Lecture I

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Contents

- Minimal logic
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- Classical logic

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- (individual) variables $v_0, v_1, \ldots$;
- (individual) constants $c_0, c_1, \ldots$;
- predicate (relation) symbols $R_0, R_1, \ldots$;
- function symbols $f_0, f_1, \ldots$;
- primitive logical operators $\wedge, \vee, \rightarrow, \bot, \forall, \exists$.

Terms

Terms are defined inductively by

- variables and constants are terms;
- if $t_1, \ldots, t_n$ are terms and f is an (n-ary) function symbol, then $f(t_1, \ldots, t_n)$ is a term.

Examples:

- x, c are terms.
- $f(x)$, $f(c)$ are terms.
- $g(f(x), c)$ is a term.

Formulae

Formulae are defined inductively by

- $\bot$ is a formula;
- if $t_1, \ldots, t_n$ are terms and R is an (n-ary) predicate symbol, then $R(t_1, \ldots, t_n)$ is an (atomic) formula;
- if A and B are formulae, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae;
- if A is a formula and x is a variable, then $(\forall x A)$ and $(\exists x A)$ are formulae.

We introduce the abbreviations

- $\neg A \equiv A \rightarrow \bot$;
- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

Examples

- $P(x)$, $Q(f(x), y)$, $R(f(x), c)$ are atomic formulae.
- $<(x, y)$ or $x < y$, $=(f(x), y)$ or $f(x) = y$ are atomic formulae.
- $P(x) \rightarrow Q(f(x), y)$, $f(x) = y \wedge R(f(x), c)$, $x < y \vee f(x) = y$, $\neg R(f(x), c)$ are forumulae.
- $\forall x P(x)$, $\forall x(f(x) = y)$, $\exists y(y < y)$ are formulae.
- $\forall x P(x) \rightarrow \exists y Q(f(x), y)$, $\exists x(Q(f(x), y) \wedge \forall x R(f(x), c))$ are formulae.

Free variables

The set $\mathrm{FV}(t)$ of free variables of a term t is defined inductively by

- $\mathrm{FV}(x) := \{x\}$ and $\mathrm{FV}(c) := \emptyset$;
- $\mathrm{FV}(f(t_1, \ldots, t_n)) := \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$.

The set $\mathrm{FV}(A)$ of free variables of a formula A is defined inductively by

- $\mathrm{FV}(\bot) := \emptyset$;
- $\mathrm{FV}(R(t_1, \ldots, t_n)) := \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$;
- $\mathrm{FV}(A \circ B) := \mathrm{FV}(A) \cup \mathrm{FV}(B)$, where $\circ \in \{\wedge, \vee, \to\}$;
- $\mathrm{FV}(\forall x A) := \mathrm{FV}(\exists x A) := \mathrm{FV}(A) \setminus \{x\}$.

For a (multi) set Γ of formulae, let $\mathrm{FV}(\Gamma) := \bigcup\{\mathrm{FV}(A) \mid A \in \Gamma\}$.

Examples

- $\mathrm{FV}(f(x)) = \mathrm{FV}(x) = \{x\}$
- $\mathrm{FV}(x = c)) = \mathrm{FV}(x) \cup \mathrm{FV}(c) = \mathrm{FV}(x) \cup \emptyset = \{x\}$
- $\mathrm{FV}(f(x) = y)) = \mathrm{FV}(f(x)) \cup \mathrm{FV}(y) = \{x, y\}$
- $\mathrm{FV}(x < y \vee f(x) = y) = \mathrm{FV}(x < y) \cup \mathrm{FV}(f(x) = y) = \{x, y\}$
- $\mathrm{FV}(\forall x(f(x) = y)) = \mathrm{FV}(f(x) = y) \backslash \{x\} = \{x, y\} \backslash \{x\} = \{y\}$
- $\mathrm{FV}(\forall x P(x) \to \exists y Q(f(x), y)) = \mathrm{FV}(\forall x P(x)) \cup \mathrm{FV}(\exists y Q(f(x), y)) = (\{x\} \setminus \{x\}) \cup (\{x, y\} \setminus \{y\}) = \{x\}$

Substitution

Let s and t be terms, and let x be a variable. Then define a term $s[x/t]$ by

- $x[x/t] \equiv t$, $y[x/t] \equiv y$ $(x \not\equiv y)$, and $c[x/t] \equiv c$;
- $(f(t_1, \ldots, t_n))[x/t] \equiv f(t_1[x/t], \ldots, t_n[x/t])$.

