in $\mathbb{N}^{\mathbb{N}}$, there exists a Σ_1^1 (resp. Π_1^1) formula $\eta(x, f)$ such that RCA₀ proves that player I (resp. II) wins $\psi(f)$ at $s \in \mathbb{N}^{\mathbb{N}}$ iff $\eta(s, f)$ holds.*

First, we see the implication from Π_1^1 -TR ω to determinacy.

Theorem 3.2. *Π_1^1 -CA₀⁺ proves $(\Sigma_1^0)_\omega$ -Det.*

Sketch of proof: Let $\langle \varphi(n, f) : n \in \mathbb{N} \rangle$ be an $(\mathbb{N}, <)$ -increasing sequence of Σ_1^0 formulae. We show that a game $\eta(f) \equiv \exists n(\varphi(2n+1, f) \wedge \neg \varphi(2n, f))$ is determinate. By Normal form theorem [10, Lemma II.2.7] for Σ_1^0 , there exists a Π_0^0 formula θ such that $\exists m \theta(n, f[m]) \leftrightarrow \varphi(n, f)$ for all m and $f \in \mathbb{N}^{\mathbb{N}}$. Then, Σ_1^1 -TR ω , which is equivalent to Π_1^1 -TR ω by Remark 2.2, yields the following set W :

$$\begin{aligned} \langle s, 2n+1 \rangle \in W &\leftrightarrow \theta(2n+1, s) \wedge \text{player I wins } \varphi_1(2n+1, f), \\ \langle s, 2n+2 \rangle \in W &\leftrightarrow \text{player II wins } \varphi_2(2n, f), \text{ and} \end{aligned}$$

where $\varphi_1(2n+1, f) \equiv \forall m(\neg \theta(2n, f[m]) \vee \forall k < n(\langle 2k+2, f[m] \rangle \notin (W)^{2n+1}))$ and where $\varphi_2(2n, f) \equiv \exists m \exists k < n(\langle 2k+1, f[m] \rangle \in (W)^{2n})$. Note that this construction can be done by Σ_1^1 by Lemma 3.1. Define new Σ_1^0 game $\psi(f) \equiv \exists n \exists m(\langle f[m], 2n+1 \rangle \in W)$. Then, we can prove that

- If player I wins $\psi(f)$, player I also wins $\eta(f)$.
- If player II wins $\psi(f)$, player II also wins $\eta(f)$.

By Σ_1^0 -Det, which is proved in Π_1^1 -CA₀, the determinacy of $\eta(f)$ is implied. \square

Next, we see the converse.

Theorem 3.3. *Over RCA₀, $(\Sigma_1^0)_\omega$ -Det implies Π_1^1 -TR ω .*

Sc-ketch of proof: Instead of Π_1^1 -TR ω , we prove that $(\Sigma_1^1)_\omega$ -Det implies Σ_1^1 -TR ω . Since Δ_1^0 -Det implies arithmetical comprehension by [10, Lemma V.8.5], we can work in ACA₀. Let $\phi(x, X)$ be a Σ_1^1 formula. By [10, Lemma V.1.4], there exists a Π_0^0 formula $\theta(x, f, X)$ such that $\forall X \forall x (\psi(x, X) \leftrightarrow \exists f \theta(x, f[m], X[m]))$. We consider a game with the following property:

- Player II wins it;
- Player II's winning strategy in it yields Z such that $H_\theta((\mathbb{N}, <), Z)$.

Such a game can be defined as follows:

1. At the first step, player I asks whether $\langle n, i \rangle \in Z$ or not by playing $\langle n, i, 0 \rangle$.
2. Set $\langle m, j \rangle := \langle n, i \rangle$. Then, player II answers “yes” by playing $\langle m, j, 2 \rangle$, or “no” by playing $\langle m, j, 1 \rangle$.
If her answer is yes, go to 3, and otherwise go to 4.
3. Player II constructs the characteristic function $\chi_{(Z)^j}$ of $(Z)^j$ and a witness f such that $\forall k \theta(m, f[m], (Z)^j[m])$.

player I	player II
*	$\langle \chi_{(Z)^j}(0), f(0) \rangle$
*	$\langle \chi_{(Z)^j}(1), f(1) \rangle$
*	$\langle \chi_{(Z)^j}(2), f(2) \rangle$
\vdots	\vdots

If player I attacks player II's decision on the value of $\chi_{(Z)^j}(\langle m_0, j_0 \rangle)$ with $j_0 < j$ by playing $\langle m_0, j_0, 0 \rangle$,

- If $\chi_{(Z)^j}(\langle m_0, j_0 \rangle) = 0$, set $\langle m, j \rangle := \langle m_0, j_0 \rangle$ and go to 4.
- If $\chi_{(Z)^j}(\langle m_0, j_0 \rangle) = 1$, set $\langle m, j \rangle := \langle m_0, j_0 \rangle$ and go to 3.

If there is no such $\langle m_0, j_0 \rangle$, then

- Player I wins if f and $\chi_{(Z)^j}$ satisfy $\exists k \neg \theta(m, f[k], (Z)^j[m])$.
- Player II wins if f and $\chi_{(Z)^j}$ satisfy $\forall k \theta(m, f[k], (Z)^j[m])$.

4. The same as 3, but the roles of player I and player II in 3 are exchanged.

We can see that the winning condition of this game can be expressed by $(\Sigma_1^0)_\omega$, by describing the argument between players on membership of $(Z)^n$ with Σ_1^0 formula. Player I does not win this game, because if he had a winning strategy, then it would yield a winning strategy for player II in the same game, which is impossible. Therefore player II has a winning strategy. For any player II's winning strategy τ , it is also provable that τ always gives a correct information on the membership of Z , i. e., $\tau(\langle n, i, 0 \rangle) = \langle n, i, 2 \rangle$ iff $\exists f \forall m \theta(\langle n, i \rangle, f[m], (Z)^i[m])$, which means Z is yielded by τ by arithmetical comprehension. \square

Corollary 3.4. *Over RCA₀, Π_1^1 -TR ω and $(\Sigma_1^0)_\omega$ -Det are equivalent.*

4 Related & future works

As mentioned in Introduction, in [8], it is proved that $\Pi_1^0\text{-Det}\omega$ is equivalent to $(\Sigma_1^0)_\omega\text{-Det}^*$, i. e., ACA_0^+ over RCA_0 . In [2], a system $\Pi_\beta^0\text{-CA}_0$ which consists of RCA_0 plus “ β is well-ordered” and $\Pi_1^0\text{-TR}\beta$ for a presentation β of computable ordinal. It is natural to expect that the results in [8] and this article are generalized to such systems and $(\Sigma_1^0)_\beta$ determinacy as follows:

Conjecture Let β be a presentation of limit computable ordinal. In the system $\text{RCA}_0 + “\beta$ is well-ordered”, the following are provable.

1. $\Pi_1^0\text{-TR}\beta$ is equivalent to $(\Sigma_1^0)_\beta\text{-Det}^*$.
2. $\Pi_1^1\text{-TR}\beta$, $(\Sigma_1^0)_\beta\text{-Det}$ and $\text{Bisep}((\Sigma_1^0)_\beta, \Sigma_2^0)\text{-Det}^*$ are pairwise equivalent.

For the definition of $\text{Bisep}(-, -)$, see [6].

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