Determinacy of Wadge classes and subsystems of second order arithmetic

Takako Nemoto*

E-mail: sa4m20@math.tohoku.ac.jp

Mathematical Institute, Tohoku University,
Sendai, 980-8578, Japan

Abstract

In this paper we study the logical strength of the determinacy of infinite binary games in terms of second order arithmetic. We define new determinacy schemata inspired by the Wadge classes of Polish spaces and show the following equivalences over the system RCA_0^* , which consists of the axioms of discrete ordered semi-ring with exponentiation, Δ_1^0 comprehension and Π_0^0 induction, and which is known as a weaker system than the popular base theory RCA_0 :

- $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^* \leftrightarrow \text{WKL}_0$;
- $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{ATR}_0 + \Sigma_1^1 \text{ induction}$;
- $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \Pi_1^1\text{-CA}_0$;
- $\text{Bisep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \Pi_1^1\text{-TR}_0$;

where Det^* stands for the determinacy of infinite games in the Cantor space.

1 Introduction

We consider the following type of game: Two players, say player I and player II, alternatively choose an element of X to form an infinite sequence f . Player I wins if and only if a given formula $\varphi(f)$ of f holds. Player II wins if and only if I does not win. The formula $\varphi(f)$ is called a winning condition for player I. A determinacy statement asserts that one of the players has a winning strategy in such games. Within the framework of second order arithmetic, the strength of determinacy of games in the Baire space, i.e., the above games with $X = \mathbb{N}$, has been previously investigated in [3], [4], [8], [10], [11] and [12], and that of games in the Cantor space, i.e., the games with $X = 2 = \{0, 1\}$, has been investigated in [5]. Their results are summarized in Table 1, where a subsystem of second order arithmetic and types of determinacy in the same line are pairwise equivalent over a suitable base theory (RCA_0 ; except for the last line), and where $(\Sigma_2^0)_2$ is the class $\Sigma_2^0 \wedge \Pi_2^0$.

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Subsystem of second order arithmetic	Determinacy in the Cantor space	Determinacy in the Baire space
WKL_0	Δ_1^0 Σ_1^0	
ACA_0	$(\Sigma_1^0)_2$	
ATR_0	Δ_2^0 Σ_2^0	Δ_1^0 Σ_1^0
$\Pi_1^1\text{-CA}_0$		$(\Sigma_1^0)_2$
$\Pi_1^1\text{-TR}_0$		Δ_2^0
$\Sigma_1^1\text{-ID}_0$	$(\Sigma_2^0)_2$	Σ_2^0
$\vdots$	$\vdots$	$\vdots$
$[\Sigma_1^1]^k\text{-ID}_0$	$(\Sigma_2^0)_{k+1}$	$(\Sigma_2^0)_k$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma_1^1]^{\text{TR}}\text{-ID}_0$	Δ_3^0	Δ_3^0

Table 1: Results of earlier researches

In Table 1, we can find a large gap between Σ_2^0 and $(\Sigma_2^0)_2$ determinacy in the Cantor space. Thus, the aims of this paper are to find (1) a class whose determinacy in the Cantor space is equivalent to either $\Pi_1^1\text{-CA}_0$ or $\Pi_1^1\text{-TR}_0$ and (2) a finer hierarchy of determinacy in the Cantor space.

To find such a hierarchy, we consider the determinacy schemata which formalize, in the language of second order arithmetic, the determinacies whose winning conditions are in Borel Wadge classes in Polish spaces and also investigate the strengths of the schemata. Borel Wadge classes, which are defined as those classes of Borel sets that are closed under continuous pre-images, are known to form a finer hierarchy than the Hausdorff difference hierarchy (cf. [2], [13]). Since continuous pre-images preserve Boolean operations, this hierarchy seems to be the finest among those hierarchies to which we can define corresponding determinacy schemata at least in an obvious manner.

If we continue to work over RCA_0 , we will find that weak König's lemma and $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$, Σ_1^0 and Δ_1^0 determinacies in the Cantor space are pairwise equivalent over RCA_0 , where $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ determinacy formalizes the determinacy of the Wadge class immediately above the class Σ_1^0 of open sets (see Figure 2 and [2]). However, the proof of $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ determinacy essentially needs Σ_1^0 induction, while that of Σ_1^0 and Δ_1^0 determinacies does not (in [5]). To shed light on the differences among the strengths of the determinacies of those classes, we replace the base theory with RCA_0^* , which lacks Σ_1^0 induction. Although almost all the results in Table 1 were equivalences over RCA_0 , it is routine to check that the base theory can be replaced with RCA_0^* .

This paper proves the following theorem, and as a result, Table 1 is enhanced into Table 2.

Theorem RCA_0^* proves the following four equivalences, where Det^* (resp. Det) represents the determinacy of infinite games in the Cantor space (resp. the Baire space).

- $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^* \leftrightarrow \text{WKL}_0$
- $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{ATR}_0 + \Sigma_1^1$ induction
- $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det} \leftrightarrow \Pi_1^1\text{-CA}_0$.
- $\text{Bisep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \Pi_1^1\text{-TR}_0$.

Subsystem of second order arithmetic	Determinacy in the Cantor space	Determinacy in the Baire space
WKL_0^*	Δ_1^0 Σ_1^0	-
WKL_0	$\text{Bisep}(\Delta_1^0, \Sigma_1^0)$	-
ACA_0	$(\Sigma_1^0)_2$	-
ATR_0	Δ_2^0 Σ_2^0	Δ_1^0 Σ_1^0
$\text{ATR}_0 + \Sigma_1^1$ induction	$\text{Bisep}(\Delta_1^0, \Sigma_2^0)$	?
$\Pi_1^1\text{-CA}_0$	$\text{Bisep}(\Sigma_1^0, \Sigma_2^0)$ $\vdots$ $\text{Sep}(\Sigma_1^0, \Sigma_2^0)$	$\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ $(\Sigma_1^0)_2$
$\Pi_1^1\text{-TR}_0$	$\text{Bisep}(\Delta_2^0, \Sigma_2^0)$	Δ_2^0
$\Sigma_1^1\text{-ID}_0$	$(\Sigma_2^0)_2$	Σ_2^0
$\vdots$	$\vdots$	$\vdots$
$[\Sigma_1^1]^k\text{-ID}_0$	$(\Sigma_2^0)_{k+1}$	$(\Sigma_2^0)_k$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma_1^1]^{\text{TR}}\text{-ID}_0$	Δ_3^0	Δ_3^0

Table 2: Results of the present study

Table 2 indicates that a subsystem of second order arithmetic and types of determinacy in the same line are pairwise equivalent over RCA_0^* (except for the last line).

The proof of each equivalence is as follows. When we prove Γ determinacy in some system Sys , we reduce a Γ game $\varphi(f)$ to an “easier” one $\varphi^*(f)$ such that the determinacy of $\varphi^*(f)$ can be proved in Sys . The key point in this direction is how to reduce the game within a restricted comprehension axiom. Conversely, we prove that Γ determinacy implies the set comprehension axiom that characterizes Sys as follows. For any formula $\varphi(k)$ in a given class for which comprehension axiom characterizes the system Sys , we construct the following game whose determinacy is implied by Γ determinacy:

- Player I chooses k and asks whether $\varphi(k)$.
- Player II answers yes or no.
- Player II wins if and only if her answer is correct.

In such a game, player I cannot win if player II answers correctly. By Γ determinacy, player II has a winning strategy, which provides the desired set $\{n : \varphi(n)\}$.

The outline of this paper is as follows. Subsection 2.1 introduces basic definitions and notations. Subsection 2.2 is a short survey of fundamental results of the new base theory RCA_0^* and related systems. Subsection 2.3 formulates determinacy in second order arithmetic and overviews some basic properties. Subsection 2.4 overviews previously known results of Wadge classes in descriptive set theory. Inspired by this, Subsection 2.5 defines new determinacy schemata, e.g., $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ determinacy, which play important roles in this paper. Section 3 proves the equivalence over RCA_0^* between WKL_0 and $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ determinacy in the Cantor space. This result shows that, to

investigate the strengths of determinacies beyond $\text{Bisep}(\Delta_1^0, \Sigma_1^0)$ determinacy, it is irrelevant whether the base theory contains Σ_1^0 induction. The following sections find several determinacies in the Cantor space that are equivalent to $\text{ATR}_0 + \Sigma_1^1$ induction, $\Pi_1^1\text{-CA}_0$, and $\Pi_1^1\text{-TR}_0$.

2 Preliminaries

In this section, we recall some basic definitions and facts about second order arithmetic and about determinacy. We also define various determinacy schemata.

2.1 Basic definitions and notations in second order arithmetic

The language L_1 of first order arithmetic consists of $+, \cdot, 0, 1, =, <$, number variables $x, y, \dots$, propositional connectives and number quantifiers. The language L_2 of second order arithmetic consists of L_1 plus set variables $X, Y, \dots$, set quantifiers and $\in$. Terms and formulae are defined in the usual way. A formula is Π_0^0 , Σ_0^0 or Δ_0^0 if it is built up from atomic formulae by propositional connectives and bounded number quantifiers $\forall x < t$ and $\exists x < t$. A Σ_n^0 (resp. Π_n^0) formula is one consisting of n number quantifiers beginning with an existential (resp. universal) one followed by a Π_0^0 formula. A formula is Σ_0^1 or Π_0^1 if it does not contain set quantifiers. A Σ_n^1 (resp. Π_n^1) formula is one consisting of n set quantifier beginning with an existential (resp. universal) one followed by a Π_0^1 formula.

In what follows, Γ is a class of formulae, e.g., Σ_n^i and Π_n^i .

Definition 2.1 We consider the following schemata.

Γ comprehension: $\exists Y \forall x (x \in Y \leftrightarrow \varphi(x))$, where $\varphi(x)$ is a Γ formula in which Y does not occur freely.

Bounded Γ comprehension: $\forall n \exists Y \forall i (i \in Y \leftrightarrow (i < n \wedge \varphi(i)))$, where $\varphi(x)$ is a Γ formula in which Y does not occur freely.

Γ induction: $(\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n)$, where $\varphi(x)$ is a Γ formula.

Γ transfinite induction: $\text{WO}(Z, <_Z) \wedge \forall j (\forall i (i <_Z j \rightarrow \varphi(i)) \rightarrow \varphi(j)) \rightarrow \forall j \varphi(j)$, where $\varphi(x)$ is a Γ formula and where $\text{WO}(Z, <_Z)$ is the Π_1^1 formula which asserts that $(Z, <_Z)$ is well-ordered, i.e., $(Z, <_Z)$ is a linear ordering without infinite $<_Z$ -descending chains.

Definition 2.2 (RCA_0) RCA_0 is the formal system in the language L_2 which consists of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of Σ_1^0 induction and of Δ_1^0 comprehension:

$$\forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists Y \forall n (\varphi(n) \leftrightarrow n \in Y),$$

where $\varphi(x)$ is a Σ_1^0 formula without free occurrences of Y , and where $\psi(x)$ is a Π_1^0 formula.

Over RCA_0 , Γ comprehension implies not only Γ induction but also Γ transfinite induction.

Lemma 2.3 Over RCA_0 , Γ comprehension implies Γ transfinite induction.

Proof. Let $(Y, \prec_Y)$ be a well-ordering and $\varphi(x)$ a Γ formula. Assume $\forall j (\forall i (i \prec_Y j \rightarrow \varphi(i)) \rightarrow \varphi(j))$ and $\exists k \neg \varphi(k)$. By Γ comprehension and Δ_1^0 comprehension, we have the set $Z = \{k : \neg \varphi(k)\}$. Since Z is non-empty, we can take $k \in Z$. Note that if $l \in Z$, then $m \prec_Y l$ for some $m \in Z$ by the assumption. Define $f : \mathbb{N} \rightarrow Z$ by (unbounded) primitive recursion as follows:

$$f(0) = k \quad \text{and} \quad f(n+1) = \text{the } <\text{-minimum } l \in Z \text{ such that } l \prec_Y f(n).$$

Then $f(n+1) \prec_Y f(n)$ holds for all n , which contradicts the well-orderedness of $(Y, \prec_Y)$. $\square$

As mentioned in Introduction, we need a weaker base theory RCA_0^* .

Definition 2.4 (RCA_0^*) Let $L_i(\text{exp})$ ($i = 1, 2$) be L_i augmented by a binary operation symbol $\text{exp}(m, n)$, which is intended to denote exponentiation. For each $n < \omega$, Σ_n^i and Π_n^i formulae of $L_2(\text{exp})$ are defined accordingly. RCA_0^* is the $L_2(\text{exp})$ -theory consisting of the axiom of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus exponentiation axioms, $m^0 = 1$, $m^{n+1} = m^n \cdot m$, where $t_1^{t_2}$ is an abbreviation for $\text{exp}(t_1, t_2)$, and the schemata of Δ_1^0 comprehension and of Π_0^0 induction.

Note that the L_2 -theory RCA_0 can be regarded as the $L_2(\text{exp})$ -theory RCA_0^* plus Σ_1^0 induction, since exp can be defined in RCA_0 .

Notation 2.5 (sequences) The set of all infinite sequences from X (i.e., functions from $\mathbb{N}$ to X) is denoted by $X^\mathbb{N}$. We call $2^\mathbb{N} = \{0, 1\}^\mathbb{N}$ the *Cantor space* and $\mathbb{N}^\mathbb{N}$ the *Baire space*. The set of all finite sequences from X is denoted by $X^{<\mathbb{N}}$. Note that the *empty sequence*, denoted by $\langle \rangle$, belongs to $X^{<\mathbb{N}}$.

