Non-deterministic inductive definitions and Fullness

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Abstract

In this paper, we deal with the non-deterministic inductive definition principle NID with the weak notion of a set-generated class introduced by van den Berg and with the strong notion of a set-generated class adopted by Aczel et al.. We introduce a principle, called nullary NID, and prove that nullary NID is equivalent to Fullness in a subsystem of the constructive Zermelo-Fraenkel set theory.

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1 Introduction

The notion of a *set-generated class* was introduced in Aczel [4] for dcpos using some terminology from domain theory. A partially ordered class is a *directed complete partial order* (dcpo) if each directed subset has a least upper bound, where a subset is *directed* if any pair of elements of the subset has an upper bound in the subset. A dcpo X is *set-generated* if there is a subset G of X such that, for each $a \in X$, $\{x \in G \mid x \leq a\}$ is a directed subset whose least upper bound is a. If we restrict our attention to a class X of subsets of a set with the inclusion $\subseteq$ as a partial order, then we may say that X is *set-generated* if there exists a subset G of X such that

$$\forall \alpha \in X \forall \tau \in \mathrm{Fin}(\alpha) \exists \beta \in G[\tau \subseteq \beta \subseteq \alpha],$$

where $\mathrm{Fin}(\alpha)$ is the set of finitely enumerable subsets of α.

This definition was adopted in Aczel et al. [5], and it was shown that the notion of a set-generated class plays crucial roles in predicative constructive mathematics. However, in an early draft of [5] and in van den Berg [8], a weaker notion of a set-generated class was employed: a class X of subsets of a set is set-generated if there exists a subset G of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G[x \in \beta \subseteq \alpha].$$

Note that the strong notion of a set-generated class is crucial in some applications, for example, [9].

In [8], van den Berg introduced the principle NID on non-deterministic inductive definitions and set-generated classes using the weaker notion of a set-generated class in the constructive set theory **CZF**. A *rule* on a set S is a pair (a, b) of subsets a and b of S, and a rule is called *elementary* if a is a singleton and *finitary* if a is finitely enumerable. A subset α of S is *closed under* the rule (a, b) if

$$a \subseteq \alpha \Rightarrow b \between \alpha,$$

where $b \between \alpha \Leftrightarrow \exists x \in b(x \in \alpha)$, that is, the intersection of b and α is inhabited. For a set R of rules on S, we call a subset α of S R-*closed* if it is closed under each rule in R. The NID principle is that for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated. If we restrict rules in NID to elementary and finitary rules, we call them elementary NID and finitary NID, respectively.

On the other hand, in [5], Aczel et al. characterized set-generated classes with the strong notion using generalized geometric theories and the set generation axiom (SGA) in **CZF**. In [8], van den Berg discussed on a relation between finitary NID and SGA, and revealed some aspect between the weak notion and the strong notion of a set-generated class. He also showed that elementary NID implies Fullness, a theorem in **CZF**, which is an important axiom in a subsystem of **CZF** that implies Exponentiation (the class of functions between sets is a set).

In this paper, we introduce another NID principle, called *nullary* NID, which is weaker than finitary NID, and prove that nullary NID is equivalent to Fullness in a subsystem of **CZF**, that is, the elementary constructive set theory **ECST**. We also show that elementary NID implies nullary NID, and that nullary, elementary and finitary NID with the weak notion of a set-generated class are equivalent to respectively nullary, elementary and finitary NID with the strong notion.

2 The elementary constructive set theory

The constructive Zermelo-Fraenkel set theory **CZF**, founded by Aczel [1, 2, 3], grew out of Myhill's constructive set theory [11] as a formal system for Bishop's constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory [10]. Aczel and Rathjen introduced the elementary constructive set theory **ECST** which is a subsystem of **CZF** in their book draft [7] written by extending their research report [6].

Definition 1. The language of a constructive set theory contains variables for sets and the binary predicates $=$ and $\in$. The axioms and rules are those of intuitionistic predicate logic with equality. In addition, **ECST** has the following set theoretic axioms:

Extensionality: $\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b)$.

Pairing: $\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \vee x = b)$.

Union: $\forall a \exists b \forall x (x \in b \leftrightarrow \exists y \in a (x \in y))$.

Restricted Separation:

$$\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for every *restricted* formula $\varphi(x)$. Here a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

Replacement:

$$\forall a (\forall x \in a \exists! y \varphi(x, y) \rightarrow \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y)))$$

for every formula $\varphi(x, y)$.

Strong Infinity:

$$\exists a [0 \in a \wedge \forall x (x \in a \rightarrow x + 1 \in a) \\ \wedge \forall y (0 \in y \wedge \forall x (x \in y \rightarrow x + 1 \in y) \rightarrow a \subseteq y)],$$

where $x + 1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset$.

