Approximate minimum spanning ellipse in the streaming model

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1 Introduction

In [?] Zarrabi-Zadeh and Chan proposed a very simple streaming algorithm for computing an approximate spanning ball of a set of n points $P = \{p_1, p_2, p_3, ..., p_n\}$ and gave an elegant proof that the radius of the approximate ball is to within 3/2 of the exact one. In their model only the center and radius of the current ball are stored.

In this paper we explore the extension of their algorithm to computing an approximate spanning ellipse of P in the same model. Our main results are:

- A novel algorithm to find a minimum spanning ellipse that spans an ellipse E and a point p outside it.
- Construct an input point sequence to show that the approximation ratio can become unbounded.

Here are the main ingredients of our algorithm.

On the first input point, the approximate minimum ellipse, E_A, is set to be this point. On the second input point, E_A is set to the line segment joining this to the first one. As long as subsequent points are on the supporting line, l, of this segment, E_A continues to be a line segment joining two extreme points on l. On the first input point that is not incident on l, E_A is computed as the non-degenerate ellipse defined by the new point and the endpoints of the current segment. So far, E_A is thus identical with the exact ellipse. For a subsequent input point, p, that does not lie inside E_A, we solve the problem of finding an ellipse of minimum area that spans E_A and p.

We do this by finding an elliptic transformation (nomenclature due to Post [?]), T, that transforms E_A to a unit circle; the same transformation is applied to p. Then, in the transformed plane, we solve the easier problem of finding a minimum spanning ellipse of the unit circle and $T(p)$. Finally, we apply an inverse transformation T^{-1} to the minimum spanning ellipse in the transformed plane to find the desired minimum spanning ellipse in the original plane. That the transformed ellipse in the original plane is of minimum area is due to the fact that the elliptic transformation T preserves relative areas. We will prove this.

2 Finding T

T is required to have the following two necessary properties:

P1 For any ellipse E, both $T(E)$ as well as its inverse image $T^{-1}(E)$ is an ellipse;

P2 T preserves relativity of ellipse areas, so that $\text{area}(E_1) \leq \text{area}(E_2)$ if and only if $\text{area}(T(E_1)) \leq \text{area}(T(E_2))$.

The following lemma shows that a scaling in any direction (in particular the x-axis, say) satisfies $P1$ and $P2$.

Lemma 1. *Let $p = (x, y)$ and $\alpha > 0$. Then* $T(p) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ *is an x-scaling that satisfies $P1$ and $P2$.*

The next lemma provides the necessary transformation.

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Lemma 2. *The transformation $T = T^y(1/r_2)T^x(1/r_1)T^r(-\phi)[(x,y)-(x_0,y_0)]^t$, where T^y is an y-scaling by $1/r_2$, T^x is an x-scaling by $1/r_1$, $T^r(-\phi)$ is a rotation by $-\phi$, and $(x,y)-(x_0,y_0)$ is a translation, takes the ellipse E into a unit circle.*

Proof: Since rotations and translations satisfy P_1 and P_2, and by Lemma 1, both T^x and T^y does as well, T satisfies them too. Also, from the choice of the scalings it is clear that $T(E)$ is a unit circle. □

3 Finding E'_A

Lemma 3. *Let E and $\overline{E}$ be two ellipses that are the reflections of each other in the x-axis. Then the area of the (strict) convex combination $\lambda E + (1-\lambda)\overline{E}$ (where $0 < \lambda < 1$) is smaller than the areas of E and $\overline{E}$.*

Lemma 4. *The smallest ellipse E'_A containing p' and the unit circle will have a principal axis oriented along the x-axis.*

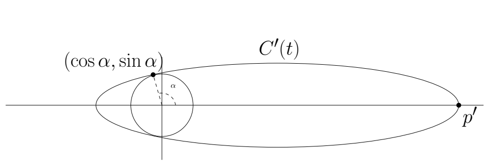

Figure 1: An ellipse containing and tangent to the unit circle, and passing through p'

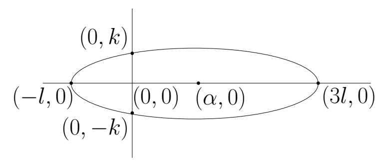

Figure 2: Exact ellipse through $(0,k)$, $(0,-k)$ and $(3l,0)$

4 Area Approximation Ratio

We can construct an example of input sequence for which the approximation ratio of the area of the approximate ellipse to the exact ellipse is unbounded. To construct the example we need the following important lemma.

Lemma 5. *The exact ellipse incident on $(0,k)$, $(0,-k)$ and $(3l,0)$ must also pass through $(-l,0)$. Moreover, the length of the minor axis of this exact ellipse is $\frac{4k}{\sqrt{3}}$.*

Theorem 1. *There exists an input point sequence of points for which the approximation ratio of the ellipse areas becomes unbounded.*

We can prove this theorem by starting with two points $(0,1)$, $(0,-1)$ and then taking a sequence $\{(x_0,0),(x_1,0),(x_2,0),\cdots\}$ where $1 < x_0 < x_2 < x_3 < \cdots$

5 Conclusions

The unboundedness of the ratio of areas was surprising as we believed otherwise in light of [**?**]. Does this diminishe the practical utility of this algorithm for finding approximate ellipses? To check we implemented this algorithm (click on the link **software** `http://cs.uwindsor.ca/~asishm` to view). Experiments with this implementation for randomly input point sets seem to suggest that the "average behaviour" of this algorithm may still be very good. While the ratio of areas is not be bounded in the worst case, we have shown that that there exists a direction in which the ratio of the widths is bounded by 2. We have also extended the results of this paper to d-dimensions.

References

[1] Mark J. Post. A minimum spanning ellipse algorithm. In *FOCS*, pages 115–122. IEEE, 1981.

[2] Hamid Zarrabi-Zadeh and Timothy Chan. A simple streaming algorithm for minimum enclosing balls. In *Proceedings of the 18th Canadian Conference on Computational Geometry (CCCG'06)*, pages 139–142, 2006.