Let A be a formula, let t be a term, and let x be a variable. Then define a formula $A[x/t]$ by

- $\bot[x/t] \equiv \bot$;
- $R(t_1, \ldots, t_n)[x/t] \equiv R(t_1[x/t], \ldots, t_n[x/t])$;
- $(A \circ B)[x/t] \equiv (A[x/t] \circ B[x/t])$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- $(\forall y A)[x/t] \equiv \forall y(A[x/t])$ and $(\exists y A)[x/t] \equiv \exists y(A[x/t])$, if $x \not\equiv y$, and $(\forall y A)[x/t] \equiv \forall y A$ and $(\exists y A)[x/t] \equiv \exists y A$, otherwise.

Examples

- $g(f(x), c)[x/y] \equiv g(f(x)[x/y], c[x/y]) \equiv g(f(x[x/y]), c) \equiv g(f(y), c)$.
- $Q(f(x), y)[x/c] \equiv Q(f(x)[x/c], y[x/c]) \equiv Q(f(x[x/c]), y) \equiv Q(f(c), y)$.
- $(x < y \lor f(x) = y)[y/g(x)] \equiv (x < y)[y/g(x)] \lor (f(x) = y)[y/g(x)] \equiv x < g(x) \lor f(x) = g(x)$.
- $(\forall x(f(x) = y))[x/c] \equiv \forall x(f(x) = y)$.
- $(\exists x(f(x) = y))[y/c] \equiv \exists x((f(x) = y)[y/c]) \equiv \exists x(f(x) = c)$.

- $(\forall y(x = y))[x/y] \equiv \forall y((x = y)[x/y]) \equiv \forall y(y = y)$.
- $(\exists y(x < y))[x/y] \equiv \exists y((x < y)[x/y]) \equiv \exists y(y < y)$.

Free for

Let A be a formula, let t be a term, and let x be a variable. Then define a predicate t is free for x in A by

- t is free for x in $\bot$;
- t is free for x in $R(t_1, \ldots, t_n)$;
- if t is free for x in A and B, then t is free for x in $(A \circ B)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- if $x \equiv y$, or t is free for x in A and $y \notin \mathrm{FV}(t)$, then t is free for x in $\forall y A$ and $\exists y A$.

Examples

- c is free for x in $Q(f(x), y)$.
- $g(x)$ is free for y in $(x < y \vee f(x) = y)$.
- y is free for x in $\forall x(f(x) = y)$.
- $f(y)$ is free for y in $\forall x(f(x) = y)$.

- y is not free for x in $\forall y(x = y)$.
- y is not free for x in $\exists y(x < y)$.

Natural Deduction System

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{array}{c} \Gamma \\ \mathcal{D} \\ A \end{array}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Deduction (Basis)

For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Deduction (Induction step, $\to$I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

is a deduction with conclusion $A \to B$ and assumptions $\Gamma \setminus \{A\}$.
We write

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I}$$

Deduction (Induction step, $\to$E)

If

$$\begin{array}{ccc} \Gamma_1 & \quad & \Gamma_2 \\ \mathcal{D}_1 & & \mathcal{D}_2 \\ A \to B & & A \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A \to B & A \end{array}}{B} \to\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      [\neg\neg A] \qquad
      \dfrac{
        \dfrac{
          [\neg\neg(A \to B)] \qquad
          \dfrac{
            \dfrac{
              [\neg B] \qquad
              \dfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}
            }{\bot}\,{\to}\mathrm{E}
          }{\neg(A \to B)}\,{\to}\mathrm{I}
        }{\bot}\,{\to}\mathrm{E}
      }{\neg A}\,{\to}\mathrm{I}
    }{\bot}\,{\to}\mathrm{E}
  }{
    \dfrac{\neg\neg B}{\neg\neg A \to \neg\neg B}\,{\to}\mathrm{I}
  }\,{\to}\mathrm{I}
}{\neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)}\,{\to}\mathrm{I}
$$

Deduction (Induction step, $\wedge I$)

If

$$\begin{array}{ccc} \Gamma_1 & \quad & \Gamma_2 \\ \mathcal{D}_1 & & \mathcal{D}_2 \\ A & & B \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A & B \end{array}}{A \wedge B}\ \wedge I$$

is a deduction with conclusion $A \wedge B$ and assumptions $\Gamma_1 \cup \Gamma_2$.

