On the independence of premiss axiom and rule

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Abstract

In this paper, we deal with a relationship among the law of excluded middle, the double negation elimination and the independence of premiss rule (IPR) for (many-sorted) intuitionistic of predicate logic. After giving a general machinery, as corollaries, we give several examples of extensions of **HA** and $\mathbf{HA}^\omega$ which are closed under IPR but do not derive the independence of premiss axiom (IP).

Keywords: independence of premiss axiom, independence of premiss rule, non-classical axioms, non-constructive axioms, functional realizability interpretation
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1 Introduction

It has been proved that intuitionistic first order arithmetic **HA** is closed under the *independence of premiss rule* (IPR)

$$\vdash A \to \exists x B \Longrightarrow \vdash \exists x(A \to B),\ (x \text{ not free in } A,\ A \text{ Rasiowa-Harrop}),$$

using, for example, Kleene or Aczel slash relation for sentence A and $\exists x B$ (cf. [8, Ch.3.5.16]). Troelstra [7, 3.7] showed that $\mathbf{HA}^\omega$ is closed under the following IPR^ω in $\mathscr{L}(\mathbf{HA}^\omega)$

$$\vdash A \to \exists x^\sigma B \Longrightarrow \vdash \exists x^\sigma(A \to B),\ (x \text{ not free in } A,\ A \text{ Rasiowa-Harrop}),$$

for negative A using the modified realizability interpretation. In [2], we gave the class $\mathcal{S}$ of schemata such that $\mathbf{HA} + \Lambda$ and $\mathbf{HA}^\omega + \Lambda$ are closed under IPR and IPR^ω for any set Λ of schemata in $\mathcal{S}$. It is trivial that if an extension $\mathbf{T}$ of **HA** or $\mathbf{HA}^\omega$ derives the following *independence of premiss axiom*

$\mathrm{IP}^{(\omega)}$**:** $(A \to \exists x^{(\sigma)} B) \to \exists x^{(\sigma)}(A \to B)$ (x is not free in A, A Rasiowa-Harrop),

then $\mathbf{T}$ is closed under $\mathrm{IPR}^{(\omega)}$. So a natural question arises: What theory $\mathbf{T}$ is closed under $\mathrm{IPR}^{(\omega)}$, but does not derive $\mathrm{IP}^{(\omega)}$?

In this paper, we give several examples of extensions of $\mathbf{HA}$ and $\mathbf{HA}^{\omega}$ which is closed under $\mathrm{IPR}^{(\omega)}$, but does not derive $\mathrm{IP}^{(\omega)}$. In Section 2, we explain a general relationship among the *law of excluded middle* (LEM) and the *double negation elimination* (DNE) for a class of formulae, and IPR. We show that

> if $\mathbf{T} \nvdash \Gamma$-LEM and $\mathbf{T} \vdash \Gamma$-DNE then $\mathbf{T}$ is not closed under IPR, and hence, in particular, $\mathbf{T}$ does not derive IP,

for a class Γ of formulae and a theory $\mathbf{T}$ with a certain property. We also show that Σ_n-LEM is a set of formulae in $\mathcal{S}$, and hence $\mathbf{HA} + \Sigma_n$-LEM is closed under IPR. Since Σ_{n+1}-DNE does not imply Σ_{n+1}-LEM over $\mathbf{HA}$ (cf. [1]), we have that $\mathbf{HA} + \Sigma_{n+1}$-DNE is not closed under IPR. As a corollary, we have that $\mathbf{HA} + \Sigma_n$-LEM, which is a subtheory of $\mathbf{HA} + \Sigma_{n+1}$-DNE (cf. [1]), does not derive IP. In Section 3, we prove that extensions of $\mathbf{HA}^{\omega}$ with several non-classical axiom schemata in $\mathcal{S}$ (cf. [2]) do not derive IP by combining the interpretation $[\![\cdot]\!]$ from $\mathscr{L}(\mathbf{HA}^{\omega})$ to $\mathscr{L}(\mathbf{EL})$ defined in [7, 2.6.2], and the functional realizability interpretation [7, 3.3.1–3.3.13]. As a by-product, we show that $[\![\cdot]\!]$ interprets any instance of a closed schema in $\mathscr{L}(\mathbf{EL})$, as a schema in $\mathscr{L}(\mathbf{HA}^{\omega})$, into an instance of the same schema in $\mathscr{L}(\mathbf{EL})$.

2 A general machinery and semi-classical principles

2.1 A general machinery

We use the standard language of (many-sorted) first-order predicate logic containing $\wedge, \vee, \to, \bot, \forall$ and $\exists$ as primitive logical operators.

Definition 1. We introduce certain predicate symbols $\nu_1, \nu_2, \nu_3, \ldots$ (being outside of our standard language), called *place holders*, to deal with schemata as syntactic objects similar to formulae. *Schemata* are inductively defined by

1. a prime formula is a schema;
2. if ν is an n-ary place holder and $t_1, \ldots, t_n$ are terms, then $\nu(t_1, \ldots, t_n)$ is a schema;
3. if α and β are schemata, then $\alpha \circ \beta$ is a schema for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. if α is a schema, then $Qx\alpha$ is a schema for $Q \in \{\forall, \exists\}$.

Formulae are schemata without place holders.

For example, the *induction schema* is given by a schema

$$\nu(0) \wedge \forall x(\nu(x) \rightarrow \nu(Sx)) \rightarrow \forall x \nu(x),$$

where ν is a unary place holder.

Definition 2. Let α be a schema, and let $B_1, \ldots, B_k$ be formulae. Let $\nu_1, \ldots, \nu_k$ be place holders, and let $\vec{x_1}, \ldots, \vec{x_k}$ be sequences of variables with lengths the arities of $\nu_1, \ldots, \nu_k$, respectively. Then a schema $\alpha[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k]$ is defined by

1. $P[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k] \equiv P$ for P prime;
2. $\nu(t_1, \ldots, t_n)[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k] \equiv B_i[y_1/t_1, \ldots, y_n/t_n]$ if $\nu \equiv \nu_i$ and $\vec{x_i} \equiv y_1, \ldots, y_n$, and $\nu(t_1, \ldots, t_n)$ otherwise;
3. $(\alpha \circ \beta)[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k]$
$\equiv \alpha[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k] \circ \beta[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k]$
for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. $(Qx\alpha)[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k] \equiv Qx(\alpha[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k])$ for $Q \in \{\forall, \exists\}$.

