Classification of the congruent embeddings of a tetrahedron into a triangular prism

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Problems related to embedding or inscribing simplices into circular cylinders are considered by many authors, mostly to study the outer j-radii of simplices, or to compute cylinders through the vertices of a simplex. See, e.g., Brandenberg, et al. [2, 3], Devillers, et al. [4], Pukhov [7], Schömer, et al. [8]. Maehara [6] considers embedding itself, and proved that all congruent embeddings of a regular tetrahedron in a circular cylinder are equivalent modulo rigid motions within the cylinder.

In this paper, we classify the congruent embeddings of a regular tetrahedron in a right prism whose base is an equilateral triangle. This study arouse from the investigation of the minimum size of an equilateral triangular hole in a plane through which a regular tetrahedron of unit edge can pass.

A regular tetrahedron with unit edge is simply called a *unit tetrahedron*. A right triangular prism $\mathbf{P} = \Delta \times \mathbb{R}$ with equilateral triangular base Δ is called simply a *prism*. The *size* of a prism $\mathbf{P}$ is represented by the length of the edge of Δ, and $\mathbf{P}(t)$ stands for a prism of size t. An *embedding* of a unit tetrahedron in $\mathbf{P}$ means such a subset of $\mathbf{P}$ that is congruent to a unit tetrahedron. Two embeddings $T_1, T_2 \subset \mathbf{P}$ of a unit tetrahedra in $\mathbf{P}$ are said to be *equivalent* (written as $T_1 \sim T_2$ in $\mathbf{P}$) if it is possible to superpose T_1 on T_2 by a continuous rigid motion of T_1 within $\mathbf{P}$. More precisely, $T_1 \sim T_2$ in $\mathbf{P}$ if there is a continuous map $F : T_1 \times [0,1] \to \mathbf{P}$ such that

(1) for every $t \in [0,1]$, the map $f_t : T_1 \to \mathbf{P}$ defined by $f_t(x) = F(x,t)$ gives an isometry from T_1 to $f_t(T_1)$, and

(2) f_0 is the inclusion map, and $f_1(T_1) = T_2$.

The relation $\sim$ in $\mathbf{P}$ is clearly an equivalence relation. Let $\nu(t)$ denote the maximum number of mutually non-equivalent embeddings of T in $\mathbf{P}(t)$. Our result is the following.

Theorem.
$$\nu(t) = \begin{cases} 0 & \text{for} \quad t < t_0 := \frac{1+\sqrt{2}}{\sqrt{6}} \\ 6 & \text{for} \quad t_0 \le t < t_1 := \frac{\sqrt{3}+3\sqrt{2}}{6} \\ 18 & \text{for} \quad t_1 \le t < 1 \\ 1 & \text{for} \quad 1 \le t. \end{cases}$$

Thus, a unit tetrahedron can be embedded in $\mathbf{P}(t)$ if and only if $t \ge \frac{1+\sqrt{2}}{\sqrt{6}}$. This fact is used in [1] to prove that a unit tetrahedron can pass through an equilateral triangular hole in a plane if and only if the edge length of the triangular hole is at least $\frac{1+\sqrt{2}}{\sqrt{6}}$.

Let $\nu_\circ(t)$ denote the number of equivalence classes of the embeddings of a unit tetrahedron into an infinite circular cylinder of diameter t modulo rigid motions within the cylinder. The number $\nu_\circ(t)$ is determined in [6]: $\nu_\circ(t) = 0$ for $r < 1$, and $\nu_\circ(t) = 1$ for $r \ge 1$. Let $\nu_\square(t)$ be the number of equivalence classes of all embeddings of a unit tetrahedron into a square prism whose base is a square with diameter t, modulo rigid motions within the prism. The fact $\nu_\circ(t) = 0$ for $r < 1$ implies that $\nu_\square(t) = 0$ for $t < 1$, see also Itoh, et al. [5].

Problem. Determine $\nu_\square(t)$ for $t \ge 1$.

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