Deduction (Induction step, $\wedge\mathrm{E}$)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A \wedge B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A \wedge B\end{array}}{A}\ \wedge\mathrm{E}_r \qquad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A \wedge B\end{array}}{B}\ \wedge\mathrm{E}_l$$

are deductions with conclusions A and B, respectively, and assumptions Γ.

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{[(A \to B) \wedge (A \to C)]}{A \to B}\,\wedge\mathrm{E} \qquad [A]
      }{B}
      \qquad
      \dfrac{
        \dfrac{[(A \to B) \wedge (A \to C)]}{A \to C}\,\wedge\mathrm{E} \qquad [A]
      }{C}
    }{B \wedge C}\,\wedge\mathrm{I}
  }{A \to B \wedge C}
}{(A \to B) \wedge (A \to C) \to (A \to B \wedge C)}
$$

Deduction (Induction step, $\vee$I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}}{A \vee B}\ \vee\mathrm{I}_r \qquad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}}{B \vee A}\ \vee\mathrm{I}_l$$

are deductions with conclusions $A \vee B$ and $B \vee A$, respectively, and assumptions Γ.

Deduction (Induction step, $\vee$E)

If

$$\begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ A \vee B & C & C \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{c} \Gamma_1 \\ \mathcal{D}_1 \\ A \vee B \end{array} \quad \begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ C \end{array} \quad \begin{array}{c} \Gamma_3 \\ \mathcal{D}_3 \\ C \end{array}}{C} \ \vee\mathrm{E}$$

is a deduction with conclusion C and assumptions $\Gamma_1 \cup (\Gamma_2 \setminus \{A\}) \cup (\Gamma_3 \setminus \{B\})$.
We write

$$\dfrac{\begin{array}{c} \mathcal{D}_1 \\ A \vee B \end{array} \quad \begin{array}{c} [A] \\ \mathcal{D}_2 \\ C \end{array} \quad \begin{array}{c} [B] \\ \mathcal{D}_3 \\ C \end{array}}{C} \ \vee\mathrm{E}$$

Example

$$\dfrac{\dfrac{[A \lor B] \qquad \dfrac{\dfrac{[(A \to C) \land (B \to C)]}{A \to C}\,\land\mathrm{E} \qquad [A]}{C} \qquad \dfrac{\dfrac{[(A \to C) \land (B \to C)]}{B \to C}\,\land\mathrm{E} \qquad [B]}{C}}{\dfrac{C}{A \lor B \to C}}\,\lor\mathrm{E}}{(A \to C) \land (B \to C) \to (A \lor B \to C)}$$

Deduction (Induction step, $\forall$I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$$

is a deduction, $x \notin \mathrm{FV}(\Gamma)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}}{\forall y A[x/y]}\ \forall\mathrm{I}$$

is a deduction with conclusion $\forall y A[x/y]$ and assumptions Γ.

Deduction (Induction step, $\forall$E)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \forall xA\end{array}$$

is a deduction, and t is free for x in A, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \forall xA\end{array}}{A[x/t]}\ \forall\mathrm{E}$$

is a deduction with conclusion $A[x/t]$ and assumptions Γ.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \to \forall x B] \quad [A]}{\forall x B}}{B}\ \forall\mathrm{E}}{A \to B}}{\dfrac{\forall x (A \to B)}{(A \to \forall x B) \to \forall x (A \to B)}}\ \forall\mathrm{I}$$

where $x \notin \mathrm{FV}(A)$.

Bad Examples

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x = c]}{\forall y(y = c)}\ \forall I}{x = c \rightarrow \forall y(y = c)}\ \rightarrow I}{\forall x(x = c \rightarrow \forall y(y = c))}\ \forall I}{c = c \rightarrow \forall y(y = c)}\ \forall E
$$

$x \in \mathrm{FV}(x = c)$.

$$
\dfrac{\dfrac{\dfrac{[\forall x(f(x) = y)]}{f(x) = y}\ \forall E}{\forall y(f(y) = y) \equiv \forall y((f(x) = y)[x/y])}\ \forall I}{\forall x(f(x) = y) \rightarrow \forall y(f(y) = y)}
$$

$y \in \mathrm{FV}(f(x) = y)$.

Bad Examples

$$\dfrac{\dfrac{[\forall x \exists y (x < y)]}{\exists y (y < y) \equiv (\exists y (x < y))[x/y]}\ \forall\mathrm{E}}{\forall x \exists y (x < y) \to \exists y (y < y)}$$

y is not free for x in $\exists y (x < y)$.