We simply write $\alpha[\nu_1/B_1, \ldots, \nu_k/B_k]$ or even $\alpha[B_1, \ldots, B_k]$ for $\alpha[\nu_1/\lambda\vec{x_1}.B_1, \ldots, \nu_k/\lambda\vec{x_k}.B_k]$ whenever possible, if it does not cause confusion. An *instance* of a schema α with place holders exactly $\vec{\nu}$ is a formula $\alpha[\vec{\nu}/\vec{B}]$ or $\alpha[\vec{B}]$.

Remark 3. In using the substitution notation $\alpha[\vec{\nu}/\vec{B}]$, we shall assume that $\vec{B}$ are free for $\vec{\nu}$ in α, respectively, or we assume that a suitable renaming of bound variables is carried out.

For a set Λ of schemata and a formula A, we write $\Lambda \vdash A$ to express that a set of instances of schemata in Λ derives A.

Definition 4. The class $\mathcal{RH}$ of *Rasiowa-Harrop schemata* is defined as follows. Let P range over prime formulae, η and η' over $\mathcal{RH}$, and α over arbitrary schemata. Then $\mathcal{RH}$ is inductively generated by the clause

$$P, \eta \wedge \eta', \forall x \eta, \alpha \to \eta \in \mathcal{RH}.$$

A theory $\mathbf{T}$ is closed under the *independence of premiss rule* (IPR) if

$$\mathbf{T} \vdash A \to \exists B \Longrightarrow \mathbf{T} \vdash \exists x(A \to B),$$

where A is a Rasiowa-Harrop formula and x does not occur freely in A. The *independence of premiss axiom* (IP) is given as follows:

$$(A \to \exists x B) \to \exists x(A \to B),$$

where A is a Rasiowa-Harrop formula and x does not occur freely in A. Note that a theory $\mathbf{T}$ deriving IP is closed under IPR.

Let Γ be a class of formulae. Then semi-classical principles Γ-LEM and Γ-DNE for Γ are defined as follows:

Γ-LEM: $A \vee \neg A$, where $A \in \Gamma$;

Γ-DNE: $\neg\neg A \to A$, where $A \in \Gamma$.

In what follows, we assume that a class Γ of formulae satisfies the following two conditions:

1. $A \in \Gamma \Rightarrow A[x/y] \in \Gamma$
2. $\exists x A \in \Gamma \Rightarrow \exists y A[x/y] \in \Gamma$

For each class Γ of formulae, the class $\exists\Gamma$ consists of formulae of the form $\exists x A$ with a formula A in Γ.

Theorem 5. *If* $\mathbf{T}$ *is a theory such that* $\mathbf{T} \vdash \Gamma$-LEM, $\mathbf{T} \vdash \exists\Gamma$-DNE, *and that* $\mathbf{T}$ *is closed under* IPR, *then* $\mathbf{T} \vdash \exists\Gamma$-LEM.

Proof. Assume that $\mathbf{T}$ is a theory such that $\mathbf{T} \vdash \Gamma$-LEM, $\mathbf{T} \vdash \exists\Gamma$-DNE and that $\mathbf{T}$ is closed under IPR. Then, for each formula A in Γ, we have $\mathbf{T} \vdash \neg\neg\exists x A(x) \to \exists x A(x)$. Since $\neg\neg\exists x A(x)$ is Rasiowa-Harrop and $\mathbf{T}$ is closed under IPR, we have $\mathbf{T} \vdash \exists x(\neg\neg\exists y A[x/y] \to A(x))$. Note that $\mathbf{T} \vdash A(x) \vee \neg A(x)$. Then, since $A(x)$ implies $\exists x A(x)$ and $\neg A(x)$ implies $\neg\exists y A[x/y]$, we have $\mathbf{T} \vdash \exists x A(x) \vee \neg\exists x A(x)$. □

Corollary 6. *Let* $\mathbf{T}$ *be a theory such that* $\mathbf{T} \vdash \Gamma$-LEM.

1. *If* $\mathbf{T} \nvdash \exists\Gamma$-LEM *and* $\mathbf{T} \vdash \exists\Gamma$-DNE*, then* $\mathbf{T}$ *is not closed under* IPR*, and hence, in particular, such* $\mathbf{T}$ *does not derive* IP*.*

2. *If* $\mathbf{T} \nvdash \exists\Gamma$-LEM *and* $\mathbf{T}$ *is closed under* IPR*, then* $\mathbf{T} \nvdash \exists\Gamma$-DNE*.*

2.2 Heyting arithmetic and semi-classical principles

In this subsection, we apply the results of the previous subsection to intuitionistic first order arithmetic $\mathbf{HA}$ (cf. [7, 1.3.3–1.3.4] and [8, 3.3.1]) and some semi-classical axioms in [1].

In the language $\mathscr{L}(\mathbf{HA})$ of $\mathbf{HA}$, a Δ_0 formula is built up from atomic formulae by means of $\wedge$, $\vee$, $\to$, $\neg$, $\forall x < t$ and $\exists x < t$. A Δ_0 formula is sometimes called also a Σ_0 or Π_0 formula. A Σ_{n+1} formula has the form $\exists x A$ with a Π_n formula A, and a Π_{n+1} formula has the form $\forall x A$ with a Σ_n formula A.

Akama et al. [1, §3] proved the following.

Lemma 7.

1. $\mathbf{HA} \vdash \Sigma_n$-LEM $\to \Sigma_n$-DNE*;*

2. $\mathbf{HA} \vdash \Sigma_{n+1}$-DNE $\to \Sigma_n$-LEM*;*

3. $\mathbf{HA} \nvdash \Sigma_n$-DNE $\to \Sigma_n$-LEM*;*

4. $\mathbf{HA} \nvdash \Sigma_n$-LEM $\to \Sigma_{n+1}$-DNE*.*

By the results of the previous section, we have the following:

Corollary 8. $\mathbf{HA} + \Sigma_{n+1}$-DNE *is not closed under* IPR*.*

Proof. Let Γ be the class of Π_n formulae. Then $\exists\Gamma$-LEM and $\exists\Gamma$-DNE are Σ_{n+1}-LEM and Σ_{n+1}-DNE, respectively. By Lemma 7 (3), $\mathbf{HA} + \Sigma_{n+1}$-DNE does not prove Σ_{n+1}-LEM. Hence, by Corollary 6 (1), $\mathbf{HA} + \Sigma_{n+1}$-DNE is not closed under IPR. □

Definition 9. The classes $\mathcal{S}$ and $\mathcal{W}$ of schemata are simultaneously defined as follows. Let η range over $\mathcal{RH}$, ν over expressions $\nu_k(t_1 \dots t_n)$ (ν_k an n-ary place holder symbol), γ and γ' over $\mathcal{S}$, and δ and δ' over $\mathcal{W}$. Then $\mathcal{S}$ and $\mathcal{W}$ are inductively generated by the clauses

$$\eta, \nu, \gamma \wedge \gamma', \gamma \vee \gamma', \forall x\gamma, \exists x\gamma, \delta \to \gamma \in \mathcal{S};$$
$$\nu, \delta \wedge \delta', \delta \vee \delta', \forall x\delta, \exists x\delta, \gamma \to \delta \in \mathcal{W}.$$

Lemma 10. *If Λ is a set of schemata in $\mathcal{S}$, then* $\mathbf{HA} + \Lambda$ *is closed under* IPR.

