A closure concept for spanning $k$-tree of $n$-connected graphs

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We consider a connected simple graph $G$ with order $|G|$, and let $k \geq 2$ be an integer. A tree is called a $k$-tree if its maximum degree is at most $k$. In particular, a spanning 2-tree is nothing but a Hamilton path. The following theorems are related to our theorem and give sufficient conditions for a graph to have a spanning $k$-tree.

\textbf{Theorem 1 (Ore [6])} Let $G$ be a connected graph. If every pair of nonadjacent vertices $u$ and $v$ of $G$ satisfies $\deg_G(u) + \deg_G(v) \geq |G| - 1$, then $G$ has a Hamilton path.

\textbf{Theorem 2 (Win [7])} Let $G$ be a connected graph and $k \geq 2$ be an integer. If $\sum_{x \in S} \deg_G(x) \geq |G| - 1$ for every independent set $S$ of $G$ with size $k$, then $G$ has a spanning $k$-tree.

Theorem 1 can be proved by using the closure $\text{cl}(G)$ of $G$ for Hamilton path, which is obtained from $G$ by recursively joining pairs of nonadjacent vertices with degree sum at least $|G| - 1$ until no such pair remains. The closure is useful based on the following theorem.

\textbf{Theorem 3 (Bondy and Chvátal [1])} Let $G$ be a connected graph, and $u$ and $v$ be a pair of nonadjacent vertices of $G$ such that $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then $G$ has a Hamilton path if and only if $G + uv$ has a Hamilton path.

In this paper, we prove the following theorem, by which we can define the closure of a graph for spanning $k$-tree.

\textbf{Theorem 4} Let $k \geq 2$ and $n \geq 1$ be integers. Let $G$ be an $n$-connected graph, and $u$ and $v$ be a pair of nonadjacent vertices of $G$ such that

\[ \deg_G(u) + \deg_G(v) \geq |G| - 1 - (k - 2)n. \]  \hspace{1cm} (1)

Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.
It is shown that the condition (1) on degree sum in Theorem 4 is best possible, that is, there exist $n$-connected graphs $G$ such that $\deg_G(u) + \deg_G(v) = |G| - 2 - (k - 2)n$ and $G + uv$ has a spanning $k$-tree but not $G$.

Notice that the above theorem is a generalization of Theorem 3 since a Hamilton path is a spanning 2-tree, and that for Hamilton path, the connectivity $n$ of a graph does not contribute to the condition (1) on degree sum. The following Corollary 5 is Theorem 4 with $n = 1$, and Corollary 6 is immediate from Theorem 4 since the closure of a graph given in Corollary 6 becomes a complete graph, which has a spanning $k$-tree.

**Corollary 5** Let $k \geq 2$ be an integer and $G$ be a connected graph. Let $u$ and $v$ be a pair of nonadjacent vertices of $G$ such that $\deg_G(u) + \deg_G(v) \geq |G| - k + 1$. Then $G$ has a spanning $k$-tree if and only if $G + uv$ has a spanning $k$-tree.

**Corollary 6** Let $k \geq 2$ and $n \geq 1$ be integers, and $G$ be an $n$-connected graph. If every pair of nonadjacent vertices $u$ and $v$ of $G$ satisfies

$$\deg_G(u) + \deg_G(v) \geq |G| - 1 - (k - 2)n,$$

then $G$ has a spanning $k$-tree.

**References**


