Discrete Geometry on Red and Blue Points on the Plane Lattice

Mikio Kano
Department of Computer and Information Sciences
Ibaraki University, Hitachi, Ibaraki 316-8511 Japan
e-mail: kano@mx.ibaraki.ac.jp
URL: http://gorogoro.cis.ibaraki.ac.jp

1 Extended abstract

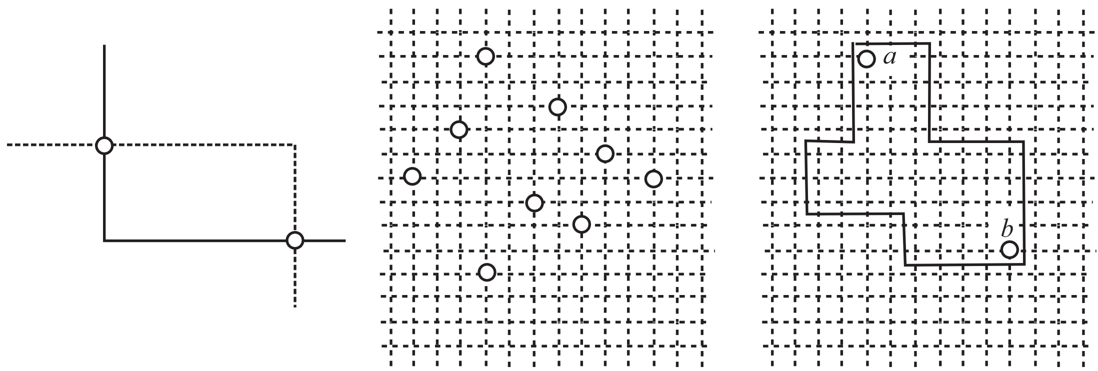
In this talk, we consider some problems on discrete geometry on the plane lattice Z^2 in the plane R^2 motivated by some results in the plane. For any point x in the plane, an L shaped line consisting of one vertical half-line and one horizontal half-line emanating from x is called an *L-line* with *corner x*. Notice that for every point x, there are exactly four L-lines with center x. A vertical line and a horizontal line are considered as a special L-line (with corner ∞).

We regard L-lines as "lines" on the plane lattice, and consider some problems from this point of view. For example, for any two points on the plane lattice, which do not lie on the same vertical or horizontal line, there are two L-lines passing through these two points. On the other hand, for any two points in the plane, there exists exactly one line that passes through these two points. So there is a difference between lines in the plane and L-lines on the plane lattice. However, as we shall show, they have some nice common properties.

Remark 1. Let S be a set of points on the plane lattice. Usually, S is defined to be in general position if for every vertical or horizontal line ℓ, ℓ contains at most one point of S. By using L-lines, S is defined to be in general position if no three points of S lie on the same L-line. These two definitions are slightly different as shown in Figure 1, but they are almost the same for many point sets S since the difference is small.

Remark 2. Let S be a set of points on the plane lattice. Usually a set S is defined to be convex if for every vertical or horizontal line ℓ, $S \cap \ell$ consists of at most one set of vertical or horizontal consecutive points. By using L-lines, we can define S to be convex if for any two points in S, there is at least one L-line that connects these two points and is contained in S. These two definitions are fairly different, and we can easily show that if S is convex in the latter sense, then S is convex in the former sense.

In order to avoid confusion, we use "being in general position" or "convex" by usual definitions.



(a) Two L-lines. (b) A set S of points on the plane lattice. (c) A convex set T in the plane lattice.

Figure 1: (a) Two L-lines passing through two given points. (b) A set of points on the plane lattice in general position by new definition. (c) A convex set by old definition, which contains no L-line connecting two points a and b.

It is well-known that if n red points and n blue points are given in the plane in general position, then there exists a non-crossing perfect matching joining the red points and the blue points, which consists of n disjoint line segments. We start with a similar result on the plane lattice using L-lines.

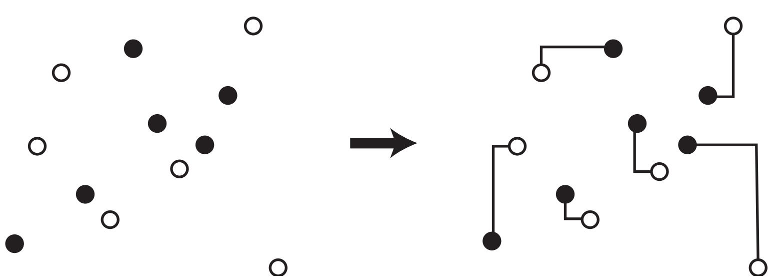


Figure 2: A non-crossing perfect matching joining red points and blue points that consists of n disjoint L-lines.

Theorem 1 ([4]) *Suppose that n red points and n blue points are given on the plane lattice in general position, where $n \geq 1$ is an integer. Then there exists a non-crossing perfect matching joining the red points and the blue points that consists of n disjoint L-lines (Figure 2).*

This theorem can be proved in the same way as the old theorem by using induction.

We now turn our attention to another well-known theorem so called Ham-sandwich Theorem, which says that if $2m$ red points and $2n$ blue points are given in the plane in general position, then there exists a line that bisects both red points and blue points.

