Colorful Strips*

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Hypergraph coloring problems have been studied for a long time, but there is a currently renewed interest in coloring problems on *geometric* hypergraphs, that is, set systems defined by geometric objects. This interest is motivated by applications to wireless and sensor networks [4]. Conflict-free colorings [5], chromatic numbers [8], covering decompositions (see [7, 6, 2] and references therein), or polychromatic (*colorful*) colorings of geometric hypergraphs [3] have been extensively studied in this context.

In this paper, we consider a natural problem in the latter category. We define a *strip* in $\mathbb{R}^d$ as the area enclosed between two parallel axis-aligned hyperplanes. We are interested in k-coloring finite point sets in $\mathbb{R}^d$ so that any strip containing at least some fixed number of points also contains a point of each color. More precisely, we define the function $p(k, d)$ as the minimum number for which there always exists a k-coloring of any point set in $\mathbb{R}^d$ such that every strip containing at least $p(k, d)$ points is polychromatic. This is a particular case of the general framework proposed in [3].

Note that this is in fact a combinatorial problem: an axis-aligned strip isolates a contiguous subsequence of the points in sorted order with respect to one of the axes. Therefore, the only thing that matters is the order in which the points appear along each axis. We can rephrase our problem, considering d-dimensional points sets, as finding the minimum value $p(k, d)$ such that the following holds: For d permutations of a set of items S, it is always possible to color the items with k colors, so that in all d permutations every subsequence of at least $p(k, d)$ contiguous items contains one item of each color.

We also study *circular* permutations, in which the first and the last elements are contiguous. We consider the problem of finding a minimum value $p'(k, d)$ such that, for any d circular permutations of a set of items S, it is possible to k-color the items so that in every permutation, every subsequence of $p'(k, d)$ contiguous items contains all colors.

A restricted geometric version of this problem in $\mathbb{R}^2$ consists of coloring a point set S with respect to wedges. For our purposes, a wedge is any area delimited by two half-lines with common endpoint at one of d given apices. Each apex induces a circular ordering of the points in S. This is illustrated in Figure 1. We wish to color S so that any wedge containing at least $p'(k, d)$ points is polychromatic. In $\mathbb{R}^2$, the non-circular case corresponds to wedges with apices at infinity. In that sense, the wedge coloring problem is more difficult than the strip coloring problem.

Finally, we study a dual problem, in which a set of strips is to be colored, so that sufficiently covered points are contained in strips from all color classes. Let H be a k-colored set of strips in $\mathbb{R}^d$. A point is said to be polychromatic with respect to H if it is contained in strips of all k color classes. We study the function $\overline{p}(k, d)$, which is the minimum number for which there always exists a k-coloring of any set of strips in $\mathbb{R}^d$ such that every point of $\mathbb{R}^d$ contained in at least $\overline{p}(k, d)$ strips is polychromatic. This function was shown to be unbounded if the strips can have arbitrary orientations [6].

*The full version of this paper is available on arXiv [1].

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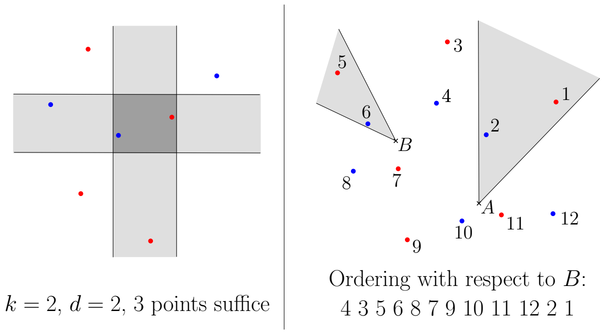

Figure 1: Illustration of the definitions of $p(k,2)$ and $p'(k,2)$. On the left, points are 2-colored so that any axis-aligned strip containing at least three points is bichromatic. On the right, two points A and B define two circular permutations of the point set. In this case, we wish to color the points so that there is no long monochromatic subsequence in either of the two circular orderings.

Our results are summarized in Table 1. The upper bounds on $p(k,2)$ and $p'(k,2)$ are obtained using Vizing's theorem. The upper bounds for larger d are obtained by the *probabilistic method*. We also consider the computational complexity of the coloring problem, and prove that the following problem is *NP*-complete:

Input: 3 permutations π_1, π_2, π_3 of an n-element set S.
Question: Can S be 2-colored so that, in each π_i, every 3 consecutive elements are not monochromatic?

	$p(k,d)$	$p'(k,d)$	$\overline{p}(k,d)$
upper bound	$k(4\ln k + \ln d)$ ($2k{-}1$ for $d{=}2$)	$k(4\ln k + \ln d)$ ($2k$ for $d{=}2$)	$d(k{-}1)+1$
lower bound	$2 \cdot \lfloor \frac{(2d-1)k}{2d} \rfloor - 1$	$2 \cdot \lfloor \frac{(2d-1)k}{2d} \rfloor - 1$	$2 \cdot \lfloor \frac{(2d-1)k}{2d} \rfloor - 1$

Table 1: Bounds on p, p' and $\overline{p}$.

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