Proof. See [2, Theorem 3.1]. □

Corollary 11. $\mathbf{HA} + \Sigma_n$-LEM *is closed under* IPR*, but does not derive* IP.

Proof. It is easy to see that for each Δ_0 formula $A(x_1, ..., x_n)$, there exists a term $t(x_1, ..., x_n)$ such that

$$\mathbf{HA} \vdash \forall x_1, ..., x_n(A(x_1, ..., x_n) \leftrightarrow t(x_1, .., x_n) = 0)$$

Hence an instance of Σ_n-LEM is equivalent to a formula of the form

$$\exists x_n \forall x_{n-1} ... Qx_1(t(x_1, ..., x_n) = 0) \vee \neg \exists x_n \forall x_{n-1} ... Qx_1(t(x_1, ..., x_n) = 0),$$

which is in the class $\mathcal{S}$. Therefore $\mathbf{HA} + \Sigma_n$-LEM is closed under IPR by Lemma 10.

By Corollary 8, $\mathbf{HA} + \Sigma_{n+1}$-DNE, which derives Σ_n-LEM by Lemma 7 (2), is not closed under IPR, and hence does not derive IP. If $\mathbf{HA} + \Sigma_n$-LEM derived IP, then $\mathbf{HA} + \Sigma_{n+1}$-DNE would derive IP, which is impossible. □

3 Non-classical axioms

In this section, we consider intuitionistic finite type arithmetic $\mathbf{HA}^\omega$, which is called $\mathbf{HA}^\omega_\to$ in [8, Ch.9 1.18] and its extension with non-classical axioms.

The following schemata are in the class $\mathcal{S}$ (cf. [2, Theorem 3.1]), and so $\mathbf{HA}^\omega$ extended by any subset of them is closed under IPR^ω.

FAN**:** $\forall a^1 \exists x^0 \nu(\overline{(a \sqcap 1)}x) \to \exists z^0 \forall a^1 \exists y^0 (y \leq z \wedge \nu(\overline{(a \sqcap 1)}y)$,
where $(a \sqcap 1)n \equiv \min\{an, 1\}$.

$\mathrm{BI_M}$: $[\forall a^1 \exists x^0 \nu(\overline{a}x) \wedge \forall n^0 \forall m^0 (\nu(n) \to \nu(n * m)) \wedge \forall n^0 (\forall y^0 \nu(n * \langle y \rangle) \to \nu(n))] \to \nu(\langle\rangle).$

$\mathrm{AC}_{\sigma,\tau}$: $\forall x^\sigma \exists y^\tau \nu(x, y) \to \exists z^{\sigma \to \tau} \forall x^\sigma \nu(x, zx)$;

DC_σ: $\forall x^\sigma \exists y^\sigma \nu(x, y) \to \forall x^\sigma \exists z^{0 \to \sigma} (z0 = x \wedge \forall n^0 \nu(zn, z(Sn)))$;

WC-N: $\forall a^1 \exists x^0 \nu(a, x) \to \forall a^1 \exists x^0 \exists y^0 \forall b^1 \nu(\overline{a}x * b, y)$;

C-N: $\forall a^1 \exists^0 x \nu(a, x) \to \exists c^1 [c \in K_0 \wedge \forall a^1 \exists m^0 \exists n^0 (c(\overline{a}m) = n + 1 \wedge \nu(a, n))]$, where $c \in K_0 \Leftrightarrow \forall a^1 \exists n^0 (c(\overline{a}n) \neq 0) \wedge \forall n^0 \forall m^0 (cn \neq 0 \to cn = c(n * m))$.

Note that $\mathrm{BI_M}$ implies FAN (cf. [3, Ch.I, §6.10]) and C-N implies WC-N (cf. [8, Ch.4.6.8]).

We prove that $\mathbf{HA}^\omega$ with any subset of the above schemata does not derive IP^ω. The problem is treated as follows:

Define semi-classical principles MP and LPO, and a theory $\mathbf{T}$ by

$$\begin{aligned} &\mathrm{MP} \equiv \neg\neg\exists x^0 (a^1(x) = 0) \to \exists x^0 (a^1(x) = 0), \\ &\mathrm{LPO} \equiv \exists x^0 (a^1(x) = 0) \vee \neg\exists x^0 (a^1(x) = 0), \\ &\mathbf{T} \equiv \mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}, \end{aligned}$$

where $\mathrm{AC} \equiv \bigcup_{\sigma,\tau} \mathrm{AC}_{\sigma,\tau}$ (cf. [7, 3.4.7]) and $\mathrm{DC} \equiv \bigcup_\sigma \mathrm{DC}_\sigma$ (cf. [5, 11.2], where DC_σ is called DAC^σ). Note that MP and LPO are $\exists\Gamma$-DNE and $\exists\Gamma$-LEM, respectively, for the class Γ of formulae of the form $a^1(x^0) = 0$. If we have $\mathbf{T} \nvdash \mathrm{LPO}$, then $\mathbf{T}$ is not closed under IPR^ω by Corollary 6 (1), and it does not derive IP^ω. Therefore any subtheory of $\mathbf{T}$ does not derive IP^ω.

To prove that $\mathbf{T} \nvdash \mathrm{LPO}$, we should note that C-N refutes Π_1-LEM (cf. [8, Ch.4.6.4], where Π_1-LEM is called $\forall$-PEM), and that LPO implies Π_1-LEM. Therefore, if $\mathbf{T}$ is consistent, then $\mathbf{T} \nvdash \mathrm{LPO}$.