Fix any $s, t \in X^\mathbb{N}$. $|s|$ denotes the *length* of s and $s(i)$ denotes the $(i+1)$ -th element of s for $i < |s|$. The *concatenation* of s and t , denoted by $s * t$, is $\langle s(0), s(1), \dots, s(|s| - 1), t(0), t(1), \dots, t(|t| - 1) \rangle$. If $f \in X^\mathbb{N}$, $s * f$ denotes $\langle s(0), s(1), \dots, s(|s| - 1), f(0), f(1), \dots, f(n), \dots \rangle$. For $s \in X^{<\mathbb{N}}$ and $n \leq |s|$, $s[n]$ is the n -th *initial segment* of s , i.e., $\langle s(0), \dots, s(n-1) \rangle$. If f is an infinite sequence, $f[n]$ denotes $\langle f(0), \dots, f(n-1) \rangle$. If $s = t[k]$ for some $k \leq |t|$, s is called an *initial segment* of t , $s \subseteq t$ for notation. If $n \leq |s|$, $s \ominus n$ is the sequence with the first n elements removed from s , i.e., $\langle s(n), s(n+1), \dots, s(|s|-1) \rangle$. If f is in $X^\mathbb{N}$, $f \ominus n$ is g defined by $g(k) = f(n+k)$ for $k \in \mathbb{N}$.

Note that, since a sequence s of $X^{<\mathbb{N}}$ can be coded by a natural number in RCA_0^* , the relations $s \in X^{<\mathbb{N}}$, $s = \langle \rangle$, $|s| = n$, $s(i) = n$, $s \subseteq t$, $s = t * u$, $s = t \ominus n$, $s = t[n]$, etc., are formally defined as Π_0^0 formulae in RCA_0^* . See [8, Sections II.1-2] for more details. (Although they work in RCA_0 in [8, Sections II.1-2], the base theory can be replaced by RCA_0^* .) By coding of finite sequences by natural numbers, we can define transfinite recursion and axioms of choice.

Definition 2.6 We define the following schemata.

Γ **transfinite recursion:** $\text{WO}(Y, \prec_Y) \rightarrow \exists Z \forall j \forall k (\langle k, j \rangle \in Z \leftrightarrow \theta(k, (Z)^j))$, where $\theta(x, Y)$ is a Γ formula in which Z does not occur freely, and where $(Z)^j = \{\langle m, i \rangle \in Z : i \prec_Y j\}$.

Γ **axiom of choice:** $\forall n \exists Y \varphi(n, Y) \rightarrow \exists Z \forall n \varphi(n, (Z)_n)$, where $\varphi(x, Y)$ is a Γ formula φ in which Z does not occur freely, and where $(Z)_n = \{m : \langle m, n \rangle \in Z\}$.

2.2 The systems RCA_0^* and WKL_0^*

In the last subsection, we defined RCA_0^* . Here we see how RCA_0^* differs from RCA_0 . We also define WKL_0 and WKL_0^* by adding weak König's lemma, and investigate the relation among them.

First we overview the provable definability of functions in RCA_0^* . By Δ_1^0 comprehension, RCA_0^* proves the existence of the constant function $g_0(n) = 0$, the *successor function* $g_{\text{suc}}(n) = n + 1$, the *addition* $g_+(n, m) = n + m$, the *multiplication* $g_\cdot(n, m) = n \cdot m$, the *exponentiation* $g_{\text{exp}}(n, m) = n^m$ and the *projection functions* $p_i^n(\langle k_1, \dots, k_i, \dots, k_n \rangle) = k_i$. The universe of provably definable functions of RCA_0^* is closed under the operations of Lemma 2.7 below. Therefore every *elementary function* can be defined in RCA_0^* (cf. [9, Section 4]).

Lemma 2.7 ([9, 2.1. Lemma and 2.2. Lemma]) *The following are provable in RCA_0^* .*

Composition *If $g_0 : Y \rightarrow Z$ and $g_1 : Z \rightarrow W$, there is $h : Y \rightarrow W$ defined by $h(i) = g_1(g_0(i))$.*

Bounded primitive recursion *If $g_0 : \mathbb{N}^k \rightarrow \mathbb{N}$, $g_1 : \mathbb{N}^{k+2} \rightarrow \mathbb{N}$ and $g_2 : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$, there exists the unique $h : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ defined by*

$$\begin{aligned} h(0, n_1, \dots, n_k) &= g_0(n_1, \dots, n_k), \\ h(m+1, n_1, \dots, n_k) &= \min\{g_1(m, h(m, n_1, \dots, n_k), n_1, \dots, n_k), g_2(m, n_1, \dots, n_k)\}. \end{aligned}$$

Minimization If $g : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ and if all $\langle n_1, \dots, n_k \rangle \in \mathbb{N}^k$ have $m \in \mathbb{N}$ with $g(m, n_1, \dots, n_k) = 1$, there is $h : \mathbb{N}^k \rightarrow \mathbb{N}$ defined by $h(n_1, \dots, n_k) =$ the least m with $g(m, n_1, \dots, n_k) = 1$.

The following lemma shows that the first order part of RCA_0^* contains Σ_1^0 collection.

Lemma 2.8 (Σ_1^0 collection, [9, 4.1. Lemma]) RCA_0^* proves Σ_1^0 collection, i.e.,

$$\forall n \exists m \varphi(n, m) \rightarrow \forall n \exists m \forall k < n \exists l < m \varphi(k, l), \quad \text{for any } \Sigma_1^0 \text{ formula } \varphi(x, y).$$

Second, we consider weak König's lemma and add this to our systems.

A set $T \subseteq X^{<\mathbb{N}}$ is a *tree* over X if it is closed under initial segments, i.e., $t \in T$ and $s \subseteq t$ imply $s \in T$. A tree is *infinite* if, for any n , there exists $s \in T$ with $|s| = n$. $f : \mathbb{N} \rightarrow X$ is a *path* through a tree T if each initial segment $f[m]$ of f is in T .

Definition 2.9 *Weak König's lemma* asserts that every infinite tree over $\{0, 1\}$ has a path. WKL_0 and WKL_0^* are the systems RCA_0 and RCA_0^* , respectively, plus weak König's lemma.

Clearly WKL_0 includes RCA_0 . Actually, this inclusion is strict (see [8, Section VIII. 2]). This implies that WKL_0^* also strictly includes RCA_0^* .

Informally, we have the following equivalences. (See [8, Section X.4])

$$\text{RCA}_0 \equiv \text{RCA}_0^* + \Sigma_1^0 \text{ induction}, \quad \text{and} \quad \text{WKL}_0 \equiv \text{WKL}_0^* + \Sigma_1^0 \text{ induction}.$$

Let us consider the first order part and the consistency strength of these systems.

IS_1 is the L_1 theory consisting of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ and of Σ_1 induction. $\text{ID}_0(\text{exp})$ is the $L_1(\text{exp})$ theory consisting of the axioms of discrete ordered semi-ring with exponentiation for $(\mathbb{N}, +, \cdot, \text{exp}, 0, 1, <)$ and of Δ_0 induction. $\text{BS}_1(\text{exp})$ is the system $\text{ID}_0(\text{exp})$ plus Σ_1 collection. By the same proof as in [6] we can see that $\text{ID}_0(\text{exp})$ is strictly included in $\text{BS}_1(\text{exp})$ although the original proof in [6] is for the language L_1 .

Fact 2.10 (conservation theorems, [8, Sections IX.1-IX.3] and [9]) *The first order part of RCA_0 and of WKL_0 is IS_1 . WKL_0 is conservative over RCA_0 for Π_1^1 sentences, i.e., for any Π_1^1 sentence φ , $\text{WKL}_0 \vdash \varphi$ if and only if $\text{RCA}_0 \vdash \varphi$. WKL_0 , RCA_0 and IS_1 have the same consistency strength as PRA (See [8, Section IX. 3] for its definition and basic properties.) and are conservative over PRA for Π_2^0 sentences.*

Analogously, the first order part of RCA_0^* and of WKL_0^* is $\text{BS}_1(\text{exp})$. WKL_0^* is conservative over RCA_0^* for Π_1^1 sentences. WKL_0^* , RCA_0^* and $\text{BS}_1(\text{exp})$ have the same consistency strength as $\text{ID}_0(\text{exp})$ and are conservative over $\text{ID}_0(\text{exp})$ for Π_2^0 sentences.

It is shown in [8, Theorem II.8.11] that RCA_0 proves the consistency of $\text{ID}_0(\text{exp})$, and so does IS_1 . On the other hand, $\text{ID}_0(\text{exp})$ cannot prove the consistency of itself and so neither can $\text{BS}_1(\text{exp})$, since the consistency statement is Π_1^0 . Therefore IS_1 strictly includes $\text{BS}_1(\text{exp})$, and the implications from RCA_0 to RCA_0^* and from WKL_0 to WKL_0^* are also strict. Moreover the consistency strength of WKL_0^* , RCA_0^* , BS_1 and $\text{ID}_0(\text{exp})$ is strictly weaker than that of WKL_0 , RCA_0 , IS_1 and PRA.

The relations among these weak systems are described in Figure 1, where usual (one-head) arrows indicate strict implication with conservation for indicated classes of sentences, where two-head arrows indicate strict implication with different consistency strength and where lines without heads indicate full conservation between systems in $L_1(\text{exp})$ and their counterparts in $L_2(\text{exp})$ (e.g., for any $L_1(\text{exp})$ sentence φ , $\text{WKL}_0^* \vdash \varphi$ if and only if $\text{BS}_1(\text{exp}) \vdash \varphi$).

Figure 1: Relations among weak systems of arithmetic

2.3 Determinacy in second order arithmetic

For a given formula $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$, a *game* $\varphi(f)$ in $X^{\mathbb{N}}$ is defined as follows: Two players, say player I and player II, alternately choose an element x in X to form $f \in X^{\mathbb{N}}$ which is called the resulting play. Player I wins if and only if $\varphi(f)$ holds. Player II wins if and only if player I does not win. In this paper, we assume that player I is male and that player II is female.

We regard a class Γ of formulae with a distinguished function variable as a class of games.

Remark 2.11 L_2 does not formally have function variables. Since the assertion “ G is the graph of a function $f : \mathbb{N} \rightarrow \mathbb{N}$ ” is Π_2^0 , this point matters in a system without arithmetical (Π_1^1) comprehension. However, we treat only the case where $X = \{0, 1\}$ in such systems, and so we can regard f as a set variable by identifying $f(n) = 0$ and $f(n) = 1$ with $n \notin f$ and $n \in f$ respectively. Therefore we do not care about this difference.

Notation 2.12 (strategy) For a game in $X^{\mathbb{N}}$, a *strategy* σ for player I (resp. II) is a function assigning an element of X to each finite sequence from X of even (resp. odd)-length. $\mathcal{S}_I^X$ (resp. $\mathcal{S}_{II}^X$) is the set of all the strategies for player I (resp. II) in a game in $X^{\mathbb{N}}$. Note that $\mathcal{S}_I^X$ and $\mathcal{S}_{II}^X$ can be regarded as $\mathbb{N}^{\mathbb{N}}$ if $X = \mathbb{N}$ and $2^{\mathbb{N}}$ if $X = \{0, 1\}$ in RCA_0^* , by a suitable coding of finite sequences. If players I and II follow strategies σ and τ respectively, the resulting play is uniquely determined and denoted by $\sigma \otimes \tau$.

For any strategy τ for player II, k^τ is the finite play of length $2k$ in which player I plays 0 at all of his turns and in which player II plays following τ . For example, 2^τ is the sequence $\langle 0, \tau(\langle 0 \rangle), 0, \tau(\langle 0, \tau(\langle 0 \rangle), 0) \rangle$.

A strategy σ for a player is a *winning strategy* if the player wins $\varphi(f)$ as long as he or she plays following it. The assertion that σ is a winning strategy for player I (resp. II) in game $\varphi(f)$ in $X^{\mathbb{N}}$ can be written $\forall \tau \in \mathcal{S}_{II}^X \varphi(\sigma \otimes \tau)$ (resp. $\forall \tau \in \mathcal{S}_I^X \neg \varphi(\tau \otimes \sigma)$). A game $\varphi(f)$ is *determinate* if one of the players has a winning strategy. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, we use the following abbreviation:

$$\text{Det}^X[\varphi] \equiv \exists \sigma \in \mathcal{S}_I^X \forall \tau \in \mathcal{S}_{II}^X \varphi(\sigma \otimes \tau) \vee \exists \tau \in \mathcal{S}_{II}^X \forall \sigma \in \mathcal{S}_I^X \neg \varphi(\sigma \otimes \tau),$$

which asserts that $\varphi(f)$ is determinate. The following schema of Γ *determinacy* asserts that all the Γ games are determinate.

Γ **determinacy in $X^{\mathbb{N}}$** : $\text{Det}^X[\varphi]$ for any Γ game $\varphi(f)$ in $X^{\mathbb{N}}$.

The following Δ_n^i *determinacy* asserts the determinacies of games both Σ_n^i and Π_n^i , in a sense.

Δ_n^i **determinacy in $X^{\mathbb{N}}$** : $\forall f \in X^{\mathbb{N}} (\varphi(f) \leftrightarrow \psi(f)) \rightarrow \text{Det}^X[\varphi]$,
 where $\varphi(f)$ is Σ_n^i game and where $\psi(f)$ is a Π_n^i games in $X^{\mathbb{N}}$.

