Recent Advances on the $L(2,1)$-labeling Problem

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Let G be an undirected graph. An $L(2,1)$-*labeling* of a graph G is an assignment f from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0, \ldots, k\}$, and the $L(2,1)$-*labeling problem* asks the minimum k among all possible assignments. We call this invariant, the minimum value k, the $L(2,1)$-*labeling number* and is denoted by $\lambda(G)$. Notice that we can use $k+1$ different labels when $\lambda(G) = k$ since we can use 0 as a label for conventional reasons.

The original notion of $L(2,1)$-labeling can be seen in the context of frequency assignment, where 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart so that they can avoid interference. Due to its practical importance, the $L(2,1)$-labeling problem has been widely studied. From the structural graph theoretical point of view, since this is a kind of vertex coloring problem, it has attracted a lot of interest [4, 10, 13, 16]. Especially, deriving upper bounds on $\lambda(G)$ for general graphs is one of the main concerns. Griggs and Yeh [10] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and they proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ at the same time. After that, it was shown that $\lambda(G) \leq \Delta^2 + \Delta$ by Chang and Kuo [4], $\lambda(G) \leq \Delta^2 + \Delta - 1$ by Král' and Škrekovski [14], and then $\lambda(G) \leq \Delta^2 + \Delta - 2$ by Gonçalves [9], however, the conjecture is still open. In this context, $L(2,1)$-labeling is generalized into $L(p,q)$-labeling for arbitrary nonnegative integers p and q, and in fact, we can see that $L(1,0)$-labeling ($L(p,0)$-labeling, actually) is equivalent to the classical vertex coloring. We can find a lot of related results on $L(p,q)$-labelings in comprehensive surveys by Calamoneri [2] and by Yeh [17].

There are also a number of studies on the $L(2,1)$-labeling problem from the algorithmic point of view [1, 8, 15]. It is known to be NP-hard for general graphs [10], and it still remains NP-hard for some restricted classes of graphs, such as planar graphs, bipartite graphs, chordal graphs [1], and it turned out to be NP-hard even for graphs of treewidth 2 [5]. In contrast, only a few graph classes are known to have polynomial time algorithms for this problem, e.g., we can determine the $L(2,1)$-labeling number of paths, cycles, wheels within polynomial time [10].

As for trees, Griggs and Yeh [10] showed that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$ for any tree T, and also conjectured that determining $\lambda(T)$ is NP-hard, however, Chang and Kuo [4] disproved this by presenting a polynomial time algorithm for computing $\lambda(T)$. Their algorithm exploits the fact that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$ for any tree T. Its running time is $O(\Delta^{4.5} n)$, where $n = |V(T)|$. This result has a great importance because it initiates to cultivate polynomially solvable classes of graphs for the $L(2,1)$-labeling problem and related problems. For example, Fiala, Kloks and Kratochvíl [8] showed that $L(2,1)$-labeling of t-almost trees can be solved in $O(\lambda^{2t+4.5} n)$ time for λ given as an input, where a t-almost tree is a graph that can be a tree by eliminating t edges. Also, it was shown that the $L(p,1)$-labeling problem for trees can be solved in $O((p+\Delta)^{5.5} n) = O(\lambda^{5.5} n)$ time [3]. Both results are based on Chang and Kuo's algorithm, which is called as a subroutine in the algorithms. Moreover, the polynomially solvable result for trees holds for more general settings. The notion of $L(p,1)$-labeling is generalized as $H(p,1)$-labeling, in which graph H defines the metric space of distances between two labels, whereas labels in $L(p,1)$-labeling (that is, in $L(p,q)$-labeling) take nonnegative integers; i.e., it is a special case that H is a path graph. In [6], it has been shown that the $H(p,1)$-labeling problem of trees for arbitrary graph H can be solved in polynomial time, which is also based on Chang and Kuo's idea. In passing, these results are unfortunately not applicable for $L(p,q)$-labeling problems for general p and q. Recently, Fiala, Golovach and Kratochvíl [7] showed that the $L(p,q)$-labeling problem for trees is NP-hard if q is not a divisor of p, which is contrasting to the positive results mentioned above.

As for $L(2,1)$-labeling of trees again, Chang and Kuo's $O(\Delta^{4.5}n)$ algorithm is the first polynomial time one. It is based on dynamic programming (DP) approach, and it checks whether $(\Delta+1)$-$L(2,1)$-labeling is possible or not from leaf vertices to a root vertex in the original tree structure. The principle of optimality requires to solve at each vertex of the tree the assignments of labels to subtrees, and the assignments are formulated as the maximum matching in a certain bipartite graph. This running time is improved into $O(\min\{n^{1.75}, \Delta^{1.5}n\})$ [11], and very recently, a linear time algorithm has been established [12]. They are based on the similar DP framework to Chang and Kuo's algorithm, but achieve their efficiency by reducing heavy computation of bipartite matching in Chang and Kuo's and by using an amortized analysis. Particularly, the latter algorithm achieves the linear running time by best utilizing a nice property of labeling, called *label compatibility*. Since this property holds for more general labelings, say $L(p,1)$-labeling, the linear time algorithm for $L(2,1)$-labeling of trees can be extended to a linear time algorithm for $L(p,1)$-labeling of trees for a fixed positive integer p.

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