Thus the problem is reduced to the consistency of $\mathbf{T}$. We show that, using two interpretations, $\mathbf{T}$ is consistent relative to an extension of the system of elementary analysis $\mathbf{EL}$ (cf. [8, Ch.3.6.2]), whose model can be constructed in a very weak fragment of $\mathbf{ZF}$.

In Subsection 3.1, we show that $\mathbf{T}$ is consistent relative to $\mathbf{EL} + \mathrm{BI_M} + \mathrm{MP} + \mathrm{GC}$ by the interpretation $[\![\cdot]\!]$ defined in [7, 2.6.2]. In Subsection 3.2, we combine the interpretation $[\![\cdot]\!]$ with the functional realizability interpretation, and show that $\mathbf{EL} + \mathrm{BI_M} + \mathrm{MP} + \mathrm{GC}$, and hence $\mathbf{T}$, is consistent relative to $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP}$.

We often use informal partially defined terms $t(s)$ and $t|s$ for terms t and s of type 1 in $\mathscr{L}(\mathbf{EL})$ which are defined as follows:

$$t(s) = t(\overline{s}k) - 1, \text{ where } k \text{ is the least } m \text{ with } t(\overline{s}m) > 0;$$
$$(t|s)(n) = t(\langle n \rangle * \overline{s}k) - 1, \text{ where } k \text{ is the least } m \text{ with } t(\langle n \rangle * \overline{s}m) > 0.$$

Note that $t(s)$ and $t|s$ are not always defined. The notations $t(s)\downarrow$ and $(t|s)\downarrow$ intuitively mean that $t(s)$ and $t|s$ are defined, respectively. Precisely, they are abbreviations of the following formulae:

$$t(s)\downarrow \equiv \exists m^0(t(\overline{s}m) > 0); \qquad (t|s)\downarrow \equiv \forall n^0 \exists m^0(t(\langle n \rangle * \overline{s}m) > 0).$$

For each formula A in $\mathscr{L}(\mathbf{EL})$, the notations $A(s(t))$ and $A(s|t)$ are abbreviations of the formulae

$$s(t)\downarrow \wedge \forall x^0(s(t) = x \to A(x)); \quad s|t\downarrow \wedge \forall u(\forall x^0(u(x) = (s|t)(x)) \to A(u)),$$

respectively.

We fix a surjective pairing function (-) : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and projections $(\text{-})_i : \mathbb{N} \to \mathbb{N}$ such that $((n)_0, (n)_1) = n$, $((m, n))_0 = m$ and $((m, n))_1 = n$ for all m^0 and n^0. For each a^1, we define $(a)_i$ by $(a)_i(x) = (a(x))_i$.

3.1 The interpretation $[\![\cdot]\!]$

In this subsection, we use the interpretation $[\cdot]$ defined in [7, 2.6.2] and write $[\![\cdot]\!]$ for $[\cdot]$.

For each type σ in $\mathscr{L}(\mathbf{HA}^\omega)$, define a type $[\![\sigma]\!]$ in $\mathscr{L}(\mathbf{EL})$ and a formula $x^{[\![\sigma]\!]} \in [\![\sigma]\!]$ as follows:

$$[\![0]\!] \equiv 0; \qquad [\![\rho \to \tau]\!] \equiv 1; \qquad x \in [\![0]\!] \equiv x = x;$$

$$x^1 \in [\![\rho \to \tau]\!] \equiv \begin{cases} \forall y^0(y \in [\![\rho]\!] \to \forall z(x(y) = z \to z \in [\![\tau]\!])) \\ \qquad \text{if } [\![\rho]\!] \equiv [\![\tau]\!] \equiv 0; \\ \forall y^0(y \in [\![\rho]\!] \to (x|(\lambda v.y)\downarrow \wedge \forall z^1(z = x|(\lambda v.y) \to z \in [\![\tau]\!]))) \\ \qquad \text{if } [\![\rho]\!] \equiv 0 \text{ and } [\![\tau]\!] \equiv 1; \\ \forall y^1(y \in [\![\rho]\!] \to (x(y)\downarrow \wedge \forall z^0(x(y) = z \to z \in [\![\tau]\!]))) \\ \qquad \text{if } [\![\rho]\!] \equiv 1 \text{ and } [\![\tau]\!] \equiv 0; \\ \forall y^1(y \in [\![\rho]\!] \to (x|y\downarrow \wedge \forall z^1(x|y = z \to z \in [\![\tau]\!]))) \\ \qquad \text{if } [\![\rho]\!] \equiv 1 \text{ and } [\![\tau]\!] \equiv 1. \end{cases}$$

Note that, for terms t of type 1 and s of type 0, a term $t(s)$ denotes usual application of t to s in $\mathscr{L}(\mathbf{EL})$, which is always defined. Therefore we always have to pay attention to the type of the term enclosed in parentheses following a term of type 1.

Fix a bijection from variables of $\mathscr{L}(\mathbf{HA}^\omega)$ to variables in $\mathscr{L}(\mathbf{EL})$ such that, for each variable x^σ in $\mathscr{L}(\mathbf{HA}^\omega)$, $[\![x^\sigma]\!]$ is a variable of type $[\![\sigma]\!]$ in $\mathscr{L}(\mathbf{EL})$. Note that each variable a of type 0 and 1 in $\mathscr{L}(\mathbf{HA}^\omega)$, the variable $[\![a]\!]$ is a variable of type 0 and 1 in $\mathscr{L}(\mathbf{EL})$, respectively.

For each constant c in $\mathscr{L}(\mathbf{HA}^\omega)$, we assign a numeral or a functional $[\![c]\!]$ in $\mathscr{L}(\mathbf{EL})$ so that the following properties hold with appropriate assumptions on types for variables, such as $[\![x^\sigma]\!] \in [\![\sigma]\!]$:

$$\begin{aligned}
&[\![0]\!] \equiv 0; && [\![Sx^0]\!] = S[\![x]\!];\\
&[\![\boldsymbol{k}^{\sigma,\tau} x^\sigma y^\tau]\!] = [\![x]\!]; && [\![\boldsymbol{s}^{\rho,\sigma,\tau} x^{\rho\to(\sigma\to\tau)} y^{\rho\to\sigma} z^\rho]\!] = [\![xz(yz)]\!];\\
&[\![\boldsymbol{r}^\sigma x^\sigma y^{\sigma\to(0\to\sigma)} 0]\!] = [\![x]\!]; && [\![\boldsymbol{r}^\sigma x^\sigma y^{\sigma\to(0\sigma)}(Sz^0)]\!] = [\![y(\boldsymbol{r}^\sigma xyz)z]\!],
\end{aligned}$$

where applications of terms are defined as follows:

$$[\![t^{\sigma\to\tau} s^\sigma]\!] \equiv \begin{cases} [\![t]\!]([\![s]\!]) & \text{if } [\![\sigma]\!] \equiv [\![\tau]\!] \equiv 0;\\ [\![t]\!]|(\lambda v.[\![s]\!]) & \text{if } [\![\sigma]\!] \equiv 0 \text{ and } [\![\tau]\!] \equiv 1;\\ [\![t]\!]([\![s]\!]) & \text{if } [\![\sigma]\!] \equiv 1 \text{ and } [\![\tau]\!] \equiv 0;\\ [\![t]\!]|[\![s]\!] & \text{if } [\![\sigma]\!] \equiv 1 \text{ and } [\![\tau]\!] \equiv 1. \end{cases}$$