Det^* and Det abbreviate determinacy in the Cantor space and that in the Baire space, respectively.

In this paper, for every game $\varphi(f)$ in $X^{\mathbb{N}}$, we say *player I (resp. II) wins $\varphi(f)$ at $s \in X^{<\mathbb{N}}$* if

- (1) $|s|$ is even and player I (resp. II) has a winning strategy in $\varphi(s * f)$, or
- (2) $|s|$ is odd and player II (resp. I) has a winning strategy in $\varphi(s * f)$.

Note that in case (2) players are exchanged.

2.4 Wadge classes in descriptive set theory

Here we review Wadge classes of Polish spaces, which gives us intuitive ideas for determinacy schemata defined in the next subsection. Note that, in this subsection, we work in the usual framework of (descriptive) set theory, not in that of second order arithmetic.

For a pair (A, B) of subsets of a Polish space X , how can we define which set is “simpler”? Wadge reducibility gives an answer: We say that A is *Wadge reducible* to B , in symbols $A \leq_W B$, if there is a continuous map $f : X \rightarrow X$ such that $f^{-1}(B) = A$. We write $A \equiv_W B$ if and only if $A \leq_W B$ and $B \leq_W A$. Clearly $\equiv_W$ is an equivalence relation, and associated equivalence classes $\mathbf{A} = [A]_W$ are called *Wadge degrees*. For a set $A \subseteq X^{\mathbb{N}}$, the *coarse degree* $\mathbf{A}^*$ of A is $[A]_W \cup [A^c]_W$, where A^c is the complement of A . WADGE denotes the set of the coarse degrees of Borel sets, and $\leq_W^*$ is defined by $\mathbf{A} \leq_W^* \mathbf{B} \leftrightarrow A \leq_W B \vee A \leq_W B^c$.

Under the assumption of Borel determinacy, which is proved in ZFC , $(\text{WADGE}, \leq_W^*)$ is well-ordered. Actually it gives a finer hierarchy than *Hausdorff and Kuratowski's difference hierarchy* from [1, §37. III. Theorem], the finite level of which is defined as follows:

- $(\Sigma_n^0)_1 = \Sigma_n^0$
- $(\Sigma_n^0)_{m+1} = \{A \setminus B : A \in \Sigma_n^0 \text{ and } B \in (\Sigma_n^0)_m\}$

A concrete description of the Wadge hierarchy of Borel sets is provided in [2]. The classes in Wadge hierarchy are composed from the classes of Hausdorff and Kuratowski's difference hierarchy by means of taking (countable) unions and complements. Especially, below $(\Sigma_2^0)_2$, the construction principle is easier because we do not need countable unions.

Figure 2 is a rough sketch of Wadge hierarchy below $(\Sigma_2^0)_2$, where only one of mutually dual classes is described, where $\mathbf{Sep}(\Delta_2^0, \Sigma_2^0)$ is the union of all Wadge classes below $(\Sigma_2^0)_2$ and where Δ_2^0 is the union of all the Wadge classes below Σ_2^0 . The constructions mentioned here are as follows:

- $\Delta(\Gamma) = \{A : A, A^c \in \Gamma\}$
- $\mathbf{Sep}(\Gamma, \Gamma') = \{(A \cap B^c) \cup (A^c \cap C) : A \in \Gamma \text{ and } B, C \in \Gamma'\}$
- $\mathbf{Bisep}(\Gamma, \Gamma') = \{(A \cap B^c) \cup (C \cap D) : A, C \in \Gamma, A \cap C = \emptyset \text{ and } B, D \in \Gamma'\}$

Then we can easily construct corresponding schemata of determinacy; replacing $\cup$ with $\vee$ and replacing $(-)^c$ (complement) with $\neg$. The definitions of $\mathbf{Sep}(\Gamma, \Gamma')$ and $\mathbf{Bisep}(\Gamma, \Gamma')$ determinacies in the next subsection are motivated by these constructions of Wadge classes in this way.

In this paper, we treat $\mathbf{Bisep}(\Delta_1^0, \Sigma_1^0)$, $\mathbf{Bisep}(\Delta_1^0, \Sigma_2^0)$, $\mathbf{Bisep}(\Sigma_1^0, \Sigma_2^0)$, $\mathbf{Sep}(\Sigma_1^0, \Sigma_2^0)$ and $\mathbf{Bisep}(\Delta_2^0, \Sigma_2^0)$ determinacies, which correspond to $\mathbf{Bisep}(\Delta_1^0, \Sigma_1^0)$, $\mathbf{Bisep}(\Delta_1^0, \Sigma_2^0)$, $\mathbf{Bisep}(\Sigma_1^0, \Sigma_2^0)$, $\mathbf{Sep}(\Sigma_1^0, \Sigma_2^0)$ and $\mathbf{Bisep}(\Delta_2^0, \Sigma_2^0)$ respectively. In section 5, we show that Π_1^1 comprehension, $\mathbf{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ and $\mathbf{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ are pairwise equivalent. Although there are ω_1 different Wadge classes between $\mathbf{Bisep}(\Sigma_1^0, \Sigma_1^0)$ and $\mathbf{Sep}(\Sigma_1^0, \Sigma_2^0)$, we will show that, in the Cantor space, the strengths of the determinacies correspond to them coincide.

Figure 2: Wadge hierarchy below $(\Sigma^0_2)_2$ due to [2]

2.5 Several new determinacy schemata

In this subsection we define, in second order arithmetic, determinacy schemata which correspond to the determinacy of Wadge classes in descriptive set theory. We only consider classes of formulae $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$.

The normal form theorem for Σ^0_1 formulae over $\mathbf{RCA}_0$ is proved in [8]. Here we prove it over $\mathbf{RCA}_0^*$.

While we usually assume that formulae and terms may contain free variables other than explicitly displayed, only in the statement and the proof of the following lemma, we assume that all free variables of formulae and terms are among displayed ones.

Lemma 2.13 (normal form theorem 1) *For any Σ^0_1 formula $\varphi(f, \vec{k}, \vec{Y})$ with a distinguished function variable $f \in 2^{\mathbb{N}}$, we can find a Π^0_0 formula $\theta(s, \vec{k}, \vec{Y})$, in which f does not occur freely, such that $\mathbf{RCA}_0^*$ proves $\forall f \in 2^{\mathbb{N}} \forall \vec{k} \forall \vec{Y} (\varphi(f, \vec{k}, \vec{Y}) \leftrightarrow \exists n \theta(f[n], \vec{k}, \vec{Y}))$.*

Proof. First, we prove that, for every Π^0_0 formula $\varphi(f, \vec{k}, \vec{Y})$, we can find Π^0_0 formula $\theta(s, \vec{k}, \vec{Y})$ and a term $t(\vec{k})$ such that $\mathbf{RCA}_0^*$ proves $\varphi(f, \vec{k}, \vec{Y}) \leftrightarrow \theta(f[t(\vec{k})], \vec{k}, \vec{Y})$ by induction on construction of formula. The cases of atomic formulae, Boolean operations. The case of bounded universal quantifier can be proved as follows: Let $\varphi(f, \vec{k}, \vec{Y})$ be Π^0_0 of the form $\forall i < t'(\vec{k}) \theta'(i, f, \vec{k}, \vec{Y})$, where $\theta'(x, f, \vec{k}, \vec{Y})$ is Π^0_0 . By induction hypothesis, we can find a Π^0_0 formula $\theta''(x, s, \vec{k}, \vec{Y})$ and a term $t''(i, \vec{k})$ such that

RCA_0^* proves $\theta'(i, f, \vec{k}, \vec{Y}) \leftrightarrow \theta''(i, f[t''(i, \vec{k})], \vec{k}, \vec{Y})$. Then, we can check that RCA_0^* proves

$$\begin{aligned} & \forall i < t'(\vec{k}) \theta'(i, f, \vec{k}, \vec{Y}) \\ \leftrightarrow & \forall i < t'(\vec{k}) \theta''(i, f[t''(i, \vec{k})], \vec{k}, \vec{Y}) \\ \leftrightarrow & \forall i < t'(\vec{k}) \theta''(i, (f[t''(t'(\vec{k}), \vec{k})])[t(i, \vec{k})], \vec{k}, \vec{Y}), \end{aligned}$$

by the monotony of the term. The case of bounded existential quantifier can be proved similarly.

Now we complete the proof. Let $\varphi(f, \vec{k}, \vec{Y}) \equiv \exists n \theta'(n, f, \vec{k}, \vec{Y})$ for some Π_0^0 formula $\theta'(x, f, \vec{k}, \vec{Y})$. By the above, we can find Π_0^0 formula $\theta''(x, f, \vec{k}, \vec{Y})$ and a term $t'(\vec{k})$ such that RCA_0^* proves $\theta'(x, f, \vec{k}, \vec{Y}) \leftrightarrow \theta''(x, f[t(x, \vec{k})], \vec{k}, \vec{Y})$. Then RCA_0^* proves

$$\begin{aligned} & \varphi(f, \vec{k}, \vec{Y}) \\ \leftrightarrow & \exists n \theta''(n, f[t(n, \vec{k})], \vec{k}, \vec{Y}) \\ \leftrightarrow & \exists n \exists n' < n + 1 (n = \langle n', t(n', \vec{k}) \rangle \wedge \theta''(n', (f[n])[t(n', \vec{k})], \vec{k}, \vec{Y})). \end{aligned}$$

□

Definition 2.14 Let Γ and Γ' be classes of formulae. $\Gamma \wedge \Gamma'$ (resp. $\Gamma \vee \Gamma'$) is the class of formulae of the form $\varphi_0 \wedge \varphi_1$ (resp. $\varphi_0 \vee \varphi_1$), where φ_0 is Γ and φ_1 is Γ' . $\neg\Gamma$ is the class of formulae of the form $\neg\varphi$, where φ is Γ .

We consider determinacy schemata also for the class defined in Definition 2.14. Furthermore, we treat the following determinacy schemata, which are motivated by Borel Wadge classes, mentioned in the previous subsection.

Definition 2.15

$\Delta(\Gamma)$ **determinacy in $X^{\mathbb{N}}$** : $\forall f \in X^{\mathbb{N}} (\psi(f) \leftrightarrow \xi(f)) \rightarrow \text{Det}^X[\psi]$,
where $\psi(f)$ is Γ and where $\xi(f)$ is $\neg\Gamma$.

$\text{Sep}(\Sigma_n^0, \Sigma_m^0)$ **determinacy in $X^{\mathbb{N}}$** : $\text{Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)]$,
where $\psi(f)$ is Σ_n^0 , where $\eta_0(f)$ is Π_m^0 and where $\eta_1(f)$ is Σ_m^0 .

$\text{Sep}(\Delta_n^0, \Sigma_m^0)$ **determinacy in $X^{\mathbb{N}}$** : $\forall f \in X^{\mathbb{N}} (\psi(f) \leftrightarrow \xi(f)) \rightarrow \text{Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)]$,
where $\psi(f)$ is Σ_n^0 , where $\xi(f)$ is Π_n^0 , where $\eta_0(f)$ is Π_m^0 and where $\eta_1(f)$ is Σ_m^0 .

$\text{Bisep}(\Sigma_n^0, \Sigma_m^0)$ **determinacy in $X^{\mathbb{N}}$** : $\forall f \in X^{\mathbb{N}} \neg(\psi_0(f) \wedge \psi_1(f)) \rightarrow \text{Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)]$,
where $\psi_i(f)$'s are Σ_n^0 , where $\eta_0(f)$ is Π_m^0 and where $\eta_1(f)$ is Σ_m^0 .

$\text{Bisep}(\Delta_n^0, \Sigma_m^0)$ **determinacy in $X^{\mathbb{N}}$** :

$$\begin{aligned} & \forall f \in X^{\mathbb{N}} (\neg(\psi_0(f) \wedge \psi_1(f))) \wedge \forall f \in X^{\mathbb{N}} (\psi_0(f) \leftrightarrow \xi_0(f)) \wedge \forall f \in X^{\mathbb{N}} (\psi_1(f) \leftrightarrow \xi_1(f)) \\ & \rightarrow \text{Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)], \end{aligned}$$

where $\psi_i(f)$'s are Σ_n^0 , where $\xi_i(f)$'s are Π_n^0 , where $\eta_0(f)$ is Π_m^0 and where $\eta_1(f)$ is Σ_m^0 .

Remark 2.16 $\text{Bisep}(\Delta_n^0, \Sigma_{n+m}^0) = \text{Sep}(\Delta_n^0, \Sigma_{n+m}^0)$ in the terms of the last subsection. Inspired by this equality, we can see that $\text{Bisep}(\Delta_n^0, \Sigma_{n+m}^0)$ determinacy in $X^{\mathbb{N}}$ is equivalent to $\text{Sep}(\Delta_n^0, \Sigma_{n+m}^0)$ determinacy in $X^{\mathbb{N}}$ over RCA_0^* , since, for any formulae $\psi_0(f)$, $\psi_1(f)$, $\eta_0(f)$ and $\eta_1(f)$, RCA_0^* proves the following:

$$\begin{aligned} & \forall f \in X^{\mathbb{N}} \neg(\psi_0(f) \wedge \psi_1(f)) \rightarrow \\ & \forall f \in X^{\mathbb{N}} (((\psi_0(f) \wedge \eta_0(f)) \vee (\psi_1(f) \wedge \eta_1(f))) \leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee (\neg\psi_0(f) \wedge (\psi_1(f) \wedge \eta_1(f)))). \end{aligned}$$

The next lemma is a reformulation of a classical separation theorem: For any disjoint pair of Π_n^0 sets A and B , there exists a Δ_n^0 set C such that $A \subseteq C$ and $B \subseteq C^c$.

