Witness (Delaunay) Graphs

Boris Aronov* Muriel Dulieu* Ferran Hurtado†

Introduction. *Proximity graphs* are used in several areas in which a neighborliness relationship for input data sets is a useful tool in their analysis and use, see [6] for a survey. Examples of such areas are computer vision, geographic analysis, pattern classification, computational morphology, and spatial analysis. On the other hand, proximity graphs have also received substantial attention from the graph drawing community, as they provide a natural way of implicitly representing graphs; a survey of such results appeared in [3] and has been extended and updated in [7]. As a tool for graph representation, proximity graphs have some limitations that may be overcome with suitable generalizations. Here we introduce a generalization that encompasses both the goal of more power and flexibility for graph drawing issues and a wider spectrum for neighborhood analysis.

A *witness graph* $G = (V, E)$ is defined by a quadruple $(P, S, W, \pm)$ in which $P = V$ is the set of *vertex points* (or just *vertices*), S provides the geometric shapes, and W is a second point set, consisting of the *witness points* (or just *witnesses*). In the *positive witness* version $(+)$, the tentative adjacency between a and b is accepted if and only if a witness point is covered by at least one of the regions of influence defined by a and b. In the *negative witness* version $(-)$, a witness inside the interaction region would destroy the tentative adjacency, hence there is an adjacency between a and b if at least one of their regions of influence is free of any witness. Notice that in both cases we only pay attention to the presence of witnesses, not of points from P, in the regions of influence. In a third variation one may admit the presence of both negative and positive witnesses and use a combined decision rule; we do not pursue this possibility here. In the present paper, we consider two concrete examples, both related to Delaunay graphs, one for positive witnesses and one for negative ones. Other witness graphs such as the *witness Gabriel graph* and the *witness rectangle-of-influence graph* are studied in the companion papers [1,2]. A systematic study is developed in [5].

Witness Delaunay Graphs. We define the *witness Delaunay graph* of a point set P of *vertices* in the plane, with respect to a point set W of *witnesses*, denoted $\mathrm{DG}^-(P, W)$, as the graph with vertex set P in which two points $x, y \in P$ are adjacent when there is an open disk whose boundary passes through x and y and that does not contain any witness $w \in W$. It is a negative-witness graph in which the shapes are all the disks in the plane whose boundary contains two points from P. When $W = \varnothing$ the graph $\mathrm{DG}^-(P, \varnothing)$ is simply the complete graph $K_{|P|}$. When $W = P$ the graph $\mathrm{DG}^-(P, P)$ is precisely the Delaunay graph $\mathrm{DG}(P)$, which under standard non-degeneracy assumptions is a triangulation and is denoted $\mathrm{DT}(P)$. The latter example illustrates the fact that the use of a witness set gives a generalization of the basic Delaunay structure.

Proposition 1. *Let P and W be two point sets in the plane, and $n := |P| + |W|$. The witness Delaunay graph $\mathrm{DG}^-(P, W)$ can be computed in $O(n^2)$ time, which is worst-case optimal.*

Theorem 1. *Let P and W be two point sets in the plane, and $n := |P| + |W|$. The witness Delaunay graph $\mathrm{DG}^-(P, W)$ can be computed in time $O(k \log n + n \log^2 n)$, where k is the number of edges in the graph.*

Every graph that is realizable as a Delaunay graph $\mathrm{DG}(P)$ is also a witness Delaunay graph, because $\mathrm{DG}^-(P, P) = \mathrm{DG}(P)$. In particular, all maximal outerplanar graphs are realizable, as proved by Dillencourt in [4]. We provide additional results when witnesses are considered.