Although any detailed construction of $[\![c]\!]$ for each constant c in $\mathscr{L}(\mathbf{HA}^\omega)$ is not given in [7, 2.6.2], constructions of $[\![\boldsymbol{k}^{\sigma,\tau}]\!]$ and $[\![\boldsymbol{s}^{\rho,\sigma,\tau}]\!]$, for $[\![\rho]\!] \equiv [\![\sigma]\!] \equiv [\![\rho]\!] \equiv 1$, can be taken as $\mathsf{k}^{\mathfrak{k}}$ and $\mathsf{s}^{\mathfrak{k}}$ in [6, Definition 3.19].

For a formula A in $\mathscr{L}(\mathbf{HA}^\omega)$, a formula $[\![A]\!]$ in $\mathscr{L}(\mathbf{EL})$ is defined as follows:

$$[\![t =_\sigma s]\!] \equiv \begin{cases} [\![t]\!] = [\![s]\!] & \text{if } [\![\sigma]\!] \equiv 0;\\ \forall n([\![t]\!](n) = [\![s]\!](s)) & \text{if } [\![\sigma]\!] \equiv 1; \end{cases} \qquad [\![\bot]\!] \equiv \bot;$$

$$[\![A \circ B]\!] \equiv [\![A]\!] \circ [\![B]\!], \text{ where } \circ \in \{\wedge, \vee, \to\};$$

$$[\![\forall x^\sigma A]\!] \equiv \forall x^{[\![\sigma]\!]}(x \in [\![\sigma]\!] \to [\![A]\!]); \quad [\![\exists x^\sigma A]\!] \equiv \exists x^{[\![\sigma]\!]}(x \in [\![\sigma]\!] \wedge [\![A]\!]).$$

A schema in $\mathscr{L}(\mathbf{EL})$ and $\mathscr{L}(\mathbf{HA}^\omega)$ is a schema whose terms and quantifiers are all in $\mathscr{L}(\mathbf{EL})$ and $\mathscr{L}(\mathbf{HA}^\omega)$, respectively.

Let ι be the standard translation from the language $\mathscr{L}(\mathbf{EL})$ into the language $\mathscr{L}(\mathbf{HA}^\omega)$. Then ι can be extended on the schemata of $\mathscr{L}(\mathbf{EL})$. For each term t of type 0 and 1 and schema α in $\mathscr{L}(\mathbf{EL})$, assign a term $(t)_E$ and a schema $(\alpha)_E$ in $\mathscr{L}(\mathbf{EL})$ as follows:

$(x^i)_E \equiv [\![\iota(x^i)]\!]$; $(ft_1...t_n)_E \equiv f(t_1)_E...(t_n)_E$;
$(\boldsymbol{r}t_1t_2t_3)_E \equiv \boldsymbol{r}(t_1)_E(t_2)_E(t_3)_E$; $(a^1(t^0))_E \equiv (a)_E((t^0)_E)$;
$(\lambda x.t)_E \equiv \lambda(x)_E.(t)_E$; $(s =_0 t)_E \equiv (s)_E =_0 (t)_E$; $(\bot)_E \equiv \bot$;
$(\nu(t_1,...,t_n))_E \equiv \nu((t_1)_E,...,(t_n)_E)$; $(\beta_1 \circ \beta_2)_E \equiv (\beta_1)_E \circ (\beta_2)_E$, for $\circ \in \{\wedge, \vee, \to\}$;
$(Qx^i\beta)_E \equiv Q(x)_E(\beta)_E$.

Note that the interpretation $[\![\cdot]\!]$ satisfies the following properties (P0)-(P5).

(P0) For each variable $x^\sigma \in \mathscr{L}(\mathbf{HA}^\omega)$, $[\![x^\sigma]\!]$ is a variable of type $[\![\sigma]\!]$ in $\mathscr{L}(\mathbf{EL})$;

(P1) $\mathbf{EL} \vdash [\![\iota(t)]\!] = (t)_E$;

(P2) $\mathbf{EL} \vdash [\![t^0 = s^0]\!] \leftrightarrow [\![t^0]\!] = [\![s^0]\!]$;

(P3) $\mathbf{EL} \vdash [\![A \circ B]\!] \leftrightarrow [\![A]\!] \circ [\![B]\!]$ for $\circ \in \{\wedge, \vee, \to\}$;

(P4) $\mathbf{EL} \vdash [\![Qu^\sigma A]\!] \leftrightarrow \forall [\![u^\sigma]\!][\![A]\!]$, for $Q \in \{\exists, \forall\}$ and $\sigma \equiv 0$ or 1;

(P5) $\mathbf{EL} \vdash [\![A[u^\sigma/t^\sigma]]\!] \leftrightarrow [\![A]\!][[\![u^\sigma]\!]/[\![t^\sigma]\!]]$.

Remark 12. For the properties (P4), note that $\mathbf{EL} \vdash x^0 \in [\![0]\!] \wedge a^1 \in [\![1]\!]$. To show the property (P5), we first prove $[\![t[x^\sigma/s^\sigma]]\!] = [\![t]\!][[\![x^\sigma]\!]/[\![s^\sigma]\!]]$ by the induction on the construction of t, and then prove (P5) by the induction on the construction of A.