Let a and b be sets. Using Replacement and Union, the *cartesian product* $a \times b$ of a and b consisting of the ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in **ECST**. Similarly, we can introduce the *disjoint union* $a + b = \{(0, x) \mid x \in a\} \cup \{(1, y) \mid y \in b\}$, where $1 = \{0\}$. A *relation* r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is *total* (or is a *multivalued function*) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$. The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$, or more formally

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

A *function* from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$. The class of functions from a to b is denoted by b^a, or more formally

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \rightarrow y = z).$$

We use ω for the unique set a such that $0 \in a \wedge \forall x(x \in a \rightarrow x + 1 \in a)$, ensured by Strong Infinity. For a set S, let $\mathrm{Fin}(S)$ denote the class $\{\mathrm{ran}(f) \mid f \in S^n, n \in \omega\}$ of finitely enumerable subsets of S, and let $\mathrm{Fin}^+(S)$ denote the class $\{\sigma \in \mathrm{Fin}(S) \mid \exists x \in S(x \in \sigma)\}$ of finitely enumerable inhabited subsets of S.

The constructive set theory **CZF** is obtained from **ECST** by replacing Replacement by

Strong Collection:

$$\forall a(\forall x \in a \exists y \varphi(x, y) \rightarrow \exists b(\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y)))$$

for every formula $\varphi(x, y)$,

and adding

Subset Collection:

$$\forall a \forall b \exists c \forall u(\forall x \in a \exists y \in b \varphi(x, y, u) \rightarrow \exists d \in c(\forall x \in a \exists y \in d \varphi(x, y, u) \wedge \forall y \in d \exists x \in a \varphi(x, y, u)))$$

for every formula $\varphi(x, y, u)$, and

$\in$-Induction: $\forall a(\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a)$, for every formula $\varphi(a)$.

In **ECST**, Subset Collection implies

Fullness: $\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a,b) \wedge \forall r \in \mathrm{mv}(a,b) \exists s \in c (s \subseteq r))$,

and Fullness and Strong Collection imply Subset Collection. The notable consequence of Fullness is that b^a forms a set, that is

Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.

Note that, in the presence of Exponentiation (and hence Fullness), the classes $\mathrm{Fin}(S)$ and $\mathrm{Fin}^+(S)$ are sets for each set S, by Replacement, Union and Restricted Separation.

For a set S, we write $\mathrm{Pow}(S)$ for the power class of S which is not a set in **ECST** nor in **CZF**:

$$a \in \mathrm{Pow}(S) \Leftrightarrow a \subseteq S.$$

3 Non-deterministic inductive definitions and Fullness

In this section, we work within the subsystem **ECST** of **CZF**.

Definition 2. Let S be a set. Then a *rule* on S is a pair (a, b) of subsets a and b of S. A rule is called *nullary* if a is empty, *elementary* if a is a singleton and *finitary* if a is finitely enumerable. A subset α of S is *closed under* the rule (a, b) if

$$a \subseteq \alpha \Rightarrow b \between \alpha.$$

For a set R of rules on S, we call a subset α of S *R-closed* if it is closed under each rule in R.

Remark 3. Note that if a rule is nullary or elementary, then it is finitary.

Definition 4. Let S be a set and let X be a subclass of $\mathrm{Pow}(S)$. Then X is *set-generated* if there exists a subset G of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G (x \in \beta \subseteq \alpha),$$

and *strongly set-generated* if there exists a subset G of X such that

$$\forall \alpha \in X \forall \sigma \in \mathrm{Fin}(\alpha) \exists \beta \in G (\sigma \subseteq \beta \subseteq \alpha).$$

Definition 5. Let NID and NID* denote the principles that for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated and strongly set-generated, respectively. The principles obtained by restricting R in NID$^{(*)}$ to a set of nullary, elementary and finitary rules are called nullary NID$^{(*)}$, elementary NID$^{(*)}$ and finitary NID$^{(*)}$, respectively.

Remark 6. Trivially, NID* implies NID, and note that finitary NID$^{(*)}$ implies nullary NID$^{(*)}$ and elementary NID$^{(*)}$.

For a set S, let $*_S$ denote the set $\{x \in S \mid x \notin x\}$ which is not in S.

