Minimal Embedding of Complete Bipartite Graphs on Surfaces

Kazuaki Kobayashi and Takako Kodate
kazuaki@lab.twcu.ac.jp, kodate@lab.twcu.ac.jp
Department of Information and Sciences
Tokyo Woman's Christian University
Tokyo, Japan

1 Introduction and Definition

The paper deals with a problem in embedding graphs on an orientable closed surface. Drawing graphs on plane without crossing their edges is considered to be a "standard form" for planar graphs. On the other hand, it is not clear what the "standard form" for nonplanar graph is.

Various proposals have been made for this problem. For example, Kobayashi proposed a standard spacial graph [2]. This presentation can only fully applied to Hamiltonian graphs. Later, Otsuki showed that a canonical embedding for a complete graph is a suitable "standard embedding" which is a bud presentation (the compactification of the book presentation) with an extra condition [5]. Unfortunately this presentation seems to be difficult to generalize. So, the purpose of this paper is to find a system that can provide the "minimal embedding" of graphs on a surface.
Generally, many embedding patterns exist for one graph on a surface with the genus denoted by $\gamma(G)$. p is the smallest integer, and G can be embedded on the surface of genus p, bearing in mind that a graph with the genus $\gamma(G)$ can be embedded on surfaces of genus p where $p \geqq \gamma(G)$.

We define a value $\omega(G)$ as follows, and we propose minimal embedding by examining this value for complete bipartite graphs. In this paper, we focus on the orientable surface.

Definition 1 *For a graph G with the genus $\gamma(G)$, let $F_{\gamma(G)}$ be a Heegaard surface of S^3 with $\gamma(G)$, $\{l_i, m_i\}(i = 1, 2, ..., \gamma(G))$ be longitude, meridian systems of $F_{\gamma(G)}$. Let $\widehat{f} : G \longrightarrow F_{\gamma(G)} \subset S^3$ be an embedding G in $F_{\gamma(G)}$,*

$$\omega(G) = min_{\widehat{f}}\Big\{\sum_{i=1}^{\gamma(G)} |\widehat{f}(G) \cap l_i| + \sum_{i=1}^{\gamma(G)} |\widehat{f}(G) \cap m_i|\Big\}.$$

Here, we hypothesize that $\{l_i, m_i\}$ and G intersect each other at vertices of G, and allow no intersection of $\{l_i, m_i\}$ and $E(G)$.
This is an embedding $\widehat{f} : G \longrightarrow F_{\gamma(G)}$ which fulfills the condition where $(\gamma(G),\ \omega(G))$ is the minimal embedding of G.

Our purpose is to determine the value of $\omega(G)$ of graph G embedded on surface of $\gamma(G)$. We obtained a following result on n-dimensional hypercubic graph Q_n as $\omega(Q_n) = 8\gamma(Q_n)$[3]. In this paper, we focus on complete bipartite graphs denoted by $K_{m,n}$. $K_{m,n}$ is a graph where the vertices are partitioned into two sets V_1 and V_2 in such a way m vertices in V_1 are adjacent to all n vertices in V_2. It means that $K_{m,n}$ has $m + n$ vertices and $m \times n$ edges. We will show that the lower and upper bounds of $\omega(K_{m,n})$ and in some cases, the lower bound is attained.

Let $G = (V, E)$ be a graph, and ρ be the number of circuits induced by the rotation system (see [1] and [4] for this rotation system). We can determine the genus $\gamma(G)$ of G by Euler's formula, which is $|V(G)| - |E(G)| + \rho = 2 - 2\gamma(G)$.
For the complete bipartite graph$K_{m,n}(m \geqq 2, n \geqq 2)$, Ringel proved that $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$. We also calculate the genus of $K_{p+q-2,n}$ from two smaller complete bipartite graphs inductively as $\gamma(K_{p+q-2,n}) = \gamma(K_{p,n}) + \gamma(K_{q,n})$. This operation is shown in [4]. We call this operation *"$\diamondsuit$(diamond) operation"*.
$\diamondsuit$ operation is an operation for connecting two smaller embedding graphs by adding one handle, deleting two vertices and identifying the other set of vertices. We write this as $K_{p+q-2,n} = K_{p,n} \diamondsuit K_{q,n}$. This operation does not increase the value of the genus.