Proposition 13. *Let $\alpha[\nu_1, ...\nu_k]$ be a schema with exactly displayed place holders in $\mathscr{L}(\mathbf{EL})$ and let $B_1, \dots, B_k$ be formulae in $\mathscr{L}(\mathbf{HA}^\omega)$. Then*

$$\mathbf{EL} \vdash [\![\iota(\alpha)[\nu_1/\lambda\vec{u_1}.B_1, \dots, \nu_k/\lambda\vec{u_k}.B_k]]\!] \leftrightarrow \\ (\alpha)_E[\nu_1/\lambda[\![\vec{u_1}]\!].[\![B_1]\!], \dots, \nu_k/\lambda[\![\vec{u_k}]\!].[\![B_k]\!]].$$

Proof. We show by induction on the complexity of α. It is clear if α is $\perp$. If α is an atomic formula $P \equiv s =_0 t$, then s and t are terms in $\mathscr{L}(\mathbf{EL})$, and hence $[\![\iota(s =_0 t)]\!] \leftrightarrow [\![\iota(s)]\!] =_0 [\![\iota(t)]\!] \leftrightarrow (s)_E =_0 (t)_E \equiv (s =_0 t)_E$, by (P1), (P2) and the definition of $(\text{-})_E$. Let $\alpha \equiv \nu_i(t_1, \dots, t_n)$ for some $i = 0, ..., k$ and $\vec{u_i} \equiv v_1^{\sigma_1}, ..., v_n^{\sigma_n}$. Then

$$\begin{aligned}[\![B_i[v_1^{\sigma_1}/\iota(t_1), \dots, v_n^{\sigma_n}/\iota(t_n)]]\!] &\leftrightarrow [\![B_i]\!][[\![v_1^{\sigma_1}]\!]/[\![\iota(t_1)]\!], \dots, [\![v_n^{\sigma_n}]\!]/[\![\iota(t_n)]\!]] \\ &\leftrightarrow [\![B_i]\!][[\![v_1^{\sigma_1}]\!]/(t_1)_E, \dots, [\![v_n^{\sigma_n}]\!]/(t_n)_E],\end{aligned}$$

by (P5) and (P1). It is straightforward to see the induction steps for the connectives, by (P3) and (P4). □

For a schema α in $\mathscr{L}(\mathbf{EL})$, the set $\mathrm{FV}(\alpha)$ of free variables in α is defined in the usual way. A schema is *closed* if $\mathrm{FV}(\alpha) = \emptyset$.

Proposition 14. *Let α be a closed schema in $\mathscr{L}(\mathbf{EL})$ with place holders ν_1,..., ν_k, and let B_1,..., B_k be formulae in $\mathscr{L}(\mathbf{HA}^\omega)$. Then*

$$\mathbf{EL} \vdash [\![\iota(\alpha)[B_1, ..., B_k]]\!] \leftrightarrow \alpha[[\![B_1]\!], ..., [\![B_k]\!]].$$

Proof. Note that for any closed schema α and formulae $\vec{C}$ in $\mathscr{L}(\mathbf{EL})$, since $(\alpha)_E[\vec{C}]$ is obtained from $\alpha[\vec{C}]$ by renaming of bound variables, we have $\mathbf{EL} \vdash (\alpha)_E[\vec{C}] \leftrightarrow \alpha[\vec{C}]$. Then it follows from Proposition 13. □

The above proposition shows that $[\![\cdot]\!]$ interprets an instance of a closed schema in $\mathscr{L}(\mathbf{EL})$, as a schema in $\mathscr{L}(\mathbf{HA}^\omega)$, into an instance of the same schema in $\mathscr{L}(\mathbf{EL})$. In particular, we have the following.

Corollary 15. *For each instance A of* MP, $\mathrm{BI_M}$ *and* C-N *in $\mathscr{L}(\mathbf{HA}^\omega)$, there exists an instance B of* MP, $\mathrm{BI_M}$ *and* C-N *in $\mathscr{L}(\mathbf{EL})$ such that*

$$\mathbf{EL} \vdash [\![A]\!] \leftrightarrow B.$$

Proof. Let a schema α be one of MP, $\mathrm{BI_M}$ and C-N in $\mathscr{L}(\mathbf{EL})$. Then, since α is closed, $\iota(\alpha)$ is a closed schema in $\mathscr{L}(\mathbf{HA}^\omega)$ corresponding to MP, $\mathrm{BI_M}$ or C-N. □

The classes Δ_0, Σ_n and Π_n of formulae in $\mathscr{L}(\mathbf{HA})$ are naturally extended into ones of formulae in $\mathscr{L}(\mathbf{EL})$, which are often written Δ_0^0, Σ_n^0 and Π_n^0 in literature. A formula A in $\mathscr{L}(\mathbf{EL})$ is *almost negative* if it is built up from formulae of the form $\exists x^i B$ with a Δ_0 formula B by means of $\wedge$, $\to$, $\forall x^\sigma$ for $\sigma \equiv 0$ or 1.

The *generalized continuity* (GC) in $\mathscr{L}(\mathbf{EL})$ is as follows:

$$\forall a^1(B(a) \to \exists b^1 C(a,b)) \to \exists c^1 \forall a^1(B(a) \to c|a \downarrow \wedge C(a, c|a)),$$

where B is almost negative (cf. [7, 3.3.9] or [8, Ch.9.6.9]).

Proposition 16. $\mathbf{EL} + \mathrm{GC} \vdash$ C-N.

Proof. Let

$$\forall a^1 \exists x^0 B(a,x) \to \exists c^1[c \in K_0 \wedge \forall a^1 \exists m^0 \exists n^0 (c(\overline{a}m) = n+1 \wedge \nu(a,n))]$$

be an instance of C-N and assume that $\forall a^1 \exists x^0 B(a,x)$. Then we have $\forall a^1 \exists b^1 B(a, b(0))$. By GC, there exists d such that $\forall a^1(d|a \downarrow \wedge B(a, (d|a)(0)))$. Define c by $c(x^0) = d(\langle 0 \rangle * \overline{x}k)$ if there is $y < |x|$ such that $d(\langle 0 \rangle * \overline{x}y) \neq 0$ and k is the least such y, and $c(x^0) = 0$ if there is no such y. Then it is easy to see that $c \in K_0 \wedge \forall a^1 \exists m^0 \exists n^0 (c(\overline{m}) = n+1 \wedge B(a,n))$. □

For each type σ in $\mathscr{L}(\mathbf{HA}^\omega)$, let s_σ and t_σ be terms in $\mathscr{L}(\mathbf{EL})$ defined by

$$s_\sigma(a^1) = \begin{cases} a(0) & \text{if } \sigma \equiv 0; \\ a & \text{otherwise;} \end{cases} \qquad t_\sigma(x^{[\![\sigma]\!]}) = \begin{cases} \lambda y.x & \text{if } [\![\sigma]\!] \equiv 0; \\ x & \text{otherwise.} \end{cases}$$

Note that $t_\sigma(x^{[\![\sigma]\!]})$ is always a term of type 1 and $s_\sigma(a)$ is a term of type $[\![\sigma]\!]$ in $\mathscr{L}(\mathbf{EL})$.

