Critical independent sets and König–Egerváry graphs

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Abstract

Let $\mathrm{Ind}(G)$ be the family of all independent sets of graph G and $\alpha(G)$ be the cardinality of an independent set of maximum size, while the set $\mathrm{core}(G)$ is the intersection of all maximum independent sets. An independent set A is called *critical* if $|A| - |N(A)| = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$. Let $\mu(G)$ be the size of a maximum matching, and $def(G) = |V(G)| - 2\mu(G)$ be the *deficiency* of G.

Our main finding is the following series of equalities

$$def(G) = \alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$$

holding for every König–Egerváry graph G.

1 Introduction

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$. A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the set of all the independent sets of G. The *independence number* of G is $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$. Let us denote the set $\{S : S \textit{ is a maximum independent set of } G\}$ by $\Omega(G)$, and let $\mathrm{core}(G) = \cap\{S : S \in \Omega(G)\}$ [5]. The *matching number* $\mu(G)$ is the cardinality of a maximum matching of G. If $\alpha(G) + \mu(G) = |V|$, then G is called a *König-Egerváry graph* [3, 8]. The number $def(G) = |V(G)| - 2\mu(G)$ is known as the *deficiency* of G.

The number $d(G) = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$ is called the *critical difference* of G. An independent set A is *critical* if $|A| - |N(A)| = d(G)$. The *critical independence number* $\alpha_c(G)$ is the cardinality of a maximum critical independent set [9]. Clearly, $\alpha_c(G) \leq \alpha(G)$. In [4] it was shown that G is a König-Egerváry graph if and only if $\alpha_c(G) = \alpha(G)$, thus giving a positive answer to the Graffiti.pc 329 conjecture [2].

In this paper we give a new characterization of König-Egerváry graphs claiming that G is a König-Egerváry graph if and only if each of its maximum independent sets is critical. On the one hand, it is similar in form to Sterboul's theorem [8]. On the other hand it extends Larson's finding [4].

2 Results

We give an alternative proof of the following.

Claim 2.1 *[1] Every critical independent set is included in a maximum independent set.*

Theorem 2.2 *If G is a König-Egerváry graph, then*
(i) *[6] $G - N[core(G)]$ has a perfect matching and it is also a König-Egerváry graph.*
(ii) *[6] $N(\text{core}(G)) = \cap \{V(G) - S : S \in \Omega(G)\}$.*
(iii) *[7] $\alpha(G) + |\cap \{V(G) - S : S \in \Omega(G)\}| = \mu(G) + |\cap \{S : S \in \Omega(G)\}|$.*

Using Theorem 2.2, we deduce our main findings.

Theorem 2.3 *If G is König-Egerváry graph, then*

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G).$$

Let us notice that for non-König-Egerváry graphs every relation between $\alpha(G) - \mu(G)$ and $|\text{core}(G)| - |N(\text{core}(G))|$ is possible.

Theorem 2.4 *The following assertions are equivalent:*
(i) *G is a König-Egerváry graph;*
(ii) *there is $S \in \Omega(G)$, such that S is critical, i.e., $\alpha_c(G) = \alpha(G)$;*
(iii) *every $S \in \Omega(G)$ is critical.*

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