A Combinatorial Approach to the Tanny Sequence

Anita Das
LNM Institute of Information Technology
Jaipur - 302 031, India.
E-mail: anita@lnmiit.ac.in

Abstract

A fibonacci sequence is a sequence of numbers defined by the recurrence relation $a(n) = a(n-1) + a(n-2)$ with $a(0) = 1$ and $a(1) = 1$. In [3], Hofstadter defined the sequence $Q(n)$ by

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), \quad n > 2,$$

with $Q(1) = Q(2) = 1$. He remarked on the apparent parallel between $Q(n)$ and the usual fibonacci recursion as follows: "each new value is a sum of two previous values-but not of the immediately two values". Guy [2] reports that Malm called $Q(n)$ a 'meta-fibonacci sequence', motivated by the above observation.

A meta-fibonacci sequence is given by the recurrence $a(n) = a(x_1(n) + a'_1(n-1)) + a(x_2(n) + a'_2(n-2))$, where $x_i : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$, $i = 1, 2$, is a linear function of n and $a'_i(j) = a(j)$ or $a'_i(j) = -a(j)$, for $i = 1, 2$. Hofstadter [3] had introduced the interesting concept of meta-fibonacci sequences in his famous book "Gödel, Escher, Bach. An Eternal Golden Braid". The sequence he introduced is known as the Hofstadter sequence and most of the problems he raised regarding this sequence is still open. Since then mathematicians studied many other closely related meta-fibonacci sequences such as Tanny sequences, Conway sequences, Conolly sequences etc.

In [5], Tanny defined a sequence recursively as $T(n) = T(n-1-T(n-1)) + T(n-2-T(n-2))$, $T(0) = T(1) = T(2) = 1$. This sequence is known as Tanny sequence which is a close relative of the Hofstadter sequence. He showed that in sharp contrast to the 'chaotic' behavior of $Q(n)$, $T(n)$ behaves in a completely predictable fashion. Jackson and Ruskey [4] were the first to give a combinatorial way to interpret some special meta-fibonacci sequences.

In the first part of this paper we give combinatorial proofs for all the results regarding $T(i)$, that Tanny proved in his paper using algebraic means. (We use the combinatorial interpretation of $T(i)$ from Bharadwaj et. al. for this purpose.) In most cases our proofs turn out to be simpler and shorter. Moreover, they give a "visual" appeal to the theory developed by Tanny. We also generalize most of Tanny's results. In the second part of the paper we present many new results regarding $T(i)$ and prove them combinatorially.

The most interesting question about Tanny's sequence is to get a formula for $T(i)$. But it seems to be very difficult to get. It is proved in [1] that for any i,

$T(i) \leq \frac{i+1}{2}$ and in fact for any i, $\frac{i+1}{2} - T(i) = \frac{j}{2}$, for some j, $0 \leq j \leq \lceil \log i \rceil - 1$. For a given value of j, $0 \leq j \leq d-1$, it is interesting to characterize the values of i for which $\frac{i+1}{2} - T(i) = \frac{j}{2}$. Let S_j represent the i's such that $\frac{i+1}{2} - T(i) = \frac{j}{2}$ for a given j. We characterize the values of i such that $i \in S_j$ for a given value of j. The following theorem gives a characterization.

Theorem 0.1. *Let* $S_j = \{i : \frac{i+1}{2} - T(i) = \frac{j}{2}\}$. *For a given* j, $i \in S_j$ *if and only if there exist integers* $d, t_1, t_2, \ldots, t_j$ *such that* $d-2 \geq t_1 > t_2 > \ldots > t_{j-1} \geq t_j \geq 1$ *and* $i = 2^d - (2^{t_1} + 2^{t_2} + \ldots + 2^{t_j}) + (j-1)$. *Moreover,* $|\{i \in S_j : 2^{d-1} \leq i \leq 2^d - 1\}| = \binom{d}{j+1}$.

Though S_j is an infinite set, the distribution of the elements of S_j over the integers is nice. As shown in Theorem 0.1, $|\{i \in S_j : 2^{d-1} \leq i \leq 2^d - 1\}| = \binom{d}{j+1}$. The following theorem allows us to get the number of integers $i \in S_j$ that are strictly below 2^d.

Theorem 0.2. $|\{i \in S_j : i < 2^d\}| = \binom{d+1}{j+1}$.

Given two integers i and k it is interesting to know whether $T(i) = k$ or not. In [5], Tanny has already given a nice solution to this question (see Proposition 2.6 in [5]). In the following theorem we give a different way of looking at it. The answer to the question whether $T(i) = k$ is encoded in the bit pattern of the binary representation of $2^{\lceil \log i \rceil} - 2k$.

In the following theorem we characterize such numbers.

Theorem 0.3. *Let* i *and* k *be two given numbers and* $x = 2^d - 2k$, *where* $d = \lceil \log i \rceil$ *and let* t *be the number of* 1*'s in the bit representation of* x. *Let* t' *be the greatest integer such that* $2^{t'}$ *divides* x. *Then,* $T(i) = k$ *if and only if* $t' \geq i + 1 - 2k - t$.

References

[1] B. V. S. Bharadwaj, L. Sunil Chandran, Anita Das, *Isoperimetric Problem and Meta-Fibonacci Sequences*, In the Proceeding of 14th Annual International Computing amd Combinatorics Conference 2008, Dalian, China. LNCS 5092, pp. 22–30.

[2] R. K. Guy, *Some suspiciously simple sequences*, Amer. Math. Monthly, 93 (1986), pp. 186–190.

[3] D. Hofstadter, *Gödel, Escher, Bach. An eternal golden braid*, (Basic Books, New Work, 1979).

[4] B. Jackson, F. Ruskey, *Meta-Fibonacci sequences, binary trees and extremal compact codes*, Electron. J. Combin. 13 (2006), no. 1, Research Paper 26, 13 pp. (electronic).

[5] S. M. Tanny, *A well-behaved cousin of the Hofstadter sequence*, Discrete Mathematics, 105 (1992), pp. 227–239.