Then, the interpretation $[\![\cdot]\!]$ also satisfies the following properties:

(P6) $x \in [\![\sigma]\!]$ is almost negative;

(P7) $\mathbf{EL} \vdash \forall u \in [\![\sigma]\!](s_\sigma(t_\sigma(u)) = u)$;

(P8) $\mathbf{EL} \vdash \forall c[\forall u \in [\![\sigma]\!](c|t_\sigma(u) \downarrow \wedge s_\tau(c|t_\sigma(u)) \in [\![\tau]\!]) \to$
$\exists v \in [\![\sigma \to \tau]\!] \forall u \in [\![\sigma]\!](v \cdot u = s_\tau(c|t_\sigma(u^\sigma)))]$,
where $v \cdot u = [\![x^{\sigma\to\tau} y^\sigma]\!][[\![x]\!]/v, [\![y]\!]/u]$.

Remark 17. To prove the property (P8), we have to consider four cases depending on types $[\![\sigma]\!]$ and $[\![\tau]\!]$. If $[\![\sigma]\!] \equiv [\![\tau]\!] \equiv 0$, we have $\forall u^0 \exists y^0 (s_\sigma(c|t_\sigma(u)) = y)$ holds for c such that $\forall u \in [\![\sigma]\!](c|t_\sigma(u) \downarrow \wedge s_\tau(c|t_\sigma(u)) \in [\![\tau]\!])$. Then, by QF-AC in $\mathbf{EL}$, we have v^1 such that $\forall u^0(v(u) = s_\sigma(c|t_\sigma(u)))$, and so $\forall u \in [\![\sigma]\!](v \cdot u = s_\tau(c|t_\sigma(u^\sigma)))$. Other cases are proved easily.

Proposition 18. *For each instance A of* $\mathrm{AC}_{\sigma,\tau}$ *in* $\mathcal{L}(\mathbf{HA}^\omega)$

$$\mathbf{EL} + \mathrm{GC} \vdash [\![A]\!].$$

Proof. Let

$$\forall u^\sigma \exists v^\tau B(u,v) \to \exists w^{\sigma\to\tau} \forall u^\sigma B(u, wu)$$

be an instance of $\mathrm{AC}_{\sigma,\tau}$, and suppose that $(\forall u \in [\![\sigma]\!])(\exists v \in [\![\tau]\!])[\![B]\!](u,v)$. Then we have

$$\forall a^1[s_\sigma(a) \in [\![\sigma]\!] \to \exists v^{[\![\tau]\!]}(s_\tau(t_\tau(v)) \in [\![\tau]\!] \wedge [\![B]\!](s_\sigma(a), s_\tau(t_\tau(v))))],$$

by (P7), and hence

$$\forall a^1[s_\sigma(a) \in [\![\sigma]\!] \to \exists b^1(s_\tau(b) \in [\![\tau]\!] \wedge [\![B]\!](s_\sigma(a), s_\tau(b)))].$$

Since $s_\sigma(a) \in [\![\sigma]\!]$ is almost negative by (P6), we have c such that

$$\forall a[s_\sigma(a) \in [\![\sigma]\!] \to c|a\downarrow \wedge s_\tau(c|a) \in [\![\tau]\!] \wedge [\![B]\!](s_\sigma(a), s_\tau(c|a))].$$

by GC. Therefore

$$\forall u \in [\![\sigma]\!][s_\sigma(t_\sigma(u)) \in [\![\sigma]\!] \to c|t_\sigma(u)\downarrow \wedge\, s_\tau(c|t_\sigma(u)) \in [\![\tau]\!] \wedge \\ [\![B]\!](s_\sigma(t_\sigma(u)), s_\tau(c|t_\sigma(u)))],$$

and so

$$\forall u \in [\![\sigma]\!][c|t_\sigma(u)\downarrow \wedge s_\tau(c|t_\sigma(u)) \in [\![\tau]\!] \wedge [\![B]\!](u, s_\tau(c|t_\sigma(u)))],$$

by (P7). Thus

$$(\exists w \in [\![\sigma \to \tau]\!])(\forall u \in [\![\sigma]\!])[\![B]\!](u, w\cdot u),$$

by (P8). Note that, since $w \cdot u = [\![x^{\sigma\to\tau} y^\sigma]\!][[\![x]\!]/w, [\![y]\!]/u]$, this is equivalent to

$$[\![\exists w^{\sigma\to\tau}\forall u^\sigma B(u, wu)]\!],$$

in **EL**. □

Proposition 19. *For each instance A of* DC_σ *in* $\mathcal{L}(\mathbf{HA}^\omega)$

$$\mathbf{EL} + \mathrm{GC} \vdash [\![A]\!].$$

Proof. Let

$$\forall u^\sigma \exists v^\sigma B(u, v) \to \forall u^\sigma \exists w^{0\to\sigma}(w0 = u \wedge \forall n^0 B(wn, w(n+1)))$$

be an instance of DC_σ, and suppose that $(\forall u \in [\![\sigma]\!])(\exists v \in [\![\sigma]\!])[\![B]\!](u, v)$. As in the proof of Proposition 18, there exists w such that

$$w \in [\![\sigma \to \sigma]\!] \wedge \forall u \in [\![\sigma]\!][\![B]\!][u, w \cdot u].$$

For each $u \in [\![\sigma]\!]$, let $a = \lambda n^0.[\![\boldsymbol{r}^\sigma]\!]uwn$ if $[\![\sigma]\!] \equiv 0$, and $a = \lambda n^0.[\![\boldsymbol{r}^\sigma]\!]|u|w|(\lambda y.n)$, otherwise. Then it is easy to show that a satisfies

$$a \in [\![0 \to \sigma]\!] \wedge a \cdot 0 = u \wedge \forall n^0[\![B]\!](a \cdot n, a \cdot (n+1)),$$

which implies

$$[\![\exists a^{0\to\sigma}(a0 = u \wedge \forall n^0 B(an, a(n+1)))]\!],$$

in **EL**. □

By Corollary 15, Proposition 16, Proposition 18 and Proposition 19, we have the following:

Corollary 20. *If* $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP} \vdash A$*, then* $\mathbf{EL} + \mathrm{BI_M} + \mathrm{MP} + \mathrm{GC} \vdash [\![A]\!]$.