Lemma 2.17 *Assume $0 < n < \omega$. For any pair of Π_n^0 formulae $\varphi_0(f)$ and $\varphi_1(f)$, we can find a Σ_n^0 formula $\psi(f)$ and Π_n^0 formula $\xi(f)$ such that RCA_0^* proves*

$$\forall f \in X^{\mathbb{N}} \neg(\varphi_0(f) \wedge \varphi_1(f)) \rightarrow \forall f \in X^{\mathbb{N}} ((\psi(f) \leftrightarrow \xi(f)) \wedge (\varphi_0(f) \rightarrow \psi(f)) \wedge (\varphi_1(f) \rightarrow \neg\psi(f)))$$

Proof. Let $\varphi_i(f)$ be Π_n^0 , say $\forall m \theta_i(m, f)$, where $\theta_i(x, g)$ is Σ_{n-1}^0 . Assume $\forall f \in X^{\mathbb{N}} \neg(\varphi_0(f) \wedge \varphi_1(f))$. Set $\psi(f) \equiv \exists m (\neg\theta_1(m, f) \wedge \forall l < m \theta_0(l, f))$. Then we easily see that $\varphi_0(f) \rightarrow \psi(f)$ and $\varphi_1(f) \rightarrow \neg\psi(f)$ and that $\psi(f) \leftrightarrow \forall m (\theta_0(m, f) \vee \exists l \leq m \neg\theta_1(l, f))$. $\square$

The following lemma is useful to investigate the relationship among determinacy schemata defined in 2.15. The motivating result is $\text{Sep}(\Delta_n^0, \Sigma_n^0) = \Delta((\Sigma_n^0)_2) = \Delta(\Sigma_n^0 \wedge \Pi_n^0)$.

Lemma 2.18 *Let $0 < n < \omega$.*

1. *For any Σ_n^0 formulae $\psi_0(f)$ and $\eta_1(f)$, for any Π_n^0 formulae $\psi_1(f)$ and $\eta_0(f)$, we can find Σ_n^0 formulae $\xi_0(f)$ and $\xi_1(f)$ and Π_n^0 formulae $\zeta_0(f)$ and $\zeta_1(f)$ such that RCA_0^* proves*

$$\begin{aligned} \forall f \in X^{\mathbb{N}} (\psi_0(f) \leftrightarrow \psi_1(f)) &\rightarrow \\ \forall f \in X^{\mathbb{N}} (((\psi_0(f) \wedge \eta_0(f)) \vee (\psi_1(f) \wedge \eta_1(f))) &\leftrightarrow (\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow (\xi_1(f) \vee \zeta_1(f))) \end{aligned}$$

2. *For any Σ_n^0 formulae $\xi_0(f)$ and $\xi_1(f)$ and for any Π_n^0 formulae $\zeta_0(f)$ and $\zeta_1(f)$, we can find Σ_n^0 formulae $\psi_0(f)$ and $\eta_1(f)$ and Π_n^0 formulae $\psi_1(f)$ and $\eta_0(f)$ such that RCA_0^* proves*

$$\begin{aligned} \forall f \in X^{\mathbb{N}} ((\xi_0(f) \wedge \zeta_0(f)) &\leftrightarrow (\xi_1(f) \vee \zeta_1(f))) \rightarrow \\ \forall f \in X^{\mathbb{N}} (\psi_0(f) \leftrightarrow \psi_1(f)) \wedge \forall f \in X^{\mathbb{N}} ((\xi_0(f) \wedge \zeta_0(f)) &\leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee (\neg\psi_0(f) \wedge \eta_1(f)))) \end{aligned}$$

Proof. We work in RCA_0^* .

(1) Assume that $\forall f \in X^{\mathbb{N}} (\psi_0(f) \leftrightarrow \psi_1(f))$ for a Σ_n^0 formula $\psi_0(f)$ and a Π_n^0 formula $\psi_1(f)$. Then, it is easy to find Σ_n^0 formulae $\xi_0(f)$ and $\xi_1(f)$ and Π_n^0 formulae $\zeta_0(f)$ and $\zeta_1(f)$ such that RCA_0^* proves

$$\begin{aligned} \forall f \in X^{\mathbb{N}} (\xi_0(f) \leftrightarrow (\psi_0(f) \vee (\neg\psi_0(f) \wedge \eta_1(f))), \quad \forall f \in X^{\mathbb{N}} (\xi_1(f) \leftrightarrow (\neg\psi_0(f) \wedge \eta_1(f))), \\ \forall f \in X^{\mathbb{N}} (\zeta_0(f) \leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee \neg\psi_0(f))), \quad \forall f \in X^{\mathbb{N}} (\zeta_1(f) \leftrightarrow (\psi_0(f) \wedge \eta_0(f))), \end{aligned}$$

and they enjoy the desired property.

(2) Assume $\forall f \in X^{\mathbb{N}} ((\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow (\xi_1(f) \vee \zeta_1(f)))$ for Σ_n^0 formulae $\xi_0(f)$ and $\xi_1(f)$ and Π_n^0 formulae $\zeta_0(f)$ and $\zeta_1(f)$. Note that $\forall f \in X^{\mathbb{N}} ((\neg\xi_0(f) \vee \neg\zeta_0(f)) \leftrightarrow \neg(\xi_1(f) \vee \zeta_1(f)))$, and so there is no $f \in X^{\mathbb{N}}$ with $\neg\xi_0(f) \wedge \zeta_1(f)$. By Lemma 2.17, we can find a Σ_n^0 formula $\psi_0(f)$ and a Π_n^0 formula $\psi_1(f)$ such that

$$\forall f \in X^{\mathbb{N}} ((\psi_0(f) \leftrightarrow \psi_1(f)) \wedge (\zeta_1(f) \rightarrow \psi_0(f)) \wedge (\neg\xi_0(f) \rightarrow \neg\psi_0(f))).$$

$\psi_1(f)$

$\neg\xi_0(f)$	$\neg\xi_0(f) \wedge \neg\zeta_0(f)$	$\neg\zeta_0(f)$
$\xi_1(f)$	$\xi_1(f) \wedge \zeta_1(f)$	$\zeta_1(f)$

$\psi_0(f)$

Then we can check

$$\forall f \in X^{\mathbb{N}} ((\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow ((\psi_0(f) \wedge \zeta_0(f)) \vee (\neg\psi_0(f) \wedge \xi_1(f))))$$

and so $\psi_i(f)$'s, $\eta_0(f) \equiv \zeta_0(f)$ and $\eta_1(f) \equiv \xi_1(f)$ enjoy the desired property. $\square$

By Remark 2.16 and Lemma 2.18, $\text{Bisep}(\Delta_n^0, \Sigma_n^0)\text{-Det}$, $\text{Sep}(\Delta_n^0, \Sigma_n^0)\text{-Det}$ and $\Delta(\Sigma_n^0 \wedge \Pi_n^0)\text{-Det}$ are pairwise equivalent over RCA_0^* and $\text{Bisep}(\Delta_n^0, \Sigma_n^0)\text{-Det}^*$, $\text{Sep}(\Delta_n^0, \Sigma_n^0)\text{-Det}^*$ and $\Delta(\Sigma_n^0 \wedge \Pi_n^0)\text{-Det}^*$ are pairwise equivalent over RCA_0^* .

3 Weak König's lemma and determinacy

In this section, we investigate the relation among determinacy, weak König's lemma and Σ_1^0 induction.

[5] shows that $\Delta_1^0\text{-Det}^*$, $\Sigma_1^0\text{-Det}^*$ and weak König's lemma are pairwise equivalent over RCA_0 . However, it seems that their proofs do not essentially require Σ_1^0 induction. In order to make it precise, we need to consider the system WKL_0 minus Σ_1^0 induction, i.e., WKL_0^* . Then we have the following:

Proposition 3.1 $\text{RCA}_0^* \vdash \Delta_1^0\text{-Det}^* \leftrightarrow \Sigma_1^0\text{-Det}^* \leftrightarrow \text{WKL}_0^*$.

Proof. See [5]. Although the original proofs show the equivalence over RCA_0 , it can be replaced with RCA_0^* . Note that Σ_1^0 separation is equivalent to weak König's lemma over RCA_0^* . $\square$

Next we consider the strength of Σ_1^0 induction+weak König's lemma over RCA_0^* . For that purpose, we introduce a set comprehension axiom which is equivalent to Σ_1^0 induction.

Lemma 3.2 ([8, Theorem II.3.8, Theorem X.4.4]) Σ_1^0 induction is equivalent to bounded Σ_1^0 comprehension over RCA_0^* .

The theorems below show Σ_1^0 induction+weak König's lemma is equivalent to $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$ over RCA_0^* .

Theorem 3.3 RCA_0^* proves that $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$ implies Σ_1^0 induction and weak König's lemma.

Proof. Assume $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$. Since $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$ implies $\Sigma_1^0\text{-Det}^*$ and so weak König's lemma, it is enough to prove bounded Σ_1^0 comprehension, which is equivalent to Σ_1^0 induction by Lemma 3.2. Let $\varphi(x)$ be a Σ_1^0 formula. For fixed n , we consider the following game. Player I chooses $k < n$ and asks whether $\varphi(k)$ or not. Player II answers yes or no. She wins if and only if her answer is correct. Such a game is realized as follows:

- Player I chooses $k < n$ by playing 0 at his first k turns and 1 at his $(k+1)$ -th turn.
- When player II chooses 1 at her $(k+1)$ -th turn, she wins if $\varphi(k)$ holds.
- When player II chooses 0 at her $(k+1)$ -th turn, she wins if $\neg\varphi(k)$ holds.

The game goes as follows:

	$\overbrace{0, \dots, 0}^{k \text{ times}}$			1	...
player I					
player II	*	...	*	$\underbrace{i}_{\text{answer}}$	...

Player I wins if and only if one of the following holds:

- Player I chooses $k < n$, player II answers “yes,” and $\neg\varphi(k)$ holds.
- Player I chooses $k < n$, player II answers “no,” and $\varphi(k)$ holds.

Formally, this winning condition can be written as below:

$$(\exists k < n(\psi(f, k, 1) \wedge \neg\varphi(k))) \vee (\exists k < n(\psi(f, k, 0) \wedge \varphi(k))), \quad (b)$$

where $\psi(f, k, i)$ is the Π_1^0 formula $\forall i < k(f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i$. (b) is equivalent to the following formula:

$$\underbrace{((\exists k < n \psi(f, k, 1)) \wedge \forall k < n (\psi(f, k, 1) \rightarrow \neg \varphi(k)))}_{(1)} \vee \underbrace{((\exists k < n \psi(f, k, 0)) \wedge \exists k < n (\psi(f, k, 0) \wedge \varphi(k)))}_{(3)}.$$

Clearly (1) and (3) are Σ_1^0 , and we can easily see that (1) and (3) are equivalent to some Π_1^0 formulae. There is no $f \in 2^{\mathbb{N}}$ which satisfies both (1) and (3). (2) is equivalent to some Π_1^0 formula and (4) is equivalent to some Σ_1^0 formula. Player I has no winning strategy in (b), since for any $k < m$, player I cannot win if player II's answer is correct. By $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$, player II has a winning strategy τ in (b). We can also see that, $\tau(k^\tau * \langle 1 \rangle) = 1$ if and only if $\varphi(k)$ holds for any $k < n$, where k^τ is defined in Notation 2.12. Δ_1^0 comprehension, provided by RCA_0^* , yields a set Y such that $k \in Y \leftrightarrow k < n \wedge \tau(k^\tau * \langle 1 \rangle) = 1$. Intuitively, Y is the set of natural numbers $k < n$ such that the winning strategy τ tells player II to answer “yes” when she is asked about it. We can check that $\forall k(\varphi(k) \wedge k < n \leftrightarrow k \in Y)$. $\square$

For the converse, we must notice that the class of Π_1^0 formulae is enriched by weak König's lemma.

Lemma 3.4 *For a Π_1^0 formula $\varphi(X)$, there is a Π_1^0 formula $\widehat{\varphi}$ such that WKL_0^* proves $\widehat{\varphi} \leftrightarrow \exists X \varphi(X)$.*

Proof. See [8, Lemma VIII.2.4]. Although the original lemma is proved in WKL_0 , we can check that the proof works also in WKL_0^* . $\square$

Remark 3.5 1. For a Π_1^0 game $\varphi(f)$ in the Cantor space, the assertion “ σ is a winning strategy for player I in $\varphi(f)$ ” is equivalent to a Π_1^0 formula over RCA_0^* . Let $\varphi(f) \equiv \forall n \theta(f[n])$. Then this assertion can be written as $\forall \tau \in \mathcal{S}_{\Pi}^{\{0,1\}} \forall n \theta(\sigma \otimes \tau[n])$, which is equivalent to a Π_1^0 formula $\forall s \in 2^{<\mathbb{N}} \forall n < 2|s| \theta((\sigma \otimes s)[n])$, where $\sigma \otimes s$ is the sequence of length $2|s|$ such that, for any $2k < 2|s|$, $(\sigma \otimes s)(2k) = \sigma((\sigma \otimes s)[2k])$ and, for each $2k+1 < 2|s|$, $(\sigma \otimes s)(2k+1) = s(k)$.