Theorem 7. *Nullary* NID *is equivalent to Fullness.*

Proof. Suppose nullary NID. Let A and B be sets, and define a set R of nullary rules on $(A \times B) \cup \{*_{A\times B}\}$ by

$$R = \{(\emptyset, \{x\} \times B) \mid x \in A\} \cup \{(\emptyset, \{*_{A\times B}\})\}.$$

Then, by nullary NID, there exists a subset G of the class X of R-closed subsets of $(A \times B) \cup \{*_{A\times B}\}$ such that

$$\forall \alpha \in X \forall z \in \alpha \exists \beta \in G(z \in \beta \subseteq \alpha).$$

Let $C = \{\beta \cap (A \times B) \mid \beta \in G\}$. Then for each $\beta \in G$ and $x \in A$, since $\beta \between \{x\} \times B$, there exists $y \in B$ such that $(x, y) \in \beta$, and hence $(x, y) \in \beta \cap (A \times B)$. Therefore $C \subseteq \mathrm{mv}(A, B)$. For each $r \in \mathrm{mv}(A, B)$, since $\{x\} \times B \between r \cup \{*_{A\times B}\}$ for each $x \in A$ and $*_{A\times B} \in r \cup \{*_{A\times B}\}$, we have $r \cup \{*_{A\times B}\} \in X$, and hence there exists $\beta \in G$ such that $*_{A\times B} \in \beta \subseteq r \cup \{*_{A\times B}\}$. Thus $\beta \cap (A \times B) \subseteq (r \cup \{*_{A\times B}\}) \cap (A \times B) = r$.

Conversely, suppose Fullness. Let R be a set of nullary rules on a set S. Then, by Fullness, there exists a set $C \subseteq \mathrm{mv}(R, S)$ such that

$$\forall r \in \mathrm{mv}(R, S) \exists s \in C(s \subseteq r).$$

Let

$$G = \{\{x\} \cup \mathrm{ran}(r) \mid x \in S, r \in C, \forall((\emptyset, b), y) \in r(y \in b)\}.$$

If $r \in C$ and $\forall((\emptyset, b), y) \in r(y \in b)$, then for each $(\emptyset, b) \in R$, we have $b \between \mathrm{ran}(r)$, and hence $b \between \{x\} \cup \mathrm{ran}(r)$ for each $x \in S$. Therefore G is a subset of the class of R-closed subsets of S. For each R-closed subset α of S, since $r = \{((\emptyset, b), y) \mid (\emptyset, b) \in R, y \in b \cap \alpha\} \in \mathrm{mv}(R, S)$, there exists $s \in C$ such that $s \subseteq r$. Note that, since $\forall((\emptyset, b), y) \in r(y \in b)$ and $\mathrm{ran}(r) \subseteq \alpha$, we have $\forall((\emptyset, b), y) \in s(y \in b)$ and $\mathrm{ran}(s) \subseteq \alpha$. Therefore if $x \in \alpha$, then $\{x\} \cup \mathrm{ran}(s) \in G$ and $x \in \{x\} \cup \mathrm{ran}(s) \subseteq \{x\} \cup \alpha = \alpha$. $\square$

Proposition 8. *Elementary* NID *implies nullary* NID.

Proof. Let R be a set of nullary rules on a set S, and define a set R' of elementary rules on $S \cup \{*_S\}$ by

$$R' = \{(\{*_S\}, b) \mid (\emptyset, b) \in R\} \cup \{(\{x\}, \{*_S\})\}) \mid x \in S\}.$$

Then, by elementary NID, there exists a subset G' of the class X' of R'-closed subsets of $S \cup \{*_S\}$ such that

$$\forall \alpha' \in X' \forall z \in \alpha' \exists \beta' \in G' (z \in \beta' \subseteq \alpha').$$

Let $G = \{\beta' \cap S \mid \beta' \in G', *_S \in \beta'\}$. Then for each $\beta' \cap S \in G$ and $(\emptyset, b) \in R$, since $*_S \in \beta'$, we have $b \between \beta'$, and hence $b \between \beta' \cap S$. Therefore G is a set of R-closed subsets of S. Let α be an R-closed subset of S. Then for each $(\emptyset, b) \in R$, since $b \between \alpha$, we have $b \between \alpha \cup \{*_S\}$, and $*_S \in \alpha \cup \{*_S\}$. Hence $\alpha \cup \{*_S\} \in X'$. Therefore for each $x \in \alpha$, since $x \in \alpha \cup \{*_S\}$, there exists $\beta' \in G'$ such that $x \in \beta' \subseteq \alpha \cup \{*_S\}$, and so $*_S \in \beta'$. Thus $\beta' \cap S \in G$ and $x \in \beta' \cap S \subseteq (\alpha \cup \{*_S\}) \cap S = \alpha$. □

Remark 9. Note that, assuming nullary, elementary or finitary NID, we have Fullness, and hence the classes $\mathrm{Fin}(S)$ and $\mathrm{Fin}^+(S)$ are sets for each set S.

For a subset α of the disjoint union $A + B$, let $(\alpha)_0$ and $(\alpha)_1$ denote the sets $\{x \in A \mid (0, x) \in \alpha\}$ and $\{y \in B \mid (1, y) \in \alpha\}$, respectively.