Note that, although we have used the particular interpretation $[\![\cdot]\!]$, we can prove Proposition 14 for any interpretation with the properties (P0)–(P5), and also Proposition 18 and Proposition 19 if there exist terms s_σ and t_σ satisfying (P6)–(P8) additionally.

3.2 Combining with functional realizability interpretation

In this subsection, we combine the interpretation $[\![\cdot]\!]$ with the functional realizability interpretation defined in [7, 3.3.1–3.3.13], and show that $\mathbf{EL} + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$ is consistent relative to $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP}$.

Definition 21. For each formula A and a functional variable a^1 in $\mathscr{L}(\mathbf{EL})$, define a formula $a\,\mathbf{r}\,A$ as follows:

$a\,\mathbf{r}\,A \equiv A$ for prime A;

$a\,\mathbf{r}\,B \wedge C \equiv (a)_0\,\mathbf{r}\,B \wedge (a)_1\,\mathbf{r}\,C$;
$a\,\mathbf{r}\,B \vee C \equiv ((a)_0(0) = 0 \to (a)_1\,\mathbf{r}\,B) \wedge ((a)_0(0) \neq 0 \to (a)_1\,\mathbf{r}\,C)$;
$a\,\mathbf{r}\,B \to C \equiv \forall b^1(b\,\mathbf{r}\,B \to a|b\downarrow \wedge a|b\,\mathbf{r}\,C)$;
$a\,\mathbf{r}\,\forall x^0\,\mathbf{r}\,B(x) \equiv \forall x^0((a|\lambda y.x)\downarrow \wedge (a|\lambda y.x)\,\mathbf{r}\,B(x))$;
$a\,\mathbf{r}\,\exists x^0 B(x) \equiv (a)_1\,\mathbf{r}\,B((a)_0(0))$;
$a\,\mathbf{r}\,\forall x^1\,\mathbf{r}\,B(x) \equiv \forall b^1(a|b\downarrow \wedge a|b\,\mathbf{r}\,B(b))$;
$a\,\mathbf{r}\,\exists x^1 B(x) \equiv (a)_1\,\mathbf{r}\,B((a)_0)$.

The *bar induction for decidable bars* ($\mathrm{BI_D}$) is given as follows (cf. [7, 1.9.20]):

$$[\forall a^1\exists x^0 C(\overline{a}x) \wedge \forall n^0(C(n) \vee \neg C(n)) \wedge \forall n^0(C(n) \to D(n))\wedge \\ \forall n^0(\forall y^0 D(n * \langle y\rangle) \to D(n))] \to D(\langle\rangle).$$

A variant BI! of $\mathrm{BI_D}$ is given by replacing $\exists$ with $\exists!$ in $\mathrm{BI_D}$ (cf. [7, 3.3.4]). It is easy to see that BI! is equivalent to $\mathrm{BI_D}$ over **EL** by replacing $C(x)$ with $C(x) \wedge (\forall y < |x|)\neg C(\overline{x}y)$ in $\mathrm{BI_D}$.

Lemma 22. $\mathbf{EL} + \mathrm{BI_D} + \mathrm{GC}$ *derives* $\mathrm{BI_M}$.

Proof. $\mathbf{EL}+\text{C-N}+\mathrm{BI_D}$ derives $\mathrm{BI_M}$ (cf. [8, Ch.4, 8.13]), and since GC implies C-N by Proposition 16, $\mathbf{EL} + \mathrm{GC} + \mathrm{BI_D}$ derives $\mathrm{BI_M}$. □

Lemma 23. $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} + \mathrm{GC} \vdash A$ *if and only if* $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \exists a^1(a\,\mathbf{r}\,A)$.

Proof. Note that MP is almost negative. Then $\mathbf{EL} + \mathrm{MP} \vdash \exists a^1(a\,\mathbf{r}\,A)$ for each instance A of MP, by [7, 3.3.8 Lemma]. By [4, pp. 103–104], we have that $\mathbf{EL}+\mathrm{BI!} \vdash \exists a^1(a\,\mathbf{r}\,A)$ for each instance A of BI!, and hence $\mathbf{EL}+\mathrm{BI_D} \vdash \exists a^1(a\,\mathbf{r}\,A)$ for each instance A of $\mathrm{BI_D}$.

Therefore

$$\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash A \Longrightarrow \mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \exists a^1(a\,\mathbf{r}\,A)$$

by [7, 3.3.3 Corollary], and so

$$\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} + \mathrm{GC} \vdash A \Longleftrightarrow \mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \exists a^1(a\,\mathbf{r}\,A)$$

for each formula A in $\mathscr{L}(\mathbf{EL})$, by [7, 3.3.11 Theorem (ii)]. □

Corollary 24. $\mathbf{EL} + \mathrm{BI_M} + \mathrm{MP} + \mathrm{GC} \vdash A$ *if and only if* $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \exists a^1 (a \, \mathbf{r} \, A)$.

Finally, we have the following:

Theorem 25. *If* $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP} \vdash A$*, then* $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \exists a^1 (a \, \mathbf{r} \, [\![A]\!])$.

Proof. By Corollary 20 and Corollary 24. □

By Theorem 25, $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP} \vdash \bot$ implies $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP} \vdash \bot$, and so $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$ is consistent relative to $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP}$. Since the consistency of $\mathbf{EL} + \mathrm{BI_D} + \mathrm{MP}$ is proved in a very weak fragment of **ZF**, we have the consistency $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$. Therefore, we have the following.

Theorem 26.

1. $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$ *is not closed under* IPR^ω*, and so it does not derive* IP^ω.

2. $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC}$ *derives neither* IP^ω *nor* MP.

Proof. (1): As in the observation of the beginning of this section, $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$ does not prove LPO. Note that MP and LPO are $\exists\Gamma$-DNE and $\exists\Gamma$-LEM, respectively, for the class Γ of formulae of the form $a^1(x^0) = 0$. Therefore $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC} + \mathrm{MP}$ is not closed under IPR^ω, and does not prove IP^ω, by Corollary 6 (1).
(2): By (1), $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC}$ does not imply IP^ω. Since it consists of schemata in $\mathcal{S}$, it is closed under IPR^ω, by [2, Theorem 3.1]. Since $\mathbf{HA}^\omega + \mathrm{BI_M} + \text{C-N} + \mathrm{AC} + \mathrm{DC}$ does not derive LPO, it does not prove MP, by Corollary 6 (2). □

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