A closure concept for spanning k-tree of n-connected graphs

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We consider a connected simple graph G with order $|G|$, and let $k \geq 2$ be an integer. A tree is called a *k-tree* if its maximum degree is at most k. In particular, a spanning 2-tree is nothing but a Hamilton path. The following theorems are related to our theorem and give sufficient conditions for a graph to have a spanning k-tree.

Theorem 1 (Ore [6]) *Let G be a connected graph. If every pair of nonadjacent vertices u and v of G satisfies $\deg_G(u) + \deg_G(v) \geq |G| - 1$, then G has a Hamilton path.*

Theorem 2 (Win [7]) *Let G be a connected graph and $k \geq 2$ be an integer. If $\sum_{x \in S} \deg_G(x) \geq |G| - 1$ for every independent set S of G with size k, then G has a spanning k-tree.*

Theorem 1 can be proved by using *the closure $cl(G)$* of G for Hamilton path, which is obtained from G by recursively joining pairs of nonadjacent vertices with degree sum at least $|G| - 1$ until no such pair remains. The closure is useful based on the following theorem.

Theorem 3 (Bondy and Chvátal [1]) *Let G be a connected graph, and u and v be a pair of nonadjacent vertices of G such that $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then G has a Hamilton path if and only if $G + uv$ has a Hamilton path.*

In this paper, we prove the following theorem, by which we can define the closure of a graph for spanning k-tree.

Theorem 4 *Let $k \geq 2$ and $n \geq 1$ be integers. Let G be an n-connected graph, and u and v be a pair of nonadjacent vertices of G such that*

$$\deg_G(u) + \deg_G(v) \geq |G| - 1 - (k-2)n. \tag{1}$$

Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.

It is shown that the condition (1) on degree sum in Theorem 4 is best possible, that is, there exist n-connected graphs G such that $\deg_G(u) + \deg_G(v) = |G| - 2 - (k-2)n$ and $G + uv$ has a spanning k-tree but not G.

Notice that the above theorem is a generalization of Theorem 3 since a Hamilton path is a spanning 2-tree, and that for Hamilton path, the connectivity n of a graph does not contribute to the condition (1) on degree sum. The following Corollary 5 is Theorem 4 with $n = 1$, and Corollary 6 is immediate from Theorem 4 since the closure of a graph given in Corollary 6 becomes a complete graph, which has a spanning k-tree.

Corollary 5 *Let $k \geq 2$ be an integer and G be a connected graph. Let u and v be a pair of nonadjacent vertices of G such that $\deg_G(u) + \deg_G(v) \geq |G| - k + 1$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

Corollary 6 *Let $k \geq 2$ and $n \geq 1$ be integers, and G be an n-connected graph. If every pair of nonadjacent vertices u and v of G satisfies*

$$\deg_G(u) + \deg_G(v) \geq |G| - 1 - (k-2)n,$$

then G has a spanning k-tree.

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