2. For a Σ_1^0 game $\varphi(f)$ in the Cantor space, the assertion “ σ is a winning strategy for player I in $\varphi(f)$ ” is equivalent to a Σ_1^0 formula over WKL_0^* . Say $\varphi(f) \equiv \exists n \theta(f[n])$. For a strategy σ for player I, $T_\sigma = \{s \in 2^{<\mathbb{N}} : \forall 2n < |s| (\sigma(s[2n]) = s(2n) \wedge \forall t \subseteq s \neg \theta(t))\}$ is a tree. By weak König's lemma, the assertion is equivalent to the finiteness of T_σ , which can be written in Σ_1^0 .

3. For any Π_1^0 game $\varphi(f)$ in the Cantor space, the assertion “player I has a winning strategy in $\varphi(f)$ ” is equivalent to a Π_1^0 formula over WKL_0^* . By 1, the assertion can be written as $\exists \sigma \in \mathcal{S}_I^{\{0,1\}} \psi(\sigma)$, where $\psi(X)$ is Π_1^0 . This is equivalent to a Π_1^0 formula in WKL_0^* by Lemma 3.4.

4. For any Σ_1^0 game $\varphi(f)$ in the Cantor space, the assertion “player I has a winning strategy in $\varphi(f)$ ” is equivalent to a Σ_1^0 formula over WKL_0^* . By a similar argument to 3, the assertion “player II has a winning strategy in $\varphi(f)$ ” is equivalent to a Π_1^0 formula ψ over WKL_0^* . By Σ_1^0 determinacy, which is proved in WKL_0^* , the aimed assertion is equivalent to $\neg \psi$, and is Σ_1^0 .

Lemma 3.6 (1) RCA_0^* proves Σ_1^0 axiom of choice. (2) WKL_0^* proves Π_1^0 axiom of choice.

Proof. (1) We work in RCA_0^* . Assume $\forall n \exists Y \exists m \theta(n, Y[m])$, where $\theta(x, s)$ is Π_1^0 . Then, by Δ_1^0 comprehension, define $Z = \{\langle n, k \rangle : \theta(n, e(k))\}$, where $e : \mathbb{N} \rightarrow 2^{<\mathbb{N}}$ is the canonical enumeration of $2^{<\mathbb{N}}$. By the assumption $\forall n \exists Y \exists m \theta(n, Y[m])$, for each n , there is k with $\langle n, k \rangle \in Z$. Take $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n)$ is the least k such that $\langle n, k \rangle \in Z$ holds for all n . Clearly $Y = \{\langle n, m \rangle : m < |e(f(n))| \wedge e(f(n))(m) = 1\}$ enjoys the desired property.

(2) We work in WKL_0^* . Assume $\forall n \exists Y \forall m \theta_1(n, Y[m])$, where $\theta_1(x, s)$ is Π_1^0 . Define T by $\{s \in 2^{<\mathbb{N}} : \forall \langle n, m \rangle < |s| \theta_1(n, s_{n,m})\}$, where $s_{n,m} = \langle s(\langle n, 0 \rangle), s(\langle n, 1 \rangle), \dots, s(\langle n, m-1 \rangle) \rangle$. Then T is infinite if

and only if $\forall n \exists Y \forall m \theta_1(n, Y[m])$. Using weak König's lemma, take a path f of T . Then $Y = \{\langle n, m \rangle : f(\langle n, m \rangle) = 1\}$ enjoys the desired property. $\square$

Theorem 3.7 WKL_0 proves $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$.

Proof. By Remark 2.16, it suffices to prove $\text{Sep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$ in WKL_0 , instead of $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^*$. Let $\varphi(f)$ be a game of the form $(\psi(f) \wedge \eta_0(f)) \vee (\neg\psi(f) \wedge \eta_1(f))$, where both $\psi(f)$ and $\eta_1(f)$ are Σ_1^0 , where $\eta_0(f)$ is Π_1^0 , and where $\forall f \in 2^{\mathbb{N}} (\psi(f) \leftrightarrow \neg\psi'(f))$ for some Σ_1^0 formula $\psi'(f)$. By Lemma 2.13, there are Π_0^0 formulae $\theta_0(x)$ and $\theta_1(x)$ such that $\psi(f) \leftrightarrow \exists n \theta_0(f[n])$ and $\neg\psi(f) \leftrightarrow \exists n \theta_1(f[n])$. By Δ_1^0 comprehension, set a tree $T = \{s \in 2^{<\mathbb{N}} : \forall t \subseteq s (\neg\theta_0(t) \wedge \neg\theta_1(t))\}$. T has no infinite path, otherwise there exists an infinite sequence $f \in 2^{\mathbb{N}}$ such that $\forall n \neg\theta_0(f[n])$ and $\forall n \neg\theta_1(f[n])$, which is impossible. By weak König's lemma, T is a finite tree. Let n_T be the least even $n \in \mathbb{N}$ such that $|t| < n$ for all $t \in T$. Note that every $s \in 2^{<\mathbb{N}}$ of length n_T satisfies exactly one of $\exists t \subseteq s \theta_0(t)$ and $\exists t \subseteq s \theta_1(t)$. By 3 and 4 of Remark 3.5, bounded Σ_1^0 comprehension, which is equivalent to Σ_1^0 induction, provides

$$W_i = \{s \in 2^{\mathbb{N}} : |s| = n_T \wedge \exists t \subseteq s \theta_i(t) \wedge (\text{Player I wins } \eta_i(f) \text{ at } s)\} \quad (i < 2).$$

Then we can define a new Δ_1^0 game $\varphi^*(f) \equiv f[n_T] \in W_0 \cup W_1$. By $\Delta_1^0\text{-Det}^*$, which is provided by WKL_0 , one of the players has a winning strategy in $\varphi^*(f)$. We show the following claim, which completes the proof of this theorem.

Claim The player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$.

Proof of the claim: First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. He can win $\varphi(f)$ by playing as follows. He follows σ^* till he reaches s of length n_T . By the assumption, s satisfies $s \in W_0 \cup W_1$. If he reaches $s \in W_i$ then he switches the strategy to a winning strategy σ_s in $\eta_i(s * f)$. More precisely, by 3 and 4 of Remark 3.5 and Lemma 3.6, we have a sequence $\langle \sigma_s : s \in W_0 \cup W_1 \rangle$ of strategies for player I such that σ_s is a winning strategy in $\eta_i(s * f)$ for each $s \in W_i$. Now, a new strategy σ for player I in $\varphi(f)$ is defined as follows:

$$\sigma(t) = \begin{cases} \sigma_{t[n_T]}(t \ominus n_T) & \text{if } |t| \geq n_T \text{ and } t[n_T] \in W_0 \cup W_1, \\ \sigma^*(t) & \text{otherwise.} \end{cases}$$

It can be checked that for any play f in which player I follows σ , $f[n_T] \in W_0 \cup W_1$ and $f \ominus n_T$ is a play in which player I follows $\sigma_{f[n_T]}$, and so $\eta_i(f)$ if $f[n_T] \in W_i$. Thus player I wins $\varphi(f)$.

Next, assume that player II has a winning strategy τ^* in $\varphi^*(f)$. She can win $\varphi(f)$ by playing as follows. She follows τ^* till she reaches s of length n_T . By the assumption, s is not in $W_0 \cup W_1$. Note that s satisfies exactly one of $\exists t \subseteq s \theta_0(t)$ and $\exists t \subseteq s \theta_1(t)$ and that if $\exists t \subseteq s \theta_i(t)$ and $s \notin W_i$, then player II wins $\eta_i(f)$ at s by $\Sigma_1^0\text{-Det}^*$. Then she switches the strategy to a winning strategy τ_s in $\eta_i(s * f)$, if s satisfies $\exists t \subseteq s \theta_i(t)$. More precisely, we have a sequence $\langle \tau_s : |s| = n_T \wedge s \notin W_0 \cup W_1 \rangle$ of strategies for player II such that if $\exists t \subseteq s \theta_i(t)$, then τ_s is a winning strategy in $\eta_i(s * f)$ by Lemma 3.6 as in the previous case. Define a strategy τ for player II as follows:

$$\tau(t) = \begin{cases} \tau_{t[n_T]}(t \ominus n_T) & \text{if } |t| \geq n_T \text{ and } t[n_T] \notin W_0 \cup W_1, \\ \tau^*(t) & \text{otherwise.} \end{cases}$$

Then τ is a winning strategy for player II in $\varphi(f)$. $\square$

Finally, we have the following.

Corollary 3.8 $\text{RCA}_0^* \vdash \text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}^* \leftrightarrow \text{WKL}_0$.

4 $\text{ATR}_0 + \Sigma_1^1$ induction and determinacy

In this section, we consider the system $\text{ATR}_0 + \Sigma_1^1$ induction and determinacy in the Cantor space. First, the system ATR_0 is defined as follows.

Definition 4.1 The system ATR_0 consists of RCA_0 plus the schema of Π_0^1 transfinite recursion.

Game theoretical characterizations of ATR_0 are given as follows.

Proposition 4.2 $\text{RCA}_0^* \vdash \Delta_1^0\text{-Det} \leftrightarrow \Sigma_1^0\text{-Det} \leftrightarrow \Delta_2^0\text{-Det}^* \leftrightarrow \Sigma_2^0\text{-Det}^* \leftrightarrow \text{ATR}_0$

Proof. For the equivalences between $\Delta_1^0\text{-Det}$, $\Sigma_1^0\text{-Det}$ and Π_0^1 transfinite recursion, see [8, Section V.8] or [10]. For the equivalence between $\Delta_2^0\text{-Det}^*$, $\Sigma_2^0\text{-Det}^*$ and Π_0^1 transfinite recursion, see [5]. The base theory RCA_0 can be replaced with RCA_0^* by Theorem 3.3. $\square$

Using the proposition, we can prove the following:

Lemma 4.3 ([8, Theorem V.8.3], [11, Corollary 3.2]) ATR_0 proves Σ_1^1 axiom of choice.

Remark 4.4 1. For any Π_2^0 game $\varphi(f)$ in the Cantor space, the assertion “ σ is a winning strategy for player I in $\varphi(f)$ ” is equivalent to a Π_2^0 formula over WKL_0^* . Let $\varphi(f) \equiv \forall n \exists m \theta(n, f[m])$. The assertion can be written as $\forall \tau \in \mathcal{S}_{\Pi}^{\{0,1\}} \forall n \exists m \theta(n, (\sigma \otimes \tau)[m]) \leftrightarrow \forall n \forall \tau \in \mathcal{S}_{\Pi}^{\{0,1\}} \exists m \theta(n, (\sigma \otimes \tau)[m])$. The underlined part is Σ_1^0 by Lemma 3.4, and so the whole statement is Π_2^0 .

2. For any Σ_2^0 game $\varphi(f)$ in the Cantor space, the assertion “ σ is a winning strategy for player I in $\varphi(f)$ ” is equivalent to a Π_1^1 formula over RCA_0^* . This is straightforward.

3. For any Π_2^0 game $\varphi(f)$ in the Cantor space, the assertion “player I has a winning strategy in $\varphi(f)$ ” is equivalent to a Σ_1^1 formula over WKL_0^* . This follows from 1.

4. For any Σ_2^0 game $\varphi(f)$ in the Cantor space, the assertion “player I has a winning strategy in $\varphi(f)$ ” is equivalent to a Π_1^1 formula over ATR_0 . By a similar argument to 3, the assertion “player II has a winning strategy in $\varphi(f)$ ” can be written as a Σ_1^1 formula ψ . By $\Sigma_2^0\text{-Det}^*$, which is provided by ATR_0 , the aimed assertion is equivalent to $\neg\psi$ and is Π_1^1 .

Now we consider the system $\text{ATR}_0 + \Sigma_1^1$ induction.

Remark 4.5 (1) [8, Corollary IX. 4. 7] shows that $\text{ATR}_0 + \Sigma_1^1$ induction proves the consistency of ATR_0 . This means that Σ_1^1 induction really strengthens ATR_0 .

(2) $\text{ATR}_0 + \Pi_1^1$ transfinite induction proves the consistency of $\text{ATR}_0 + \Sigma_1^1$ induction, since [8, Theorem VIII.3.15] shows that ATR_0 can be axiomatized by ACA_0 plus a single Π_2^1 formula and since [8, Theorem VIII.5.12] shows that $\text{ATR}_0 + \Pi_1^1$ transfinite induction implies the existence of a countable coded ω -model of ATR_0 and hence the consistency of $\text{ATR}_0 +$ full induction.

We have the following equivalence on Σ_1^1 induction.

Lemma 4.6 ([7, 2.1 Lemma]) Σ_1^1 induction is equivalent to bounded Σ_1^1 comprehension over ACA_0 , i.e., the system RCA_0 plus Σ_1^0 comprehension.

[7] shows that Σ_1^1 induction is equivalent to Σ_1^1 transfinite induction over ATR_0 .

Although the intuitive idea of the following proof is the same as Theorem 3.7, the lack of Π_1^1 axiom of choice makes the proof slightly complicated.

Theorem 4.7 $\text{ATR}_0 + \Sigma_1^1$ induction proves $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$.