Deduction (Induction step, $\exists\mathrm{I}$)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A[x/t]\end{array}$$

is a deduction, and t is free for x in A, then

$$\cfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A[x/t]\end{array}}{\exists x A}\ \exists\mathrm{I}$$

is a deduction with conclusion $\exists x A$ and assumptions Γ.

Deduction (Induction step, $\exists\mathrm{E}$)

If

$$\begin{array}{ccc} \Gamma_1 & & \Gamma_2 \\ \mathcal{D}_1 & & \mathcal{D}_2 \\ \exists y A[x/y] & & C \end{array}$$

are deductions, $x \notin \mathrm{FV}(C)$, $x \notin \mathrm{FV}(\Gamma_2 \setminus \{A\})$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$, then

$$\dfrac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \exists y A[x/y] & C \end{array}}{C}\ \exists\mathrm{E}$$

is a deduction with conclusion C and assumptions $\Gamma_1 \cup (\Gamma_2 \setminus \{A\})$.
We write

$$\dfrac{\begin{array}{cc} & [A] \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \exists y A[x/y] & C \end{array}}{C}\ \exists\mathrm{E}$$

Example

$$
\dfrac{\dfrac{[\exists x(A \to B)] \qquad \dfrac{\dfrac{[A \to B] \quad [A]}{B}}{\exists x B}\,\exists\mathrm{I}}{\exists x B}\,\exists\mathrm{E}}{\dfrac{A \to \exists x B}{\exists x(A \to B) \to (A \to \exists x B)}}
$$

where $x \notin \mathrm{FV}(A)$.

Bad Examples

$$\dfrac{\dfrac{[\forall y(y = y) \equiv (\forall y(x = y))[x/y]]}{\exists x \forall y(x = y)}\ \exists\mathrm{I}}{\forall y(y = y) \to \exists x \forall y(x = y)}$$

y is not free for x in $\forall y(x = y)$.

$$\dfrac{\dfrac{[\exists x(x = c)] \quad \dfrac{[x = c]}{}}{\dfrac{x = c}{\forall x(x = c)}\ \forall\mathrm{I}}\ \exists\mathrm{E}}{\exists x(x = c) \to \forall x(x = c)}$$

$x \in \mathrm{FV}(x = c)$.

Bad Examples

$$\dfrac{\dfrac{[\exists x \neg(x = c)] \qquad \dfrac{[\neg(x = c)] \quad [x = c]}{\bot}}{\bot}\ \exists \mathrm{E}}{\dfrac{\dfrac{\neg(x = c)}{\forall x \neg(x = c)}\ \forall \mathrm{I}}{\exists x \neg(x = c) \to \forall x \neg(x = c)}}$$

$x \in \mathrm{FV}(x = c)$.

$$\dfrac{\dfrac{\dfrac{\exists y(y\,R\,y) \equiv \exists y((x\,R\,y)[x/y]) \qquad \dfrac{[x\,R\,y]}{\exists x(x\,R\,y)}\ \exists \mathrm{I}}{\exists x(x\,R\,y)}\ \exists \mathrm{E}}{\forall y \exists x(x\,R\,y)}\ \forall \mathrm{I}}{\exists y(y\,R\,y) \to \forall y \exists x(x\,R\,y)}$$

$y \in \mathrm{FV}(x\,R\,y)$.

Minimal logic (summary)

$$\frac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\;{\to}\mathrm{I} \qquad\qquad \frac{\begin{array}{c}\mathcal{D}_1\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\;{\to}\mathrm{E}$$

$$\frac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A \wedge B}\;{\wedge}\mathrm{I} \qquad\qquad \frac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{A}\;{\wedge}\mathrm{E}_r \qquad \frac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{B}\;{\wedge}\mathrm{E}_l$$

$$\frac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{A \vee B}\;{\vee}\mathrm{I}_r \qquad \frac{\begin{array}{c}\mathcal{D}\\ B\end{array}}{A \vee B}\;{\vee}\mathrm{I}_l \qquad\qquad \frac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]\\ \mathcal{D}_3\\ C\end{array}}{C}\;{\vee}\mathrm{E}$$

