Factorization of Generalized de Bruijn and Kautz Digraphs by Loop-Rooted Trees

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1 Abstract

Generalized de Bruijn digraphs and Kautz digraphs are known to have rich structures such as a small diameter, high connectivity, and other properties. Because of these properties, de Bruijn and Kautz digraphs have been studied as the models for interconnection networks. In this paper, we present several factorizations of these digraphs based on loops. We provide a necessary and sufficient condition so that these digraphs are factorable into loop-rooted trees. For $G_B(d,n)$, we prove that if $G_B(d,n)$ satisfies certain conditions, it can be isomorphically factorized into loop-rooted trees. Moreover, in case where $G_B(d,n)$ and $G_K(d,n)$ are not factorable into loop-rooted trees, we obtain another isomorphic factorization of these digraphs. In this extended abstract, we present only for generalized de Bruijn digraphs.

2 Preliminaries

We deal with simple digraphs which admit loops but multiple arcs. We name the vertex having a loop as the *loop vertex*. Let G be a digraph. We denote the vertex set of G by $V(G)$ and its arc set by $A(G)$.

If $G_1, G_2, \ldots, G_t$ are the *factors* of G such that these factors are pairwise arc disjoint and $\bigcup_{i=1}^{t} A(G_i) = A(G)$, G is said to be *factorable* into $G_1, G_2, \ldots, G_t$. If G is factorable into $G_1, G_2, \ldots, G_t$, we represent this by $G_1 \oplus G_2 \oplus \cdots \oplus G_t$, which is called a *factorization* of G. In particular, if G is factorable into $G_1, G_2, \ldots, G_t$ such that $G_i \cong H$ for some digraph H, we say that G has an *isomorphic factorization* into the factor H.

Let d, n be two positive integers. The *generalized de Bruijn digraph* $G_B(d,n)$ and the *generalized Kautz digraph* $G_K(d,n)$ have n vertices, labeled by integers modulo n. The vertex x in $G_B(d,n)$ is adjacent to vertices $y \equiv dx + r \pmod{n}$ with $0 \le r \le d-1$, and the vertex x in $G_K(d,n)$ is adjacent to vertices $y \equiv -d(x+1) + r \pmod{n}$ with $0 \le r \le d-1$. In this paper, we will deal with d and n satisfying $2 \le d \le n$.

A *cycle-rooted tree* is defined by [2] as follows.

Definition 1 [2] A digraph G is a cycle-rooted tree if and only if G is weakly connected and every vertex of G has indegree one.

A cycle-rooted tree has the only one cycle. We call this cycle the *root-cycle* and whose vertices are called *cycle vertices*. A leaf of a cycle-rooted tree is a vertex of outdegree 0. A cycle-rooted tree is a *complete d-ary* cycle-rooted tree of height h if all leaves have depth h and all vertices except leaves have outdegree d. In this paper, we name a cycle-rooted tree whose root-cycle is a loop as a *loop-rooted tree*. Figure 1 shows an example of a cycle-rooted tree and a loop-rooted tree.

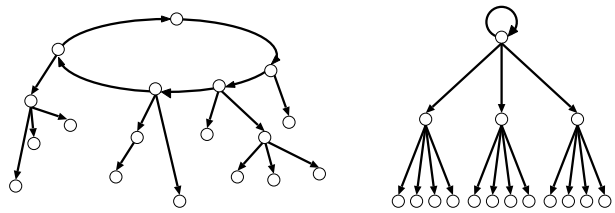

Figure 1: an example of a cycle-rooted tree and a loop-rooted tree

3 The number of loops in $G_B(d,n)$

For the generalized de Bruijn digraph $G_B(d,n)$, the number of closed walks is obtained in [3].

Theorem 1 [3] The total number of closed walks of length l in $G_B(d,n)$ is given by $d^l - 1 + g_l$, where $g_l = \gcd(d^l - 1, n)$.

Let $g = \gcd(d-1, n)$. From Theorem 1, the number of closed walks of length 1, namely, the number of loops is obtained by $d - 1 + g$. This result is equivalemt to the result shown in [1]. Let $d - 1 = gs_g$, $n = gt_g$. Then, the value of each loop vertex is shown as follows.