Proof. By Remark 2.16, it suffices to prove $\text{Sep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$ in $\text{ATR}_0 + \Sigma_1^1$ induction. Let $\varphi(f)$ be a game of the form $(\exists n\theta_0(f[n]) \wedge \eta_0(f)) \vee (\exists n\theta_1(f[n]) \wedge \eta_1(f))$, where $\theta_0(s)$ and $\theta_1(s)$ are Π_0^0 such that $\forall f(\exists n\theta_0(f[n]) \leftrightarrow \neg\exists n\theta_1(f[n]))$, where $\eta_0(f)$ is Π_2^0 , and where $\eta_1(f)$ is Σ_2^0 . As in the proof of Theorem 3.7, define a tree $T = \{s \in 2^{<\mathbb{N}} : \forall t \subseteq s (\neg\theta_0(t) \wedge \neg\theta_1(t))\}$ and take the least even n_T with $|s| < n_T$ for all $s \in T$. Bounded Π_1^1 comprehension with help of 3 and 4 of Remark 4.4 yields

$$\begin{aligned} W_i &= \{s \in 2^{<\mathbb{N}} : |s| = n_T \wedge \exists t \subseteq s \theta_i(t) \wedge \text{player I wins } \eta_i(f) \text{ at } s\}, \\ W'_i &= \{s \in 2^{<\mathbb{N}} : |s| = n_T \wedge \exists t \subseteq s \theta_i(t) \wedge s \notin W'_i\}, \end{aligned}$$

for $i < 2$. Note that, by $\Sigma_1^0\text{-Det}^*$, player II wins $\eta_i(f)$ at $s \in W'_i$.

Define a new Δ_1^0 game $\varphi^*(f) \equiv f[n_T] \in W_0 \cup W_1$. By $\Delta_1^0\text{-Det}^*$, it is determinate.

Claim The player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$.

Proof of the claim. First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. Then consider the following game: $\psi_0(f) \equiv (f[n_T] \in W_0 \cup W_1) \wedge (f[n_T] \in W_1 \rightarrow \eta_1(f))$. Player II has no winning strategy in $\psi_0(f)$ as shown below. Suppose, for contradiction, player II has a winning strategy ν in $\psi_0(f)$. Then $(\sigma^* \otimes \nu)[n_T] \in W_1$, for otherwise, $(\sigma^* \otimes \nu)[n_T] \in W_0$ would hold and player II would lose $\psi_0(f)$. By the definition of W_1 , player I has a winning strategy ν in $\eta_1((\sigma^* \otimes \nu)[n_T] * f)$. Then define a strategy ρ for player I in $\psi_0(f)$ as follows:

$$\rho(t) = \begin{cases} \sigma^*(t) & \text{if } |t| < n_T, \\ \nu(t \oplus n_T) & \text{otherwise.} \end{cases}$$

Since $(\rho \otimes \nu)[n_T] = (\sigma^* \otimes \nu)[n_T]$ and $(\rho \otimes \nu) \oplus n_T$ is a resulting play of ν , $\eta_1(\rho \otimes \nu)$ holds, and so $\psi_0(\rho \otimes \nu)$ holds, which contradicts the assumption that ν is a winning strategy for player II in $\psi_0(f)$.

Since $\psi_0(f)$ is equivalent to some Σ_2^0 formula, player I has a winning strategy σ_0 in $\psi_0(f)$ by $\Sigma_2^0\text{-Det}^*$. Let $\langle \sigma_s : s \in W_0 \rangle$ be a sequence of winning strategies for player I in $\eta_0(s * f)$, which is yielded by Lemma 4.3. Define a strategy σ for player I in $\varphi(f)$:

$$\sigma(t) = \begin{cases} \sigma_{t[n_T]}(t \oplus n_T) & \text{if } t[n_T] \in W_0, \\ \sigma_0(t) & \text{otherwise.} \end{cases}$$

Then, for any strategy τ for player II in $\varphi(f)$, $(\sigma \otimes \tau)[n_T] \in W_0 \cup W_1$ holds and, for any $i < 2$, if $(\sigma \otimes \tau)[n_T] \in W_i$, then $\eta_i(\sigma \otimes \tau)$ holds, and so $\varphi(\sigma \otimes \tau)$.

Next, assume that player II has a winning strategy τ^* in $\varphi^*(f)$. Note that $\varphi^*(f) \leftrightarrow \neg(f[n_T] \in W'_0 \cup W'_1)$. We can prove that player II has a winning strategy τ_0 in $\psi_1(f) \equiv \neg((f[n_T] \in W'_0 \cup W'_1) \wedge (f[n_T] \in W'_0 \rightarrow \neg\eta_0(f)))$ as in the previous case. By Lemma 4.3, we have a sequence of winning strategies $\langle \tau_s : |s| = n_T \wedge \exists t \subseteq s \theta_1(t) \wedge s \notin W_1 \rangle$ for player II in $\eta_1(s * f)$. Then the strategy τ for player II defined by

$$\tau(t) = \begin{cases} \tau_{t[n_T]}(t \oplus n_T) & \text{if } t[n_T] \in W'_1, \\ \tau_0(t) & \text{otherwise.} \end{cases}$$

is a winning strategy for player II in the original game $\varphi(f)$. $\square$

Lemma 4.8 (normal form theorem 2, [8, Theorem V.1.4]) For a Σ_1^1 formula $\varphi(X)$, we can find a Π_0^0 formula $\theta(X, f)$ such that ACA_0 proves $\forall Y(\varphi(Y) \leftrightarrow \exists g \in \mathbb{N}^{\mathbb{N}} \forall m \theta(Y[m], g[m]))$.

To investigate the other direction, some definitions and notations are needed.

Definition 4.9 A finite sequence $s \in 2^{<\mathbb{N}}$ is *regular* if:

(r1) $s = \langle \rangle$ or $|s| > 1 \wedge s(|s| - 1) \neq s(|s| - 2)$,

(r2) for all $i < 2$, $2k + i < |s| - 1$ and $s(2k + i) = i$ imply $s(2k + i + 1) = i$.

If $s \in 2^{<\mathbb{N}}$ is a non-empty regular sequence, $\bar{s} = \langle n_0, \dots, n_k \rangle \in \mathbb{N}^{<\mathbb{N}}$ is the sequence such that

$$s = \langle \underbrace{0, \dots, 0}_{2n_0 \text{ times}}, 1, \underbrace{1, \dots, 1}_{2n_1 \text{ times}}, 0, \dots, \underbrace{i_k, \dots, i_k}_{2n_k \text{ times}}, 1 - i_k \rangle,$$

where i_k is 0 if k is even, $i_k = 1$ otherwise. The assertion “ s is regular” is equivalent to a Π_0^0 formula over RCA_0^* .

An infinite sequence $f \in 2^{\mathbb{N}}$ is *totally regular* if it satisfies

(r2') for all $i < 2$ and k , $s(2k + i) = i$ implies $s(2k + i + 1) = i$,

(r3) for all $i < 2$ and k , if $f(k) = i$, then there exists $m > k$ with $f(m) = 1 - i$.

For a totally regular sequence $g \in 2^{\mathbb{N}}$, $\bar{g} = \langle n_0, n_1, n_2, \dots \rangle \in \mathbb{N}^{\mathbb{N}}$ is the sequence such that

$$g = \langle \underbrace{0, \dots, 0}_{2n_0 \text{ times}}, 1, \underbrace{1, \dots, 1}_{2n_1 \text{ times}}, 0, \underbrace{0, \dots, 0}_{2n_2 \text{ times}}, 1, \dots \rangle.$$

The assertion “ f is totally regular” is equivalent to a Π_2^0 formula over RCA_0^* .

Theorem 4.10 RCA_0^* proves that $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$ implies Σ_1^1 induction.

Proof. Assume that $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$. By Lemma 4.6, it suffices to prove bounded Σ_1^1 comprehension. Let $\varphi(x)$ be a Σ_1^1 formula. We can find a Π_0^0 formula $\theta(x, s)$ such that ACA_0 proves $\varphi(x) \leftrightarrow \exists f \forall m \theta(x, f[m])$ for all $x \in \mathbb{N}$.

For any fixed n , consider the following game. First, player I chooses $k < n$ and asks whether $\varphi(k)$ or not. Then player II answers “Yes” or “No.” If she answers “Yes,” she has to construct a witness g of $\varphi(k)$. If she answers “No,” player I has to construct a witness of $\varphi(k)$. Such a game is realized as follows:

- Player I chooses $k < n$ by playing 0 at his first k -th turns and 1 at his $(k + 1)$ -th turn.
- Player II chooses 0 or 1 at her $(k + 1)$ -th turn.
- If player II answers 1 at her $(k + 1)$ -th turn, she wins by constructing g such that $\forall m \theta(k, \bar{g}[m])$. (case 1)
- If player II chooses 0 at her $(k + 1)$ -th turn, player I wins by constructing g such that $\forall m \theta(k, \bar{g}[m])$. (case 2)

To illustrate, the game goes as follows:

Case 1

$$\begin{array}{llllllllll} \text{player I} & \underbrace{0, \dots, 0}_{k \text{ times}} & 1, & *, \dots, *, & *, & *, \dots, *, & *, & *, \dots, *, & *, & \dots \\ \text{player II} & *, \dots, *, & 1, & \underbrace{0, \dots, 0}_{2\bar{g}(0) \text{ times}} & 1, & \underbrace{1, \dots, 1}_{2\bar{g}(1) \text{ times}} & 0, & \underbrace{0, \dots, 0}_{2\bar{g}(2) \text{ times}} & 1, & \dots \end{array}$$

Case 2

$$\begin{array}{llllllllll} \text{player I} & \underbrace{0, \dots, 0}_{k \text{ times}} & 1, & \underbrace{0, \dots, 0}_{2\bar{g}(0) \text{ times}} & 1, & \underbrace{1, \dots, 1}_{2\bar{g}(1) \text{ times}} & 0, & \underbrace{0, \dots, 0}_{2\bar{g}(2) \text{ times}} & 1, & \dots \\ \text{player II} & *, \dots, *, & 0, & *, \dots, *, & *, & *, \dots, *, & *, & *, \dots, *, & *, & \dots \end{array}$$

Player I wins if and only if one of the following holds:

- Player I chooses $k < n$ and player II answers “Yes” and player II fails to construct totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \bar{g}[m])$.

- Player I chooses $k < n$ and player II answers “No” and player I succeeds to construct totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \bar{g}[m])$.

Formally, this winning condition can be written as below:

$$\begin{aligned} & \exists k < l(\psi(f, k, 1) \wedge ((f_{k,1} \text{ is not totally regular}) \vee \exists l((f_{k,1}[l] \text{ is regular}) \wedge \neg \theta(k, \overline{f_{k,1}[l]}))) \vee \\ & \exists k < l(\psi(f, k, 0) \wedge (f_{k,0} \text{ is totally regular}) \wedge \forall l \exists m(|\overline{f_{k,0}[m]}| = l \wedge \theta(k, \overline{f_{k,0}[m]}))), \quad (\sharp) \end{aligned}$$

where $\psi(f, k, i)$ is the Π_0^0 formula $\forall i < k(f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i$ and $f_{k,i}$ is $g \in 2^{\mathbb{N}}$ such that, for all l , $g(l) = f(2(k+1) + 2l + i)$.

Similarly to the proof of Theorem 3.7, $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$ implies that $(\sharp)$ is determinate. Moreover, player I cannot have a winning strategy in $(\sharp)$, since if $\varphi(k)$ holds, player I cannot win when player II answers “Yes” and constructs a witness, otherwise, i.e., $\varphi(k)$ holds, player I cannot win when player II answers “No.” Therefore player II has a winning strategy τ . Δ_1^0 comprehension yields $Y = \{k < n : \tau(k^* * \langle 1 \rangle) = 1\}$. Clearly, Y enjoys the desired property. $\square$

Since $\text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^*$ implies $\Sigma_2^0\text{-Det}^*$, it implies Π_0^1 transfinite recursion by Proposition 4.2. Finally, we have the following.

Corollary 4.11 $\text{RCA}_0^* \vdash \text{Bisep}(\Delta_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{ATR}_0 + \Sigma_1^1 \text{ induction}.$

5 $\Pi_1^1\text{-CA}_0$ and the determinacy of Wadge class

In this section, we find determinacy schemata which is equivalent to Π_1^1 comprehension.

Definition 5.1 $\Pi_1^1\text{-CA}_0$ is the system RCA_0 plus Π_1^1 comprehension.

By [8, Theorem V.5.1], $\Pi_1^1\text{-CA}_0$ includes ATR_0 . Moreover, since $\Pi_1^1\text{-CA}_0$ proves both Π_1^1 and Σ_1^1 transfinite induction by Lemma 2.3, $\Pi_1^1\text{-CA}_0$ includes $\text{ATR}_0 + \Sigma_1^1$ induction and, by Remark 4.5 (2), proves the consistency of $\text{ATR}_0 + \Sigma_1^1$ induction.

By determinacies in the Baire space, $\Pi_1^1\text{-CA}_0$ is characterized as follows:

Proposition 5.2 $\text{RCA}_0^* \vdash \text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det} \leftrightarrow (\Sigma_1^0)_2\text{-Det} \leftrightarrow \Pi_1^1\text{-CA}_0$, where $(\Sigma_1^0)_2$ is $\Sigma_1^0 \wedge \Pi_1^0$.

Proof. See [8, Lemma VI.5.2, Lemma VI.5.3]. Although the proof of [8, Lemma VI.5.3] shows the implication from $(\Sigma_1^0)_2\text{-Det}$ to Π_1^1 comprehension in RCA_0 , $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}$ implies the determinacy of the game defined in the proof and so we need only $\text{Bisep}(\Delta_1^0, \Sigma_1^0)\text{-Det}$ for the proof. $\square$

By the above proposition, it turns out that there is no determinacy schemata corresponding to Wadge classes which are equivalent to $\text{ATR}_0 + \Sigma_1^1$ induction.