Proposition 10. *Nullary, elementary and finitary* NID *imply nullary, elementary and finitary* NID*, *respectively.*

Proof. Let R be a set of nullary rules on a set S, and define a set R' of nullary rules on $\mathrm{Fin}(S)$ by

$$R' = \{(\emptyset, \mathrm{Fin}^+(b)) \mid (\emptyset, b) \in R\}.$$

Then, by nullary NID, there exists a subset G' of the class X' of R'-closed subsets of $\mathrm{Fin}(S)$ such that

$$\forall \alpha' \in X' \forall \sigma \in \alpha' \exists \beta' \in G' (\sigma \in \beta' \subseteq \alpha').$$

Let $G = \{\bigcup \beta' \mid \beta' \in G'\}$. Then for each $\beta' \in G'$ and $(\emptyset, b) \in R$, since $\mathrm{Fin}^+(b) \between \beta'$, there exists $\sigma \in \mathrm{Fin}^+(b) \cap \beta'$, hence there exists $x \in b$ such that $x \in \sigma \in \beta'$, and therefore $b \between \bigcup \beta'$. Thus G is a set of R-closed subsets of S. Let α be an R-closed subset of S. Then for each $(\emptyset, \mathrm{Fin}^+(b)) \in R'$, since $(\emptyset, b) \in R$, we have $b \between \alpha$, and hence $\mathrm{Fin}^+(b) \between \mathrm{Fin}(\alpha)$. Therefore $\mathrm{Fin}(\alpha) \in X'$. Thus for each $\sigma \in \mathrm{Fin}(\alpha)$ there exists $\beta' \in G'$ such that $\sigma \in \beta' \subseteq \mathrm{Fin}(\alpha)$, and so $\sigma \subseteq \bigcup \beta' \subseteq \bigcup \mathrm{Fin}(\alpha) = \alpha$.

Let R be a set of elementary or finitary rules on a set S, and define a set R' of elementary and finitary rules on $S + \mathrm{Fin}(S)$ by

$$\begin{aligned} R' = &\{(1 \times a, 1 \times b) \mid (a, b) \in R\} \\ &\cup \{(1 \times a, \{(1, \sigma) \mid \sigma \in \mathrm{Fin}(b)\}) \mid (a, b) \in R\} \\ &\cup \{(\{(1, \sigma)\}, \{(0, x)\}) \mid \sigma \in \mathrm{Fin}(S), x \in \sigma\}. \end{aligned}$$

Note that R' is elementary and finitary if R is elementary and finitary, respectively. Then, by elementary or finitary NID, there exists a subset G' of the class X' of R'-closed subsets of $S + \mathrm{Fin}(S)$ such that

$$\forall \alpha' \in X' \forall z \in \alpha' \exists \beta' \in G'(z \in \beta' \subseteq \alpha').$$

Let $G = \{(\beta')_0 \mid \beta' \in G'\}$. Then for each $\beta' \in G'$ and $(a, b) \in R$ with $a \subseteq (\beta')_0$, since $1 \times a \subseteq \beta'$, we have $1 \times b \between \beta'$, and hence $b \between (\beta')_0$. Therefore G is a set of R-closed subsets of S. Let α be an R-closed subset of S. Then $\alpha + \mathrm{Fin}(\alpha)$ is an R'-closed subset of $S + \mathrm{Fin}(S)$. In fact, for each $(a, b) \in R$, if $1 \times a \subseteq \alpha + \mathrm{Fin}(\alpha)$, then, since $a \subseteq \alpha$, we have $b \between \alpha$, and hence $1 \times b \between \alpha + \mathrm{Fin}(\alpha)$ and $(1, \sigma) \in \alpha + \mathrm{Fin}(\alpha)$ for some $\sigma \in \mathrm{Fin}(b)$. For $(\{(1, \sigma)\}, \{(0, x)\}) \in R'$ where $\sigma \in \mathrm{Fin}(S)$ and $x \in \sigma$, if $\{(1, \sigma)\} \subseteq \alpha + \mathrm{Fin}(\alpha)$, then, since $\sigma \in \mathrm{Fin}(\alpha)$, we have $x \in \sigma \subseteq \alpha$, and hence $(0, x) \in \alpha + \mathrm{Fin}(\alpha)$. Therefore $\alpha + \mathrm{Fin}(\alpha) \in X'$. Thus for each $\sigma \in \mathrm{Fin}(\alpha)$ there exists $\beta' \in G'$ such that $(1, \sigma) \in \beta' \subseteq \alpha + \mathrm{Fin}(\alpha)$, and since β' is R'-closed and $(0, x) \in \beta'$ for each $x \in \sigma$, we have $\sigma \subseteq (\beta')_0 \subseteq (\alpha + \mathrm{Fin}(\alpha))_0 = \alpha$. □

Corollary 11. *Nullary, elementary and finitary* NID *are equivalent to respectively nullary, elementary and finitary* NID*.

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