Minimal logic (summary)

$$\frac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{\forall y A[x/y]}\ \forall\mathrm{I} \qquad \frac{\begin{array}{c}\mathcal{D}\\ \forall x A\end{array}}{A[x/t]}\ \forall\mathrm{E}$$

$$\frac{\begin{array}{c}\mathcal{D}\\ A[x/t]\end{array}}{\exists x A}\ \exists\mathrm{I} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \exists y A[x/y]\end{array} \quad \begin{array}{c}[A]\\ \mathcal{D}_2\\ C\end{array}}{C}\ \exists\mathrm{E}$$

- In $\forall\mathrm{E}$ and $\exists\mathrm{I}$, t must be free for x in A.
- In $\forall\mathrm{I}$, $\mathcal{D}$ must not contain assumptions containing x free, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.
- In $\exists\mathrm{E}$, $\mathcal{D}_2$ must not contain assumptions containing x free except A, $x \notin \mathrm{FV}(C)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

Minimal logic

We denote by

$$\Gamma \vdash_m A$$

that there is a deduction in minimal logic with conclusion A and assumptions in Γ.

Exercises

- $\vdash_m \neg A \leftrightarrow \neg\neg\neg A$,
- $\vdash_m \neg\neg A \wedge \neg\neg B \leftrightarrow \neg\neg(A \wedge B)$,
- $\vdash_m \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$,
- $\vdash_m \forall x \neg\neg A \leftrightarrow \neg\neg\forall x \neg\neg A$.

- $\vdash_m \neg(\neg A \wedge \neg B) \leftrightarrow \neg\neg(A \vee B)$,
- $\vdash_m (\neg\neg A \to \neg\neg B) \leftrightarrow (A \to \neg\neg B)$,
- $\vdash_m \neg\forall x \neg A \leftrightarrow \neg\neg\exists x A$.

Place holder

We introduce

- a proposition symbol (0-ary predicate symbol) $*$ acting as a place holder.
- an abbreviation $\neg_* A \equiv A \to *$.

Substitution

Let A and C be formulae. Then define a formula $A[*/C]$ by

- $\bot[*/C] \equiv \bot$;
- $*[*/C] \equiv C$ and $(R(t_1, \ldots, t_n))[*/C] \equiv R(t_1, \ldots, t_n)$;
- $(A \circ B)[*/C] \equiv (A[*/C] \circ B[*/C])$, where $\circ \in \{\wedge, \vee, \to\}$;
- $(\forall x A)[*/C] \equiv \forall x(A[*/C])$ and $(\exists x A)[*/C] \equiv \exists x(A[*/C])$,

Free for

Let A and C be formulae. Then define a predicate C is free for $*$ in A by

- C is free for $*$ in $\bot$;
- C is free for $*$ in $*$ and $R(t_1, \ldots, t_n)$;
- if C is free for $*$ in A and B, then C is free for $*$ in $(A \circ B)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
- if C is free for $*$ in A and $x \notin \mathrm{FV}(C)$, then C is free for $*$ in $\forall x A$ and $\exists x A$.

Example

$$
\dfrac{\dfrac{\dfrac{[\neg_*\neg_* A] \qquad \dfrac{\dfrac{[\neg_*\neg_*(A \to B)] \qquad \dfrac{\dfrac{[\neg_* B] \qquad \dfrac{[A \to B] \quad [A]}{B}\ {\to}\mathrm{E}}{*}\ {\to}\mathrm{E}}{\neg_*(A \to B)}\ {\to}\mathrm{I}}{*}\ {\to}\mathrm{E}}{\neg_* A}\ {\to}\mathrm{I}}{\dfrac{*}{\neg_*\neg_* B}\ {\to}\mathrm{I}}\ {\to}\mathrm{E}}{\neg_*\neg_* A \to \neg_*\neg_* B}\ {\to}\mathrm{I}}{\neg_*\neg_*(A \to B) \to (\neg_*\neg_* A \to \neg_*\neg_* B)}\ {\to}\mathrm{I}
$$

Example

$$
\cfrac{\cfrac{\cfrac{[\neg_* \neg_* A] \qquad \cfrac{\cfrac{[\neg_* \neg (A \to B)] \qquad \cfrac{\cfrac{[\neg B] \qquad \cfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg (A \to B)}\,{\to}\mathrm{I}}{*}\,{\to}\mathrm{E}}{\neg_* A}\,{\to}\mathrm{I}}{*}\,{\to}\mathrm{E}}{\neg_* \neg B}\,{\to}\mathrm{I}}{\neg_* \neg_* A \to \neg_* \neg B}\,{\to}\mathrm{I}
$$

$$
\cfrac{\neg_* \neg_* A \to \neg_* \neg B}{\neg_* \neg (A \to B) \to (\neg_* \neg_* A \to \neg_* \neg B)}\,{\to}\mathrm{I}
$$

Intuitionistic logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_i$$

is a deduction with conclusion A and assumptions Γ.