*Polytechnic Institute of NYU, Brooklyn, New York 11201, USA. Part. supp. by a grant from the U.S.- Israel Binational Sci. Found., by NSA MSP Grant H98230-06-1-0016, and NSF Grant CCF-08-30691. aronov@poly.edu,mdulieu@gmail.com

†Dept. Matem. Aplic. II, Universitat Politècnica de Catalunya, Barcelona, Spain. Part. supp. by projects MEC MTM2006-01267 and MTM2009-07242, and Gen.Catalunya DGR 2005SGR00692 and 2009SGR1040. ferran.hurtado@upc.edu

Theorem 2. *Every tree can be realized as witness Delaunay graph* $\mathrm{DG}^-(S, W)$ *for suitable point sets* S *and* W*. The realization can be carried out in time linear in the size of the tree, in infinite-precision-arithmetic model of computation.*

Theorem 3. *A non-planar bipartite graph cannot be realized as witness Delaunay graph* $\mathrm{DG}^-(P, W)$*, for any point sets* P *and* W*.*

SQUARE GRAPHS. The *square graph* of a point set P in the plane, with respect to point set W of witnesses, denoted $\mathrm{SG}^+(P, W)$, is the graph with vertex set P, in which two points $x, y \in P$ are adjacent when there is an axis-aligned square with x and y on its boundary whose interior contains some witness point $q \in W$. It is a positive-witness graph in which the shapes are all the axis-aligned squares in the plane whose boundary contains two points from P. Observe that a negative-witness version $\mathrm{SG}^-(P, W)$ of this graph, with $W = P$, would be the standard Delaunay graph for the L_∞ metric, and hence we are studying here the positive-witness–variant of this Delaunay structure.

Theorem 4. *Let* P *and* W *be two point sets in the plane, and* $n := |P| + |W|$*. The square graph* $\mathrm{SG}^+(P, W)$ *can be computed in* $O(k + n \log n)$ *time, which is optimal, where* k *is the number of edges in* $\mathrm{SG}^+(P, W)$*.*

We have obtained a complete characterization of square graphs:

Theorem 5. *A combinatorial graph* $G = (V, E)$ *can be realized as a square graph* $\mathrm{SG}^+(P, W)$ *for suitable point sets* P *and* W *in the plane if and only if the complement of* G*,* G'*, is the disjoint union of comparability graphs of dimension 2. Moreover, any square graph can be realized using at most one witness.*

Corollary 6. *One can decide in* $O(|V|^2)$ *time whether a given combinatorial graph* $G = (V, E)$ *can be realized as a square graph* $\mathrm{SG}^+(P, W)$ *for some point sets* P, W *in the plane.*

STABBING OBJECTS. The preceding results give rise to natural formulations of problems on *stabbing* (or *piercing*) sets of geometric objects, using relatively few points.

Let P be a set of n points. Let $\mathcal{D}$ be the set of disks whose boundary contains at least two points from P and let $\mathcal{S}$ be the set of isothetic squares with at least two points of P on its boundary. A point set W *stabs* $\mathcal{D}$ (resp. $\mathcal{S}$) if every disk (resp. square) with two points from P on its boundary contains some point from W in its interior. The size of the smallest stabbing set is denoted by $st_{\mathcal{D}}(P)$ (resp. $st_{\mathcal{S}}(P)$). These numbers, and its extremal values, can be expressed in terms of witness Delaunay graphs for the metric L_2 (resp. L_∞), and we have obtained lower and upper bounds for them:

$$st_{\mathcal{D}}(n) = \max_{|P|=n} st_{\mathcal{D}}(P) = \max_{|P|=n} \min\{|W| : \mathrm{DG}^-(P, W) = \varnothing\},$$

$$st_{\mathcal{S}}(n) = \max_{|P|=n} st_{\mathcal{S}}(P) = \max_{|P|=n} \min\{|W| : \mathrm{SG}^-(P, W) = \varnothing\}.$$

Theorem 7. *For* $n \geq 2$*, the function* $st_{\mathcal{D}}(n)$ *satisfies* $n \leq st_{\mathcal{D}}(n) \leq 2n - 2$*.*

Theorem 8. *The function* $st_{\mathcal{S}}(n)$ *satisfies* $\frac{5}{4}n - \Theta(\sqrt{n}) \leq st_{\mathcal{S}}(n) \leq 2n - \Theta(\sqrt{n})$*.*

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