In the following proof, we construct a game which is similar to that in the proof of Theorem 4.10. In the following game player I can ask about any natural number k , whereas he can ask only about $k < n$ in the game defined in the proof of Theorem 4.10. This point corresponds to the difference between (unbounded) Σ_1^1 comprehension and bounded Σ_1^1 comprehension.

Theorem 5.3 RCA_0^* proves that $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ implies Π_1^1 comprehension.

Proof. Assume $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$. Since we can easily see $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ implies $\Sigma_1^0 \wedge \Pi_1^0\text{-Det}^*$, we can work in ACA_0 . We show Σ_1^1 comprehension, which is equivalent to Π_1^1 comprehension over RCA_0^* . Let $\varphi(n)$ be a Σ_1^1 formula. In ACA_0 , we can find a Π_0^0 formula $\theta(x, f)$ such that $\varphi(n) \leftrightarrow \exists f \forall m \theta(n, f[m])$, for all $n \in \mathbb{N}$. Now we consider the following game: Player I chooses $k \in \mathbb{N}$ and asks whether $\varphi(n)$ or not. Player II answers “Yes” or “No.” If she answers “Yes,” she has to construct a witness for $\varphi(n)$. If she answers “No,” player I has to construct a witness for $\varphi(n)$.

Such a game is realized as follows:

- Player I chooses $k \in \mathbb{N}$ by playing 0 his first k turns and 1 at his $(k+1)$ -th turn.
- Player II chooses 0 or 1 at her $(k+1)$ -th turn.
- If player II chooses 1, she wins by constructing totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(n, \bar{g}[m])$.
- If player II chooses 0, player I wins by constructing totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(n, \bar{g}[m])$.

Player I wins if and only if one of the following conditions holds:

- Player I chooses k and player II answers “Yes,” and she fails to construct g with $\forall m \theta(k, \bar{g}[m])$.
- Player I chooses k and player II answers “No,” and he succeeds to construct g with $\forall m \theta(k, \bar{g}[m])$.

Rigorously, the above winning condition can be written as below:

$$\begin{aligned} & \exists k (\psi(f, k, 1) \wedge ((f_{k,1} \text{ is not totally regular}) \vee \exists l ((f_{k,1}[l] \text{ is regular}) \wedge \neg \theta(k, \overline{f_{k,1}[l]}))) \vee \\ & \exists k (\psi(f, k, 0) \wedge ((f_{k,0} \text{ is totally regular}) \wedge \forall l \exists m (|\overline{f_{k,0}[m]}| = l \wedge \theta(k, \overline{f_{k,0}[m]})))) \end{aligned} \quad (\natural)$$

where $\psi(f, k, i)$ is the Π_0^0 formula $\forall i < k (f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i$ and $f_{k,i}$ is $g \in 2^{\mathbb{N}}$ such that, for all l , $g(l) = f(2(k+1) + 2l + i)$. As in the proof of Theorem 4.10, **Bisep**(Σ_1^0, Σ_2^0)-**Det**^{*} implies that player II has a winning strategy τ in $(\natural)$. Then $Y = \{k : \tau(k^\tau * \langle 1 \rangle) = 1\}$ enjoys the desired property. $\square$

Next, we consider the converse. It suffices to show the implication from Π_1^1 -**CA**₀ to **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*}, since it is easy to prove that **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*} implies **Bisep**(Σ_1^0, Σ_2^0)-**Det**^{*}. It can be proved in a similar way to Theorem 4.7. As mentioned before Theorem 4.7, if we had a sufficient fragment of the axiom of choice, it could be proved in a simpler way like Theorem 3.7. Now, although Π_1^1 -**CA**₀ does not prove Π_1^1 axiom of choice, we have the following conservation results.

Lemma 5.4 ([8, Theorem IX.4.9]) Π_1^1 -**AC**₀, the system **ACA**₀ plus Π_1^1 axiom of choice, is conservative over Π_1^1 -**CA**₀ for Π_3^1 sentence, i.e., any Π_3^1 sentence ψ which is provable in Π_1^1 -**AC**₀ is already provable in Π_1^1 -**CA**₀.

Since all the sentences in **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*} are equivalent to Π_3^1 formulae, Lemma 5.4 guarantees that if we can prove **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*} in Π_1^1 -**AC**₀, then it is already provable in Π_1^1 -**CA**₀,

Theorem 5.5 Π_1^1 -**CA**₀ proves **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*}.

Proof. As mentioned above, it suffices to prove **Sep**(Σ_1^0, Σ_2^0)-**Det**^{*} in Π_1^1 -**AC**₀. Let $\varphi(f)$ be a **Sep**(Σ_1^0, Σ_2^0) game, say $(\exists n \theta(f[n]) \wedge \psi(f)) \vee (\neg \exists n \theta(f[n]) \wedge \eta(f))$, where θ is Π_0^0 , $\psi(f)$ is Π_2^0 and $\eta(f)$ is Σ_2^0 . By 3 of Remark 4.4, Π_1^1 comprehension implies the existence of V and V' such that

$$\begin{aligned} V &= \{s : |s| \text{ is even, } \exists t \subseteq s \theta(t) \text{ and player I wins } \psi(f) \text{ at } s\}, \\ V' &= \{s : |s| \text{ is even, } \exists t \subseteq s \theta(t) \text{ and } s \notin V\}. \end{aligned}$$

Define a new game $\varphi^*(f)$ as follows: $\varphi^*(f) \equiv \exists n (f[n] \in V) \vee (\neg \exists n (f[n] \in V) \wedge \eta(f))$. By Σ_2^0 -**Det**^{*}, which is proved in **ATR**₀ (cf. Proposition 4.2), $\varphi^*(f)$ is determinate. We show the following claim:

Claim The player who wins $\varphi^*(f)$ also wins in the original game $\varphi(f)$.

Proof of the claim First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. By Π_1^1 axiom of choice, which is proved in **ATR**₀, we have a sequence $\langle \sigma_s : s \in V \rangle$ of winning strategies for player I in $\psi(s * f)$. Now we define a new strategy σ for player I in $\varphi(f)$ as follows:

$$\sigma(s) = \begin{cases} \sigma^*(s) & \text{if } t \notin V \text{ for all } t \subseteq s, \\ \sigma_t(s \ominus |t|) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in V. \end{cases}$$

We prove that σ is a winning strategy for player I in $\varphi(f)$. Let f be a play in which player I follows σ . If there is no $k \in \mathbb{N}$ with $\theta(f[k])$, then f satisfies $\neg \exists n \theta(f[n]) \wedge \eta(f)$, and so player I wins. If there is k with $\theta(f[k])$, then f satisfies $\exists m f[2m] \in V$. Take the least such m . Then $\sigma(f[2m+2l]) = \sigma_{f[2m]}(f[2m+2l] \ominus 2m)$ holds for all l , and so $\psi(f)$ holds, and so player I wins.

Next, assume that player II has a winning strategy τ^* in the game $\varphi^*(f)$. Since ATR_0 implies $\Sigma_2^0\text{-Det}^*$, player II wins $\psi(f)$ at each $s \in V'$. By Π_1^1 axiom of choice, we have a sequence $\langle \tau_s : s \in V' \rangle$ of winning strategies for player II in $\psi(s * f)$.

Now, define a new strategy τ for player II in $\varphi(f)$ as follows:

$$\tau(s) = \begin{cases} \tau^*(s) & \text{if } t \notin V' \text{ for all } t \subseteq s, \\ \tau_t(s \ominus |t|) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in V'. \end{cases}$$

We can prove that τ is a winning strategy for player II in $\varphi(f)$, similarly to the previous case. $\square$

Corollary 5.6 $\text{RCA}_0^* \vdash \text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \text{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \Pi_1^1\text{-CA}_0$.

Proof. Theorem 5.3 shows the implication of Π_1^1 comprehension from $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$. Theorem 5.5 shows the implication from Π_1^1 comprehension to $\text{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$. The implication from $\text{Sep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ to $\text{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-Det}^*$ is clear. $\square$

6 $\Pi_1^1\text{-TR}_0$ and determinacy of Wadge class

Now let us turn to considering the relation between Π_1^1 transfinite recursion and determinacy.

Definition 6.1 $\Pi_1^1\text{-TR}_0$ is the system RCA_0 plus Π_1^1 transfinite recursion.

$\Pi_1^1\text{-TR}_0$ is characterized by determinacy as follows:

Proposition 6.2 ([11, Theorem 6.1]) $\text{RCA}_0^* \vdash \Delta_2^0\text{-Det} \leftrightarrow \Pi_1^1\text{-TR}_0$.

In this section, we prove the equivalence between $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ and $\Pi_1^1\text{-TR}_0$ over RCA_0^* .

Theorem 6.3 RCA_0^* proves that $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ implies $\Delta_2^0\text{-Det}$.

Proof. We work in RCA_0^* . Let $\varphi(f)$ be a game in the Baire space such that $\forall f \in \mathbb{N}^\mathbb{N} ((\varphi(f) \leftrightarrow \forall n \exists m \theta_0(n, f[m])) \text{ and } \forall f \in \mathbb{N}^\mathbb{N} (\neg \varphi(f) \leftrightarrow \forall n \exists m \theta_1(n, f[m])))$ for some Π_0^0 formulae $\theta_0(x, s)$ and $\theta_1(x, s)$.

We shall define a game $\varphi^*(f)$ in the Cantor space such that the player who wins $\varphi^*(f)$ also wins $\varphi(f)$ and $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ proves its determinacy. The intuitive idea of $\varphi^*(f)$ is as follows. Instead of playing in the Baire space, players I and II construct a totally regular sequence $f \in 2^\mathbb{N}$ (recall Definition 4.9). The first player who gives up making f totally regular loses. If they succeed to construct totally regular f , then player I wins if and only if $\varphi(\bar{f})$. To give a rigorous formulation to this idea, first let

$$\psi_0(k, i, f) \equiv f(2k+i) = i \rightarrow f(2k+i+1) = i,$$

$$\psi_1(i, f) \equiv \exists n (\forall k < n \forall j < 2 \psi_0(k, j, f) \wedge \forall l < (1-i) \psi_0(n, l, f) \wedge \neg \psi_0(n, 1-i, f)),$$

(This means “player I (resp. II) first breaks **(r2’)** of Definition 4.9” if $i = 0$ (resp. $i = 1$).)

$$\psi_2(i, f) \equiv \forall n \forall i < 2 \psi_0(n, j, f) \wedge \exists m \forall l > m (f(2l+i) = i),$$

(This means “both players keep **(r2’)** but player I (resp. II) breaks **(r3)**” if $i = 0$ (resp. $i = 1$).)

$$\psi(i, f) \equiv \psi_1(i, f) \vee \psi_2(i, f).$$

(This means “player I (resp. II) first gives up making f totally regular” if $i = 0$ (resp. $i = 1$).)

Note that $\psi(i, f)$ is equivalent to a Σ_2^0 formula over RCA_0^* . Then, for $i < 2$, set $\eta_i(f)$ by $(f \text{ is totally regular}) \wedge \forall n \exists m (f[m] \text{ is regular} \wedge \theta_i(n, \overline{f[m]}))$. Clearly η_i is equivalent to a Π_2^0 formula over RCA_0^* . Then there is no $f \in 2^{\mathbb{N}}$ with $\eta_0(f) \wedge \eta_1(f)$. By Lemma 2.17, RCA_0^* yields a Σ_2^0 formula $\eta(f)$ and a Π_2^0 formula $\eta'(f)$ with $\forall f \in 2^{\mathbb{N}} ((\eta(f) \leftrightarrow \eta'(f)) \wedge (\eta_0(f) \rightarrow \eta(f)) \wedge (\eta_1(f) \rightarrow \neg \eta(f)))$. Furthermore, we can check each f satisfies exactly one of $\psi(0, f)$, $\psi(1, f)$ and $\eta_i(f)$ for each $f \in 2^{\mathbb{N}}$.

$\eta(f)$	$\neg \eta(f)$
$\psi(0, f)$	$\psi(0, f)$
$\eta_0(f)$	$\eta_1(f)$
$\psi(1, f)$	$\psi(1, f)$

Define a new game $\varphi^*(f)$ by $(\eta(f) \wedge \neg \psi(0, f)) \vee (\neg \eta(f) \wedge \psi(1, f))$. $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ implies that $\varphi^*(f)$ is determinate. Since $\varphi^*(f)$ holds if and only if $\eta_0(f) \vee \psi_1(1, f)$, we can check that player who has a winning strategy in $\varphi^*(f)$ also has a winning strategy in $\varphi(f)$. $\square$

For the converse, we need some preparations motivated by the reformulation of Δ_2^0 in [11].