Theorem 2 ([3], [5]) *Let $m \geq 1$ and $n \geq 1$ be integers. If $2m$ red points and $2n$ blue points are given on the plane lattice in general position, then there exists an L-line that bisects both red points and blue points (Figure 3).*

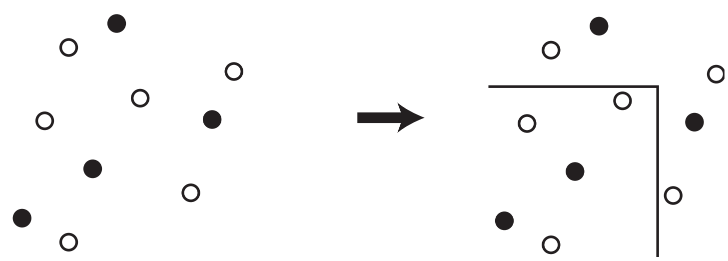


Figure 3: An L-line that bisects both red points and blue points.

Theorem 3 (Bereg, [1]) *Suppose that kn red and km blue points are given in general position on the plane lattice, where $m \geq 1$, $n \geq 1$ and $k \geq 2$ are integers. Then there exists a subdivision of the plane into k regions with at most $k-1$ horizontal line segments and at most $k-1$ vertical line segments such that every region contains precisely n red points and m blue points.*

For a set X of points in the plane, we can draw a spanning trees on X such that each edge is a line segment joining two points of X and no two line segments intersect. This graph is called a *geometric spanning tree* on X and denoted by $tree(X)$. When a set R of red points and a set B of blue points are given in the plane in general position, the minimum number of crossings of $tree(R)$ and $tree(B)$ is given in the next theorem.

Theorem 4 (Tokunaga [2]) *Let $\tau(R, B)$ denote the number of unordered pairs $\{x, y\}$ of vertices of $conv(R \cup B)$ such that one of $\{x, y\}$ is red and the other is blue, and xy is an edge of $conv(R \cup B)$. Then $\tau(B, R)$ is an even number, and the minimum number of crossings in $tree(R) \cup tree(B)$ among all pairs (tree(R), tree(B)) is equal to*

$$\max\left\{\frac{\tau(R, B) - 2}{2},\ 0\right\}.$$

In particular, we can draw red and blue geometric spanning trees without crossings if and only if $\tau(B, R) \leq 2$.

We consider a similar problem on the plane lattice. For a set X of points on the plane lattice in general position, we can draw a *geometric Hamilton path* $path(X)$ on X such that each edge is an L-line connecting two points of X and no two L-lines intersect. When we consider a set R of red points and a set B of blue points on the plane lattice, $conv(R \cup B)$ which is the minimum rectangular containing $R \cup B$ satisfies $\tau(R, B) \leq 4$, and the following theorem holds.

Theorem 5 ([4]) *Let R and B be two disjoint sets of red points and blue points on the plane lattice in general position. Then the minimum number of crossings in $path(R) \cup path(B)$ is 0 if $\tau(R, B) \leq 2$, and 1 otherwise (Figure 4)*

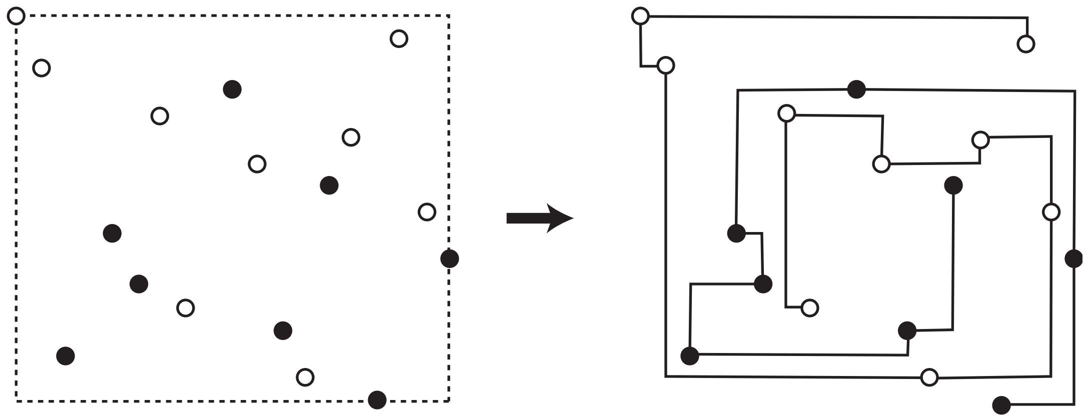


Figure 4: The $conv(R \cup B)$ and a non-crossing geometric Hamilton paths $path(R)$ and $path(B)$.

We propose some problems and other related results.

References

[1] S. Bereg, Orthogonal equipartitions. *Comput. Geom.* bf 42 (2009), 305–314.

[2] S. Tokunaga, Intersection number of two connected geometric graphs. *Inform. Process. Lett.* **59** (1996), 331–333.

[3] M. Kano, T. Kawano and M. Uno, Balanced subdivision of two sets of points in the plane 16 lattice. *In The Kyoto International Conference on Computational Geometry and Graph Theory*, 2007.

[4] M. Kano and K. Suzuki, in preparation.

[5] M. Uno, T. Kawano and M. Kano, Bisections of two sets of points in the plane lattice, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* **E92-A** (2009), 502–507.