Theorem 2 For $G_B(d,n)$, a loop vertex u is given by

$$u \equiv yt_g + x\frac{\alpha t_g - 1}{s_g} \pmod{n},\ 0 \le y < g,\ 0 \le x \le s_g,$$

where α is the minimum positive integer satisfying $\alpha t_g \equiv 1 \pmod{s_g}$.

4 Factorizations into loop-rooted trees

We show a necessary and sufficient condition so that $G_B(d,n)$ is factorable into loop-rooted trees.

From Theorem 1, the total number of loop vertices in $G_B(d,n)$ is d when $g = 1$. For the case, let d loop vertices be $l_0, l_1, \ldots, l_{d-1}$ ($l_0 < l_1 < \cdots < l_{d-1}$), and let d vertices adjacent to 0 be $x_0, x_1, \ldots, x_{d-1}$, ($x_0 < x_1 < \cdots < x_{d-1}$). Then, the next property holds.

Lemma 1 For $G_B(d,n)$, $x_0 = l_0 = 0$, $x_{d-1} \le l_{d-1} = n-1$ and, $x_i \le l_i < x_{i+1}$, where $0 < i < d-1$.

Proof: It is clear that $x_0 = l_0 = 0$, $x_{d-1} \le l_{d-1} = n-1$. We show $x_i \le l_i < x_{i+1}$ for $0 < i < d-1$. Assume that any vertex between x_i and x_{i+1} has no loop.

Let v be a vertex in the given range adjacent to a vertex whose value is larger than or equal to v. Then, some vertex between x_i and v has a loop since x_i is adjacent to 0 and each vertex is adjacent to consecutive d vertices. Thus, any vertex in the range is not adjacent to a vertex whose value is larger than or equal to that of the predecessor under the assumption. However, since x_{i+1} is adjacent to 0, either x_{i+1} or $x_{i+1}-1$ is adjacent to $n-1$. Each vertex in $G_B(d,n)$ has a value less than n, therefore at least one loop vertex definitely exists in the range, which deduces a contradiction. Moreover, since such a vertex exists for all i, only one loop vertex exists between x_i and x_{i+1} from the pigeon hole principle. □

Using these conditions, we give a method to factorize $G_B(d,n)$ into loop-rooted trees.

Theorem 3 Let $g = \gcd(d-1,n)$. Then, $G_B(d,n)$ is factorable into loop-rooted trees if and only if $g = 1$.

5 Isomorphic factorization into loop-rooted trees

In this section, we describe an isomorphic factorization of $G_B(d,n)$ into loop-rooted trees.

For integers d,n such that $d^p \le n < d^{p+1}$, the height of the complete d-ary loop-rooted tree embedded in $G_B(d,n)$ is at most p. We show that $G_B(d,n)$ contains complete d-ary loop-rooted trees of the height p when d and n have the following relation.

Theorem 4 Let d and n be integers satisfying $d-1|n-1$ and p be an integer satisfying $d^p \le n < d^{p+1}$. Then, $G_B(d,n)$ contains a complete d-ary loop-rooted tree of the height p with the root at each loop.

We represent the complete d-ary loop-rooted tree of height m whose root-cycle is l_x as x-CRT_m. Let V_x^i be the set of vertices whose distance from l_x is i. Then, from Theorem 4, we obtain

$$\begin{aligned} &V(x\text{-}CRT_m) = \textstyle\bigcup_{i=0}^{m} V_x^i \\ &= \left\{ l_x + k \,\middle|\, -x\textstyle\sum_{i=0}^{m-1} d^i \le k \le d^m - 1 - x\sum_{i=0}^{m-1} d^i \right\}, \\ &A(x\text{-}CRT_m) = \left\{ a(v;k) \,\middle|\, v \in \textstyle\bigcup_{i=0}^{m-1} V_x^i,\ 0 \le k \le d-1 \right\}. \end{aligned}$$

This means that $A(x\text{-}CRT_m)$ contains only the arcs incident from the vertex of $\bigcup_{i=0}^{m-1} V_x^i$. Thus, the relation between two distinct x-CRT_m, y-CRT_m is as follows.