2 Lower Bound

The *generalized vertex skewness* of G denoted by $gvsk(G)$ is the minimum number of vertices deleted from G to obtain a subgraph G' with genus $\gamma(G') \leqq \gamma(G) - 1$. For a graph G and its genus $\gamma(G)$, the lower bound for minimal embedding is given by Kobayashi. If $\widehat{f} : G \longrightarrow F_{\gamma(G)} \subset S^3$ is a minimal embedding, then we have

$$\omega(G) \geqq 2\gamma(G) \cdot gvsk(G) \tag{1}$$

The proof is given by the definition of $gvsk(G)$, $|\widehat{f}(G) \cap l_i| \geqq gvsk(G)$ and $|\widehat{f}(G) \cap m_i| \geqq gvsk(G)(1 \leqq i \leqq \gamma(G))$.
This lower bound is not tight for certain graphs. Therefore, we propose an additional lower bound.

Proposition 1 *Let $\widehat{f} : G \to F_{\gamma(G)}$ be an embedding. If each closed region of $F_{\gamma(G)}$ partitioned by $\widehat{f}(G)$ does not have a self-attachment (i.e. each closed region is homeomorphic to a closed disk), then we have*

$$\omega(G) \geqq 4\gamma(G). \tag{2}$$

For example, for $K_{3,3}$ that has a self-attachment, $\omega(K_{3,3}) = 2$.

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For the complete bipartite graph $K_{m,n}(m \geqq 3, n \geqq 3)$, the lower bound is obtained by (1) and (2) as follows.

$$\omega(K_{m,n}) \geqq \begin{cases} 2\gamma(K_{m,n}) & \text{if there are self-attachments} \\ 4\gamma(K_{m,n}) & \text{otherwise} \end{cases} \tag{3}$$

The upper bound is obtained inductively from the values of $\omega(K_{3,n})$ and $\omega(K_{6,n})$.

Proposition 2 *For the complete bipartite graph $K_{m,n}(m \geqq 3, n \geqq 3)$,*

$$\omega(K_{m,n}) \leqq \begin{cases} 2\left\lfloor \dfrac{n-1}{2} \right\rfloor & \textit{if } m = 3 \\ \omega(K_{m-1,6}) + \omega(K_{3,6}) & \textit{if } m \geqq 4, n = 6 \\ \omega(K_{m-4,n}) + \omega(K_{6,n}) & \textit{otherwise} \end{cases}$$

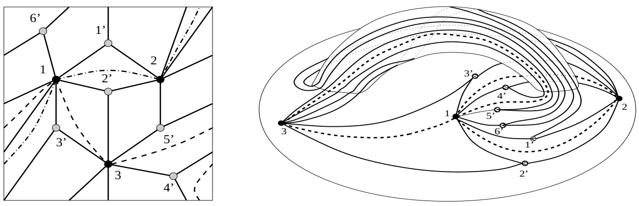

Figure 1: Embedding $K_{3,6}$ on Torus. The longitude and meridian are shown by dotted lines. Thus, $\omega(K_{3,6}) = 4$.

4 Conclusion

We proposed minimal embedding for nonplanar graphs on a surface as the “standard” embedding, and demonstrated that although the complete bipartite graphs would be ideal, they were not possible at this stage.b However, we obtained the result for the hypercubic graphs, in which the lower bound was achieved.

References

[1] Hartsfield, N. and Ringel, G., “Pearls in Graph Theory: A Comprehensive Introduction,” Dover Publications, 2003.

[2] Kobayashi, K., *Standard Spatial Graph*, Hokkaido Math. J. **23** (1992) 117–140.

[3] Kobayashi, K. and Kodate, T., *Minimal Embedding of Hypercubic Graphs on Surfaces*, preprint.

[4] Mohar, B. and Thomassen, C., “Graphs on Surfaces,” The Johns Hopkins University Press, 2001.

[5] Otsuki, T., *Knots and Links in Certain Spatial Complete Graphs*, J. Combin. Theory Ser.B, **68** (1996) 23–35.