Lemma 6.4 *For any Π_2^0 formula $\varphi(f)$ with a distinguished function variable $f \in 2^{\mathbb{N}}$, we can find, in RCA_0 , a Π_0^0 formula $\theta'(x)$ in which n does not occur such that $\forall f \in 2^{\mathbb{N}} (\varphi(f) \leftrightarrow \forall n \exists m > n \theta'(f[m]))$.*

Proof. See [3, Lemma 3.1]. Although the original proof is done in ACA_0 , it works also in RCA_0 . $\square$

Let $\psi(f)$ be a Σ_2^0 formula and $\psi'(f)$ Π_2^0 a formula such that $\forall f (\psi(f) \leftrightarrow \psi'(f))$. Then, by Lemma 6.4, we can find Π_0^0 formulae $\theta_0(x)$ and $\theta_1(x)$ such that

$$\psi(f) \leftrightarrow \forall n \exists m > n \theta_0(f[m]), \quad \neg \psi(f) \leftrightarrow \forall n \exists m > n \theta_1(f[m]).$$

We may assume that $\{s : \theta_0(s) \wedge \theta_1(s)\} = \emptyset$ and $\theta_0(\langle \rangle)$ by replacing $\theta_0(x)$ with $\theta'_0(x) \equiv (\theta_0(x) \wedge \neg \theta_1(x)) \vee (x = \langle \rangle)$ and $\theta_1(x)$ with $\theta'_1(x) \equiv \theta_1(x) \wedge \neg \theta_0(x) \wedge (x \neq \langle \rangle)$ if necessary. Define a T_{θ_0, θ_1} by

$$T_{\theta_0, \theta_1} = \{x \in (2^{<\mathbb{N}})^{<\mathbb{N}} : x(0) \subsetneq x(1) \subsetneq \dots \subsetneq x(|x| - 1),$$

$$\theta_0(x(k)) \text{ for each even } k < |x| \text{ and } \theta_1(x(k)) \text{ for each odd } k < |x|\}$$

For every $f \in 2^{\mathbb{N}}$, we can prove that there exists $x \in T$ such that x has no proper extension in T and $\bigcup_{i < |x|} x(i) = x(|x| - 1) \subseteq f$ as follows. If $\forall n \exists m > n \theta_i(f[m])$, there exists p and $r > p$ with $\forall q > p \neg \theta_{1-i}(f[q])$ and $\theta_i(f[r])$. If $i = 0$, $\langle f[r] \rangle$ enjoys the desired property. If $i = 1$, $\langle \rangle, f[r] \rangle$ enjoys the desired property.

T_{θ_0, θ_1} has no infinite path, since if F were an infinite path of T_{θ_0, θ_1} , $f \in 2^{\mathbb{N}}$ defined by $f(n) = \bigcup_{n \in \mathbb{N}} F(n) = (F(n))(n)$ would satisfy both $\psi(f)$ and $\neg \psi(f)$. The restriction $\prec_{\theta_0, \theta_1}$ of *Kleene Brouwer ordering* to T_{θ_0, θ_1} is defined as follows: For all $x, y \in T_{\theta_0, \theta_1}$,

$$x \prec_{\theta_0, \theta_1} y \iff x \subseteq y \vee \exists j < \min(|x|, |y|) (x(j) < y(j) \wedge \forall i < j (x(i) = y(i))).$$

Then ACA_0 proves that $\prec_{\theta_0, \theta_1}$ is a well ordering (cf. [8, Lemma V.1.3]).

Theorem 6.5 $\Pi_1^1\text{-TR}_0$ proves $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$.

Proof. Let $\varphi(f)$ be a game in the Cantor space such that $\forall f \in 2^{\mathbb{N}} (\varphi(f) \leftrightarrow (\psi(f) \wedge \eta_0(f)) \vee (\neg \psi(f) \wedge \eta_1(f)))$ for some Σ_2^0 formulae $\psi(f)$ and $\eta_1(f)$ and Π_2^0 formula $\eta_0(f)$. Assume that $\forall f (\psi(f) \leftrightarrow \psi'(f))$ for some Π_2^0 formula $\psi'(f)$. Then we can find Π_0^0 formula θ_0 and θ_1 such that $\forall f \in 2^{\mathbb{N}} ((\psi(f) \leftrightarrow \forall n \exists m > n \theta_0(f[m])) \wedge (\neg \psi(f) \leftrightarrow \forall n \exists m > n \theta_1(f[m])))$. Then take T_{θ_0, θ_1} as mentioned above and define T_0 and T_1 as follows:

$$T_i = \{x \in T_{\theta_0, \theta_1} : x \text{ has no proper extension in } T_{\theta_0, \theta_1} \text{ and } \theta_i(x(|x| - 1))\}.$$

Then, for each $i < 2$, the restriction $\prec_i$ of $\prec_{\theta_0, \theta_1}$ to T_i is also well orderings. Note that, for every $f \in 2^{\mathbb{N}}$, there is $x \in T_0 \cup T_1$ such that $x(|x| - 1) \subseteq f$. For any $x \in T_i$, Π_1^1 transfinite recursion yields

$$W_x^i = \{s \in 2^{<\mathbb{N}} : x(|x| - 1) \subseteq s \wedge (|s| \text{ is even}) \wedge (\text{player I wins } (\eta_i(f) \vee \exists n(f[n] \in \bigcup_{y \prec_i x} W_y^i)) \text{ at } s)\}.$$

Claim 1 Let $x \in T_i$. Then, for every $f \in 2^{\mathbb{N}}$, $\forall n \exists m > n \theta_i((x(|x| - 1) * f)[m])$.

Proof of Claim 1 Let $x \in T_i$. We prove that, for any $s \in 2^{<\mathbb{N}}$, $\neg \theta_{1-i}(x(|x| - 1) * s)$, which implies for any $f \in 2^{\mathbb{N}}$, $\neg \forall n \exists m > n \theta_{1-i}((x(|x| - 1) * f)[m])$ and so $\forall n \exists m > n \theta_0((x(|x| - 1) * f)[m])$. Suppose that $\theta_{1-i}(x(|x| - 1) * s)$ for some $s \in 2^{<\mathbb{N}}$. Then $x * \langle s \rangle \in T_{\theta_0, \theta_1}$, which contradicts to the fact that x has no proper extension in T_{θ_0, θ_1} .

Claim 2 For each even-length $s \in 2^{<\mathbb{N}}$ with $\exists x \in T_i(x(|x| - 1) \subseteq s)$, s is in $\bigcup_{x \in T_i} W_x^i$ if and only if player I wins $\eta_i(f)$ at s .

Proof of Claim 2 The “if” part is clear. For the converse, by Σ_1^1 (or Π_1^1) transfinite induction along T_i , we prove $\forall x \in T_i(\forall s \in W_x^i(\text{player I wins } \eta_i(f) \text{ at } s))$. By 3 and 4 of Remark 4.4, “player I wins $\eta_i(f)$ at s ” is equivalent to a Σ_1^1 formula or a Π_1^1 formula and, if $i = 0$, the underlined part is equivalent to a Σ_1^1 formula by Σ_1^1 axiom of choice. Assume that, for any $y \prec_i x$, for any $t \in W_y^i$, player I wins $\eta_i(f)$ at t . Since $\eta_i(f)$ is Π_2^0 or Σ_2^0 , it is determinate in ATR_0 . Suppose, for contradiction, that player II has a winning strategy ν in $\eta_i(s * f)$ for some $s \in W_x^i$. Let v be a winning strategy for player I in $\eta_i(s * f) \vee \exists n(s * f)[n] \in W_{\prec_i x}^i$. Then there exists n such that $s * (v \otimes \nu)[n] \in W_{\prec_i x}^i$, for otherwise $\eta_i(s * (v \otimes \nu))$ would hold and player II would lose $\eta_i(s * f)$. Take such least n . By inductive hypothesis, player I has a winning strategy v' in $\eta_i(s * (v \otimes \nu)[n] * f)$. Then define a new strategy ρ for player I in $\eta_i(s * f)$ by

$$\rho(u) = \begin{cases} v'(u \oplus n) & \text{if } (v \otimes \nu)[n] \subseteq u, \\ v(u) & \text{otherwise.} \end{cases}$$

Then $(\rho \otimes \nu)[n] = (v \otimes \nu)[n]$ and so $\eta_i(s * (\rho \otimes \nu))$ holds, which contradicts the assumption that ν is a winning strategy for player II in $\eta_i(s * f)$.

Then, define a game $\varphi^*(f) \equiv \exists n(f[n] \in \bigcup_{x \in T_0} W_x^0 \cup \bigcup_{y \in T_1} W_y^1)$. We prove the following claim, which completes the proof of this theorem.

Claim 3 The player who wins $\varphi^*(f)$ also wins $\varphi(f)$.

Proof of Claim 3 First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. We prove that player I has a winning strategy in the game $\varphi_0(f) \equiv \exists n(f[n] \in \bigcup_{x \in T_0} W_x^0 \cup \bigcup_{y \in T_1} W_y^1) \wedge (\exists n(f[n] \in \bigcup_{y \in T_1} W_y^1) \rightarrow \eta_1(f))$, which is determined in $\Pi_1^1\text{-TR}_0$. Suppose, for contradiction, that player II has a winning strategy ν' in $\varphi_0(f)$. Then there exists n with $(\sigma^* \otimes \nu')[n] \in \bigcup_{y \in T_1} W_y^1$, for otherwise $(\sigma^* \otimes \nu')[m] \in \bigcup_{x \in T_0} W_x^0$ would hold for some m and player II would lose $\varphi_0(f)$. Take the least such n . By Claim 2 for $i = 1$, player I has a winning strategy v' in $\eta_1((\sigma^* \otimes \nu')[n] * f)$. Then we define a new strategy ρ' for player I in $\varphi_0(f)$ as follows:

$$\rho'(u) = \begin{cases} v'(u \oplus n) & \text{if } (\sigma^* \otimes \nu')[n] \subseteq u, \\ \sigma^*(u) & \text{otherwise.} \end{cases}$$

Then $(\rho' \otimes \nu')[n] = (\sigma^* \otimes \nu')[n] \in \bigcup_{y \in T_1} W_y^1$ and so $\eta_1(\rho' \otimes \nu')$ holds, which contradicts the assumption that ν' is a winning strategy for player II in $\varphi_0(f)$. By $\Sigma_2^0\text{-Det}^*$, player I has a winning strategy σ^{**} in $\varphi_0(f)$.

By 3 of Remark 4.4 and Lemma 4.3, we have a sequence $\langle \sigma_s : s \in \bigcup_{x \in T_0} W_x \rangle$ of winning strategies for player I in $\eta_0(s * f)$. Then define σ by

$$\sigma(s) = \begin{cases} \sigma_t(s \ominus |t|) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in \bigcup_{x \in T_0} W_x^0, \\ \sigma^{**}(s) & \text{if } \forall t \subseteq s (t \notin \bigcup_{x \in T_0} W_x^0). \end{cases}$$

We can see that σ is a winning strategy for player I in $\varphi(f)$ as follows.

Since σ^{**} is a winning strategy for player I in $\varphi_0(f)$, he can reach, following σ , $f[k] \in (\bigcup_{x \in T_0} W_x^0 \cup \bigcup_{y \in T_1} W_y^1)$. Take the least such k .

If $f[k] \in \bigcup_{x \in T_0} W_x^0$, then $f \ominus k$ is a resulting play of $\sigma_{f[k]}$, and so $\eta_0(f)$ holds. By Claim 1 and the fact $f[k] \in \bigcup_{x \in T_0} W_x^0$, f also satisfies $\forall n \exists m > n \theta_0(f[m])$. Therefore player I wins $\varphi(f)$.

If $f[k] \in \bigcup_{y \in T_1} W_y^1$, then f is a resulting play of σ^{**} , and so $\eta_1(f)$ holds. Again by Claim 1 and the fact $f[k] \in \bigcup_{y \in T_1} W_y^1$, f also satisfies $\forall n \exists m > n \theta_1(f[m])$. Therefore player I wins $\varphi(f)$.

Next assume that player II has a winning strategy τ^* in the game $\varphi^*(f)$. Define $\widetilde{W}^0$ and $\widetilde{W}^1$ by

$$\widetilde{W}^i = \{s \in 2^{<\mathbb{N}} : (|s| \text{ is even}) \wedge \exists x \in T_i (x(|x| - 1) \subseteq s) \wedge s \notin \bigcup_{y \in T_i} W_y^i\}.$$

For each resulting play f of τ^* , take the least even $n \in \mathbb{N}$ and $x \in T_0 \cup T_1$ with $x(|x| - 1) \subseteq f[n]$. By the assumption for τ^* , $f[n] \in \widetilde{W}^0 \cup \widetilde{W}^1$. Note that, player II wins $\eta_0(f)$ at each $s \in \widetilde{W}^0$ and $\eta_1(f)$ at each $s \in \widetilde{W}^1$ by $\Sigma_2^0\text{-Det}^*$, which is provable in ATR_0 . Similarly to the above case, we can show that player II has a winning strategy τ^{**} in $\neg(\exists n(f[n] \in \widetilde{W}^0 \cup \widetilde{W}^1) \wedge (\exists n(f[n] \in \widetilde{W}^0) \rightarrow \neg\eta_0(f)))$. Let $\langle \tau_s : s \in \widetilde{W}^1 \rangle$ be a sequence of winning strategies for player II in $\eta_1(s * f)$ yielded by 1 of Remark 4.4 and Lemma 4.3. Then strategy τ for player II defined by

$$\tau(s) = \begin{cases} \tau_t(s \ominus |t|) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in \widetilde{W}^1, \\ \tau^{**}(s) & \text{if } \forall t \subseteq s (t \notin \widetilde{W}^1). \end{cases}$$

is a winning strategy for player II in $\varphi(f)$. $\square$

Corollary 6.6 $\text{RCA}_0^* \vdash \text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^* \leftrightarrow \Delta_2^0\text{-Det} \leftrightarrow \Pi_1^1\text{-TR}_0$.

Proof. The implication from $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ to $\Delta_2^0\text{-Det}$ is proved in Theorem 6.3. The implication from Π_1^1 transfinite recursion to $\text{Sep}(\Delta_2^0, \Sigma_2^0)\text{-Det}^*$ is proved in Theorem 6.5. The equivalence between $\Delta_2^0\text{-Det}$ and $\Pi_1^1\text{-TR}_0$ is proved in [11]. $\square$

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