Example

$$
\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \quad [A]}{\bot}}{B}\,{\color{red}\bot_i}}{A \to B}}{\bot}}{\neg\neg A}}{\neg\neg B} \qquad \dfrac{\dfrac{[\neg(A \to B)] \quad \dfrac{[B]}{A \to B}}{\bot}}{\neg B}}{\dfrac{\bot}{\neg\neg(A \to B)}}}{(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)}
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg_* \neg A \to \neg_* \neg_* B] \qquad \dfrac{\dfrac{[\neg_*(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \quad [A]}{\bot}}{B}\ \bot_i}{A \to B}}{*}}{\neg_* \neg A}}{\neg_* \neg_* B} \qquad \dfrac{\dfrac{\neg_*(A \to B)] \qquad \dfrac{[B]}{A \to B}}{*}}{\neg_* B}}{*}}{\neg_* \neg_* (A \to B)}}{(\neg_* \neg A \to \neg_* \neg_* B) \to \neg_* \neg_* (A \to B)}
$$

Intuitionistc logic

We denote by

$$\Gamma \vdash_i A$$

that there is a deduction in intuitionistic logic with conclusion A and assumptions in Γ.

Note that

$$\Gamma \vdash_m A \Rightarrow \Gamma \vdash_i A.$$

Exercises

- $\vdash_m (A \to B) \to (\neg_* \neg_* A \to \neg_* \neg_* B)$,
- $\vdash_m (\neg_* \neg_* A \to \neg_* \neg_* B) \leftrightarrow (A \to \neg_* \neg_* B)$,
- $\vdash_m \neg_* \neg (A \to B) \to (\neg_* \neg_* A \to \neg_* \neg B)$,
- $\vdash_i (\neg_* \neg A \to \neg_* \neg_* B) \to \neg_* \neg_* (A \to B)$.

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

is a deduction with conclusion A and assumption $\Gamma \setminus \{\neg A\}$.

Example (classical logic)

The double negation elimination (DNE):

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\,\to\mathrm{E}}{A}\,\bot_c}{\neg\neg A \to A}\,\to\mathrm{I}$$

Example (classical logic)

The principle of excluded middle (PEM):

$$
\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{[A]}{A \vee \neg A}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{A \vee \neg A}\,\vee\mathrm{I}_l}{\bot}\,\to\mathrm{E}}{A \vee \neg A}\,\bot_c
$$

Example (classical logic)

De Morgan's law (DML):

$$
\dfrac{
  \dfrac{
    [\neg(\neg A \vee \neg B)]
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          [\neg(\neg A \vee \neg B)]
          \qquad
          \dfrac{
            \dfrac{
              \dfrac{
                [\neg(A \wedge B)]
                \qquad
                \dfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathrm{I}
              }{\bot}\,\to\mathrm{E}
            }{\neg A}\,\to\mathrm{I}
          }{\neg A \vee \neg B}\,\vee\mathrm{I}_r
        }{\bot}\,\to\mathrm{E}
      }{\neg B}\,\to\mathrm{I}
    }{\neg A \vee \neg B}\,\vee\mathrm{I}_l
  }{
    \dfrac{\bot}{\neg A \vee \neg B}\,\bot_c
  }\,\to\mathrm{E}
}{\neg(A \wedge B) \to \neg A \vee \neg B}\,\to\mathrm{I}
$$

Classical logic

We denote by

$$\Gamma \vdash_c A$$

that there is a deduction in classical logic with the conclusion A and the assumptions in Γ.

Note that

$$\Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A.$$

RAA vs $\rightarrow$I

$\bot_c$: deriving A by deducing absurdity ($\bot$) from $\neg A$.

$$\dfrac{\begin{matrix}[\neg A]\\ \mathcal{D}\\ \bot\end{matrix}}{A}\ \bot_c$$

$\rightarrow$I: deriving $\neg A$ by deducing absurdity ($\bot$) from A.

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg A}\ \rightarrow\mathrm{I}$$

References

- D. van Dalen, *Logic and Structure*, 5th ed., Springer, London, 2013.
- A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics, An Introduction*, Vol. I, North-Holland, Amsterdam, 1988.