Corollary 1 Let d and n be integers satisfying $d-1|n-1$ and p be an integer satisfying $d^p \le n < d^{p+1}$. Then, for $0 \le x, y \le d-1$, $x \ne y$, $(\bigcup_{i=0}^{p-1} V_x^i) \cap (\bigcup_{j=0}^{p-1} V_y^j) = \emptyset$.

For any distinct two integers $0 \le x_1, x_2 \le d-1$, x_1-CRT_p and x_2-CRT_p are arc disjoint. In addition, since x-CRT_p is a complete d-ary loop-rooted tree, all vertices except leaves have d arcs incident to distinct d vertices. Therefore, the rest of the arcs in $A(G_B(d,n))$, namely $A(G_B(d,n)) - \bigcup_{i=0}^{d-1} A(x\text{-}CRT_m)$, are incident from vertices which are not included in any x-CRT_{m-1}. Thus, we can obtain an isomorphic factorization of $G_B(d,n)$ into loop-rooted trees if the results of the addition of remaining arcs to each x-CRT_p are isomorphic. Indeed, we provide an isomorphic factorization of some $G_B(d,n)$ in the following theorem.

Theorem 5 Let $d,n \ge 2$ be integers satisfying $n = d^q(d^p + c(d-1))$, $(p \ge 2, q \ge 0)$. If $c|d$ and $c \le d^{p-2}$, then $G_B(d,n)$ is isomorphically factorable into loop-rooted trees of height $q+p+1$.

6 Isomorphic factorization into the digraphs with multiple loops

In this section, we give an isomorphic factorization of $G_B(d,n)$ with $g > 1$. In such cases, it is impossible to factorize these digraphs into loop-rooted trees. Since there is no isomorphism which maps a loop vertex to a vertex without a loop, $G_B(d,n)$ and $G_K(d,n)$ are isomorphically factorable only if each factor has m loops, where m is a divisor of the number of loops in the based digraph.

For $G_B(d,n)$, we give the following automorphism.

Theorem 6 Let $d,n \ge 2$ be integers and let $g = \gcd(d-1,n)$, $d-1 = gs_g$, $n = gt_g$. Then, $\phi_i(v) = v + it_g \pmod n$, $0 \le i \le g-1$, is an automorphism of $G_B(d,n)$.

By using these automorphisms, we show that $G_B(d,n)$ and $G_K(d,n)$ are isomorphically factorable into g or h factors, respectively.

Theorem 7 Let $A_i = \{a(v;r) | v = it_g, it_g+1, \ldots, (i+1)t_g - 1, r = 0,1,\ldots,d-1\}$, $(0 \le i \le g-1)$ be arc subsets of $G_B(d,n)$. Then, $G_B(d,n)$ is isomorphically factorized into $\langle A_i \rangle$.

Proof: For any vertex v in $G_B(d,n)$ there is a vertex $v - it_g \pmod n$ such that $\phi_i(v - it_g) = v$. Thus, ϕ_i is a surjection. On the other hand, if $\phi_i(v) \equiv \phi_i(v') \pmod n$ for two distinct values v and v', then the congruence equation $v + it_g \equiv v' + it_g \pmod n$, that is, $v \equiv v' \pmod n$ is given. Therefore, ϕ_i is an injection and a bijection. We show next that $\phi_i(v)$ preserves adjacency. The vertices adjacent from v are represented as $dv + r \pmod n$, $(r = 0,1,\ldots,d-1)$. Similarly, the vertices adjacent from $\phi_i(v) = v + it_g \pmod n$ are represented as

$$dv + dit_g + r \quad = \quad dv + (d-1)it_g + it_g + r.$$

Since $d-1 = gs_g$, we obtain $(d-1)it_g = gs_g it_g = s_g in$ and,

$$\begin{aligned} dv + (d-1)it_g + it_g + r &\equiv dv + it + r \pmod n \\ &= \phi_i(dv + r), \end{aligned}$$

which preserves adjacency. Thus, ϕ_i is an automorphism of $G_B(d,n)$. □

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