A closest vector problem arising in radiation therapy planning

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1 Introduction

We deal with the following *Closest Vector Problem (CVP)*. Recall that the ℓ_1-*norm* of a vector $\mathbf{x} \in \mathbb{R}^d$ is defined by $\|\mathbf{x}\|_1 := \sum_{i=1}^d |x_i|$, and that $\mathbf{x}$ is said to be *binary* if $x_i \in \{0,1\}$ for all $i \in [d] := \{1,2,\ldots,d\}$. The CVP is stated as follows:

Input: A set $\mathcal{G} = \{\mathbf{g}_1,\ldots,\mathbf{g}_k\}$ of binary vectors in $\mathbb{R}^d$ (the *generators*) and a vector $\mathbf{a}$ in $\mathbb{R}^d$ (the *target* vector).

Goal: Find coefficients $u_j \in \mathbb{Z}_+$ for $j \in [k]$ such that $||\mathbf{a}-\mathbf{b}||_1$ is minimum, where $\mathbf{b} := \sum_{j=1}^k u_j\,\mathbf{g}_j$.

Measure: The *total change* TC $:= ||\mathbf{a}-\mathbf{b}||_1$.

The closest vector problem that is studied here differs significantly from the well-studied closest vector problem on a *lattice*. In the latter problem, the integer coefficients u_j are not assumed to be nonnegative and $\mathcal{G}$ is any set of linearly independent vectors in $\mathbb{R}^d$ (in particular, $k \leqslant d$).
The CVP arises in radiation therapy planning, where matrix decomposition problems that can be reduced to the CVP play an important role. A given nonnegative integer intensity matrix A of size $m\times n$ has to be decomposed into a nonnegative integer linear combination of 0-1-matrices that correspond to field shapes to be delivered by so called multileaf collimators [1, 3, 4]. These devices can cover parts of the irradiation field in each row of the matrix by shifting metal leaf pairs from left and right towards each other. Basically, a *segment* is a 0-1-matrix that satisfies the consecutive ones property in each row, i.e., an $m \times n$-matrix $S = (s_{ij})$ is called a segment, if there are integral intervals $[\ell_i, r_i]$ for all $i \in [m]$ ($[\ell_i, r_i] = \varnothing$ is possible and modeled by $\ell_i = r_i + 1$) such that

$$s_{ij} = \begin{cases} 1 & \text{if } \ell_i \le j \le r_i \\ 0 & \text{otherwise.} \end{cases}$$

Mostly, because of technical and dosimetric constraints in the planning process, only a subset $\mathcal{S}$ of all these matrices is allowed. If $\mathcal{S} = \{S_1,\ldots,S_k\}$ is our set of feasible segments, our aim is to find an approximation B such that there exists a decomposition $B = \sum_{j=1}^k u_j S_j$ with nonnegative integral coefficients u_j and $\|A-B\|_1 := \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} - b_{ij}|$ is minimum.

2 Main results

Theorem 1. *The restriction of CVP to instances where the set $\mathcal{G}$ of generators is formed by vectors satisfying the consecutive ones property can be solved in polynomial time.*

Sketch of the proof. In this proof, generators are denoted by $\mathbf{g}_{\ell,r}$ where $[\ell, r]$ is the interval of ones of the generator, i.e. , $\mathbf{g} = \mathbf{g}_{\ell,r}$ iff $g_i = 1$ for $i \in [\ell, r]$ and 0 otherwise. Let $\mathcal{I}$ be the set intervals such that $\mathcal{G} = \{\mathbf{g}_{\ell,r} \mid [\ell,r] \in \mathcal{I}\}$. We show that there is a cost-preserving bijection between optimal solutions of CVP and minimum cost flows in the network $D = (V, A)$ with the set of nodes $V := \{1,2,\ldots,d+1\}$ and the set of arcs $A := \big\{(j, j+1) \mid j \in [d]\big\} \cup \big\{(j+1, j) \mid j \in [d]\big\} \cup \big\{(\ell, r+1) \mid [\ell,r] \in \mathcal{I}\big\}$ and demands of the nodes defined by

$$d(j) = a_{j-1} - a_j \text{ for all } j \in V.$$

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The capacity of the arcs is not restricted and the costs are 1 for all arcs of type $(j, j+1)$ and $(j+1, j)$ and 0 for the other arcs. An example of the network is shown in Figure 1. □

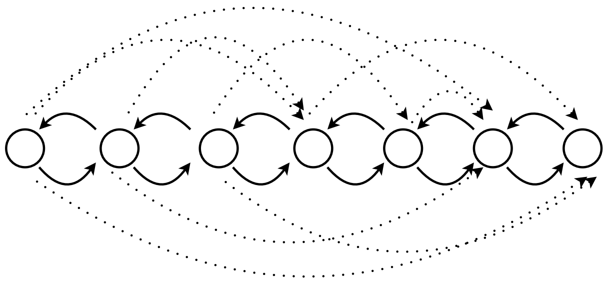

Figure 1: The network for an instance with $d = 6$ and $k = 9$.

For the general CVP, we prove the following hardness result:

Theorem 2. *For all $\varepsilon > 0$ there exists no polynomial-time approximation algorithm for CVP with an additive error of at most $(\ln 2 - \varepsilon)d$, unless $P = NP$.*

Sketch of the proof. We obtain a gap-introducing reduction from the 3SAT problem to the CVP by combining the PCP theorem and an approach from [5] (see also [2]). □

The CVP can be formulated as integer linear programming problem. We propose a randomized approximation algorithm for the CVP that consists in solving the relaxation of this ILP and rounding in a randomized way the optimal solution of the relaxed problem.

Theorem 3. *The randomized rounding algorithm approximates the CVP with an expected additive error of at most $\left(\frac{e}{4} + \frac{\ln 2}{2}\right) d^{3/2} \approx 1.026\ d^{3/2}$.*

3 Application to radiation therapy planning

For the application in radiation therapy, we consider the following CVP: Given an $m \times n$ input matrix A and a number of allowed segments $\mathcal{S} = \{S_1, \ldots, S_k\}$, find nonnegative integers u_j such that for $B = \sum_{j=1}^{k} u_j S_j$ the total change $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} - b_{ij}|$ is minimum. This is indeed an instance of the CVP, as each matrix can be considered as a vector in $\mathbb{R}^{mn}$ (with mn components).

Theorem 4. *There exists some $\epsilon > 0$ such that it is NP-hard to approximate the CVP, restricted to $2 \times n$ matrices and generators with their ones consecutive on each row, with an additive error of at most $\epsilon \cdot n$, unless $P = NP$.*

As an example, we consider the minimum separation constraint that arises in clinical application if one requires minimum leaf openings. Let $\lambda \in [n]$. This constraint requires that the rows which are not totally closed have a leaf opening of at least λ. Mathematically, the leaf positions of open rows have to satisfy $r_i - \ell_i \geq \lambda - 1$ for all $i \in [m]$. We cannot decompose any matrix A under this constraint. Indeed, the single row matrix $A = (1\ 1\ 4\ 1\ 1)$ cannot be decomposed for $\lambda = 3$.
The associated decision problem was previously studied by Kamath et al (see [6]) and they proved that this problem is polynomial. Obviously, the minimum separation constraint is a restriction on the leaf openings in each row, but does not affect the combination of leaf openings in different rows. Thus, it is sufficient to solve the CVP for single rows in this case and we know from Theorem 1 that this can be achieved in polynomial time.

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