New graph polynomials satisfying deletion-contraction relations

Yusuke Watanabe *† Kenji Fukumizu*

1 Introduction and Definitions

In this paper, we introduce two graph polynomials that satisfy contraction-deletion relations and discuss their properties. The motivation for the definition is analysis of performance of the Bethe approximation of the Ising partition function, but we do not focus on it. The most famous graph polynomial that satisfy deletion-contraction relation is the Tutte polynomial. Our polynomials are essentially different from it [1].

First we define the graph polynomial called θ. Beforehand, we prepare a set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ inductively by the relations $f_0(x) = 1, f_1(x) = 0$ and $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$. Therefore, for instance, $f_2(x) = 1, f_3(x) = x$ and so on. Note that these polynomials are transformations of the Chebyshev polynomials of the second kind: $f_{n+2}(2\sqrt{-1}z) = (\sqrt{-1})^n U_n(z)$, where $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$. For a given graph $G = (V, E)$, in which loops and multiple edges are allowed, the graph polynomial θ_G is defined by

$$\theta_G(\beta, \gamma) := \sum_{s \subset E} \beta^{|s|} \prod_{i \in V} f_{d_i(s)}(\gamma) \quad \in \mathbb{Z}[\beta, \gamma],$$

where $d_i(s)$ is the degree of vertex i in the subgraph s. The most important feature of this polynomial is

$$\theta_G(\beta, \gamma) = (1 - \beta)\theta_{G \backslash e}(\beta, \gamma) + \beta\theta_{G/e}(\beta, \gamma) \quad \text{if } e \in E \text{ is not a loop.}$$

Note that the graph $G\backslash e$ is obtained from G by deletion of the edge e, and the graph G/e is the result of contraction of e. This type of relations are called deletion-contraction relations. For the proof of the relation, the equation $f_{n+m-2}(x) = f_n(x)f_m(x) + f_{n-1}(x)f_{m-1}(x)$ plays an essential role.

We define the graph polynomial ω_G by $\omega_G(\beta) := (1-\beta)^{|V|-|E|}\theta_G(\beta, 2\sqrt{-1})$. Since $f_n(2\sqrt{-1}) = (\sqrt{-1})^n(n-1)$, we have

$$\omega_G(\beta) = (1 - \beta)^{|V|-|E|} \sum_{s \subset E} (-\beta)^{|s|} \prod_{i \in V} (1 - d_i(s)).$$

*The Institute of Statistical Mathematics 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, watay@ism.ac.jp, fukumizu@ism.ac.jp

†The Graduate University for Advanced Studies

It is not trivial by definition that $\omega_G(\beta)$ is a polynomial in β, but we can show that it is indeed a polynomial.

2 Properties of ω_G

In the rest of paper, we focus on the polynomial ω_G. The following three properties characterize the graph polynomial ω:

(a) $\omega_{G_1 \cup G_2}(\beta) = \omega_{G_1}(\beta)\omega_{G_2}(\beta)$.

(b) $\omega_G(\beta) = \omega_{G \setminus e}(\beta) + \beta\omega_{G/e}(\beta) \quad$ if $e \in E$ is not a loop.

(c) $\omega_{B_n}(\beta) = 1 + (2n-1)\beta$.

Note that $G_1 \cup G_2$ is the disjoint union of G_1 and G_2. The *bouquet graph* B_n is the graph with a single vertex and n loops.

This polynomial is related to matchings on G. A *matching* of G is a set of edges such that any edges do not occupy a same vertex. The number of edges in a matching $\mathbf{D}$ is denoted by $|\mathbf{D}|$ and the vertices covered by the edges in $\mathbf{D}$ are denoted by $[\mathbf{D}]$. We have

$$\omega_G(\beta) = \sum_{\mathbf{D}} (-\beta)^{|\mathbf{D}|} \prod_{i \in V \setminus [\mathbf{D}]} (1 + (d_i - 1)\beta),$$

where the sum is over all matchings on G and d_i is the degree of vertex i. In other words, ω_G is the monomer-dimer partition function with edge weight $-\beta$ and vertex weights $1 + (d_i - 1)\beta$. This result implies that ω_G is equivalent to the matching polynomial if the graph is regular.

The values at $\beta = 1$ has a combinatorial interpretation. For given graph G, let $G^{(2)} := (V_A \cup V_O, E^{(2)})$ be the graph obtained by adding a vertex on each edge in G, where V_O is the original vertices and V_A is the ones newly added, respectively. This graph is called the incident graph of G. Then we have

$$\omega_G(1) = |\{\mathbf{D} \subset E^{(2)}; \mathbf{D} \text{ is a matching on } G^{(2)}, [\mathbf{D}] \supset V_O\}|.$$

An example is given in Figure 1. For the graph X, we have $\omega_{X_3}(1) = 10$.

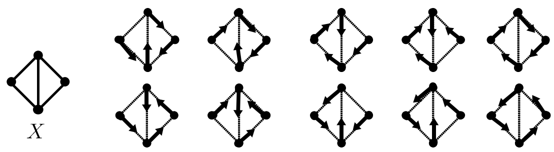

Figure 1: Graph X and configurations.

References

[1] Y. Watanabe and K. Fukumizu. New graph polynomials from the Bethe approximation of the Ising partition function. *arXiv:0908.3850*, 2009.